Lecture 13

Heterogeneous Workers and Sorting

7/21

Goals

- This is the final lecture.
- Today, we shall introduce **worker heterogeneity** and wage bargaining into the Melitz model of firm heterogeneity.

Heterogeneous Firms and Heterogeneous Workers

Helpman, Itskhoki, and Redding, “Inequality and Unemployment in a Global Economy." *Econometrica*, 2010.

Heterogeneous Firms

- Aggregate output is the same as in Melitz:

$$Y = \left(\int_{z \in Z} y(z)^{\frac{\sigma-1}{\sigma}} dz\right)^{\frac{\sigma}{\sigma-1}}$$

- Production is **with scale effects**.
- Similarly,

$$P = \left[\int_{z \in Z} p(j)^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}$$

- RHS is the **price index** of the intermediate goods.

Heterogeneous Firms

- The input demand satisfies

$$\frac{y(z)}{Y} = \left[\frac{p(z)}{P}\right]^{-\sigma}$$

- Aggregate output in terms of ρ:

$$Y = \left(\int_{z \in Z} y(z)^{\rho} dz\right)^{\frac{1}{\rho}}$$

- It is nice to remember that

$$markup = \frac{1}{\rho}$$

Heterogeneous Workers

- Worker ability a is distributed, and is drawn from a Pareto distribution on support $[a_{\min}, \infty)$:

$$A(a) = 1 - \left(\frac{a_{\min}}{a}\right)^{\eta}$$

- Worker ability is **partially** observable through **screening**: Each firm can identify whether a candidate's ability is above c or not.
- Screening cost is convex ($s_0 > 0, \tau > 1$):

$$s(c) = s_0 \frac{c^{\tau}}{\tau}$$

Heterogeneous Workers

- Let x be the measure of candidates. Then, employment of each firm satisfies

$$\ell = [1 - A(c)]x = \left(\frac{a_{\min}}{c}\right)^{\eta} x$$

- From the property of Pareto distribution, the average productivity of its employees is

$$\bar{a} = \frac{\eta c}{\eta - 1}$$

Production

- Production technology:

$$y = \varphi \bar{a} \ell^{\alpha}$$

- $0 < \alpha \leq 1$. φ is distributed.
- Thus, output is given by

$$y = \varphi \frac{\eta c}{\eta - 1}\left(\left(\frac{a_{\min}}{c}\right)^{\eta} x\right)^{\alpha}$$
$$= \varphi \kappa x^{\alpha} c^{1-\eta\alpha}$$

- $\kappa = \frac{\eta}{\eta-1}(a_{\min})^{\eta\alpha}$

Revenue

- Given ℓ, a firm's revenue is

$$\begin{aligned} py &= PY^{\frac{1}{\sigma}} y^{1-\frac{1}{\sigma}} \\ &= PY^{1-\rho} y^{\rho} \\ &= PY^{1-\rho} [\varphi \bar{a} \ell^{\alpha}]^{\rho} \\ &= PY^{1-\rho} [\varphi \bar{a}]^{\rho} \ell^{\alpha\rho} \\ &= \Phi \ell^{\alpha\rho} \\ &\equiv R(\ell, \varphi) \end{aligned}$$

- $\Phi = PY^{1-\rho}[\varphi \bar{a}]^{\rho}$

(Static) Labor Market Frictions

- u is the measure of job seekers, V is the aggregate vacancies. Let $m > 0$ be a parameter.
- Matching function: $mu^{\xi}V^{1-\xi}$
- Job-finding rate:

$$\frac{mu^{\xi}V^{1-\xi}}{u} = m\left(\frac{V}{u}\right)^{1-\xi} = m\theta^{1-\xi}$$

- Vacancy-filling rate:

$$\frac{mu^{\xi}V^{1-\xi}}{V} = m\left(\frac{V}{u}\right)^{-\xi} = m\theta^{-\xi}$$

(Static) Labor Market Frictions

- If the firm opens v units of vacancies, then,
  - Number of candidates $= m\theta^{-\xi}v$
  - Vacancy costs $= \gamma v$
- To find x candidates, the firm must post $m^{-1}\theta^{\xi}x$ units of vacancies.
- Thus, the cost of finding x candidates will be

$$\gamma m^{-1}\theta^{\xi}x = h(\theta)x$$

Wage Bargaining

- Given φ, x, c, and θ, the value of the firm (as a function of its size) is
$$J(\ell) = R(\ell) - w(\ell)\ell - s(c) - f - h(\theta)x$$
- Nash bargaining outcome in the DMP model:
$$\beta[J - V] = (1 - \beta)[W - U].$$
- One firm versus **many workers**:
$$\beta J'(\ell) = (1 - \beta)[W - U]$$

Wage Bargaining

- In a dynamic model,

$$W = w(\ell) + \lambda\delta U + (1-\lambda)\delta W$$
$$U = b + \theta q(\theta)\delta W + [1-\theta q(\theta)]\delta U$$

- We instead assume:

$$W = w(\ell)$$
$$U\text{: exogenous}$$

- Thus, the wage rate satisfies

$$\beta J'(\ell) = (1-\beta)[w(\ell) - U]$$

Wage Bargaining

- Thus,

$$\frac{\partial}{\partial \ell}[R(\ell) - w(\ell)\ell] = \frac{1-\beta}{\beta}[w(\ell) - U]$$

- Or,

$$\frac{\partial}{\partial \ell}[\Phi \ell^{\alpha\rho} - w(\ell)\ell] = \frac{1-\beta}{\beta}[w(\ell) - U]$$

- Thus,

$$\Phi\alpha\rho\ell^{\alpha\rho-1} - w(\ell) - w'(\ell)\ell = \frac{1-\beta}{\beta}w(\ell) - \frac{1-\beta}{\beta}U$$

$$\Leftrightarrow w'(\ell)\ell + \frac{1}{\beta}w(\ell) = \Phi\alpha\rho\ell^{\alpha\rho-1} + \frac{1-\beta}{\beta}U$$

Solving ODE

- Observe that

$$\frac{d\left[w(\ell)\ell^{\frac{1}{\beta}}\right]}{d\ell} = w'(\ell)\ell^{\frac{1}{\beta}} + \frac{1}{\beta}w(\ell)\ell^{\frac{1}{\beta}-1}$$

$$= \left[w'(\ell)\ell + \frac{1}{\beta}w(\ell)\right]\ell^{\frac{1}{\beta}-1}$$

$$= \left[\Phi\alpha\rho\ell^{\alpha\rho-1} + \frac{1-\beta}{\beta}U\right]\ell^{\frac{1}{\beta}-1}$$

$$= \Phi\alpha\rho\ell^{\alpha\rho-1+\frac{1}{\beta}-1} + \frac{1-\beta}{\beta}U\ell^{\frac{1}{\beta}-1}$$

Solving ODE

- Consider

$$\frac{d\left[w(\ell)\ell^{\frac{1}{\beta}}\right]}{d\ell} = \Phi\alpha\rho\ell^{\alpha\rho-1+\frac{1}{\beta}-1} + \frac{1-\beta}{\beta}U\ell^{\frac{1}{\beta}-1}$$

- Integrate LHS:

$$\int_0^\ell \frac{d\left[w(i)i^{\frac{1}{\beta}}\right]}{di}di = w(\ell)\ell^{\frac{1}{\beta}} - \lim_{\ell\to 0} w(\ell)\ell^{\frac{1}{\beta}}$$

$$= w(\ell)\ell^{\frac{1}{\beta}}$$

Solving ODE

- Integrate RHS:

$$\int_0^{\ell}\left\{\Phi\alpha\rho i^{\alpha\rho-1+\frac{1}{\beta}-1}+\frac{1-\beta}{\beta}Ui^{\frac{1}{\beta}-1}\right\}di$$
$$=\frac{\Phi\alpha\rho}{\alpha\rho-1+\frac{1}{\beta}}\ell^{\alpha\rho-1+\frac{1}{\beta}}+(1-\beta)U\ell^{\frac{1}{\beta}}$$

Solving ODE

- Finally, we obtain
$$w(\ell) = \frac{\beta\alpha\rho}{\beta\alpha\rho - \beta + 1}\Phi\ell^{\alpha\rho-1} + (1-\beta)U$$
- When $\alpha = 1$ (linear production),
$$w(\ell) = \beta\frac{\sigma - 1}{\sigma - \beta}\Phi\ell^{\alpha\rho-1} + (1-\beta)U$$
  - Ebell and Haefke:
$$w(\ell) = \beta\frac{\sigma - 1}{\sigma - \beta}\Phi\ell^{\frac{-1}{\sigma}} + (1-\beta)b + \beta c\theta$$
  - DMP: $w = \beta y + (1-\beta)b + \beta c\theta$

Wage Equation

- Because the revenue is $R(\ell,\varphi) = \Phi\ell^{\alpha\rho}$, we obtain
$$w(\ell) = \frac{\beta\alpha\rho}{\beta\alpha\rho + 1 - \beta}\frac{R(\ell,\varphi)}{\ell} + (1-\beta)U$$
- The proportion of the revenue that the worker receives is
$$\frac{\beta\alpha\rho}{\beta\alpha\rho + 1 - \beta}$$
- An increase in markup $\frac{1}{\rho}$ reduces the share of revenue for the worker.

Wage Equation in HIR

- HIR assume for simplicity that $U = 0$ and $\beta = \frac{1}{2}$. Then, the wage equation becomes

$$w(\ell) = \frac{\alpha\rho}{1 + \alpha\rho} \frac{R(\ell, \varphi)}{\ell}$$

- Thus,

$$w(\ell)\ell = \frac{\alpha\rho}{1 + \alpha\rho} R(\ell, \varphi)$$

- In what follows, we shall adopt the same assumption.

The Value of a Firm

- Given φ, x, c, and θ, the value of the firm is

$$\begin{aligned} J(\ell) &= R(\ell) - w(\ell)\ell - s(c) - f - h(\theta)x \\ &= \frac{1}{1+\alpha\rho} R(\ell, \varphi) - s(c) - f - h(\theta)x \\ &= \frac{PY^{1-\rho}[\varphi\bar{a}]^{\rho}}{1+\alpha\rho} \ell^{\alpha\rho} - s(c) - f - h(\theta)x \\ &= \frac{PY^{1-\rho}}{1+\alpha\rho} y^{\rho} - s(c) - f - h(\theta)x \end{aligned}$$

- $y = \varphi\kappa x^{\alpha} c^{1-\eta\alpha},\ s(c) = s_0 \frac{c^{\tau}}{\tau}$

Screening Decision

- The firm's problem:

$$\max_{c,x} \frac{PY^{1-\rho}}{1+\alpha\rho}(\varphi\kappa x^{\alpha}c^{1-\eta\alpha})^{\rho} - s_0\frac{c^{\tau}}{\tau} - f - h(\theta)x$$

- For screening to be profitable, we impose

$$1-\eta\alpha > 0$$

- Because (from the property of Pareto) $\eta > 1$, it must be that the production function is sufficiently concave:

$$\alpha < \frac{1}{\eta}$$

Screening Decision

- The firm's problem:

$$\max_{c,x} \frac{PY^{1-\rho}}{1+\alpha\rho}(\varphi\kappa x^{\alpha}c^{1-\eta\alpha})^{\rho} - s_0\frac{c^{\tau}}{\tau} - f - h(\theta)x$$

- FOC:

$$\frac{\alpha\rho}{1+\alpha\rho}R(\varphi) = h(\theta)x(\varphi)$$

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho}R(\varphi) = s_0 c(\varphi)^{\tau}$$

Wage Equation

- Thus,
$$w(\varphi) = \frac{\alpha\rho}{1+\alpha\rho}\frac{R(\ell,\varphi)}{\ell(\varphi)} = \frac{h(\theta)x(\varphi)}{\ell(\varphi)}$$
- Note $\ell = [1 - A(c)]x = \left(\frac{a_{\min}}{c}\right)^{\eta} x$. Thus,
$$w(\varphi) = \frac{h(\theta)x(\varphi)}{\ell(\varphi)} = h(\theta)\left(\frac{c(\varphi)}{a_{\min}}\right)^{\eta}$$
- The wage rate is increasing in $c(\varphi)$.

Job Creation

- FOCs are

$$\frac{\alpha\rho}{1+\alpha\rho}R(\varphi) = h(\theta)x(\varphi)$$

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho}R(\varphi) = s_0 c(\varphi)^{\tau}$$

- Then, we find the relationship between c and x:

$$s_0 c(\varphi)^{\tau} = \frac{1-\eta\alpha}{\alpha}h(\theta)x(\varphi)$$

Revenue

- A firm's revenue is
$$R(\varphi) = PY^{1-\rho}[\varphi\bar{a}]^{\rho}\ell^{\alpha\rho}$$
$$= PY^{1-\rho}\left[\varphi\frac{\eta c}{\eta-1}\right]^{\rho}[(a_{\min})^{\eta}c^{-\eta}x]^{\alpha\rho}$$
$$= PY^{1-\rho}\left[\frac{\eta}{\eta-1}\right]^{\rho}(a_{\min})^{\eta\alpha\rho}c(\varphi)^{(1-\eta\alpha)\rho}x(\varphi)^{\alpha\rho}\varphi^{\rho}$$
$$= Kc(\varphi)^{(1-\eta\alpha)\rho}x(\varphi)^{\alpha\rho}\varphi^{\rho}$$
- $K = PY^{1-\rho}\left[\frac{\eta}{\eta-1}\right]^{\rho}(a_{\min})^{\eta\alpha\rho}$

Revenue

- Let us solve the FOCs for $x(\varphi)$ and $c(\varphi)$:

$$\frac{\alpha\rho}{1+\alpha\rho} K c(\varphi)^{(1-\eta\alpha)\rho} x(\varphi)^{\alpha\rho} \varphi^{\rho} = h(\theta) x(\varphi)$$

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho} K c(\varphi)^{(1-\eta\alpha)\rho} x(\varphi)^{\alpha\rho} \varphi^{\rho} = s_0 c(\varphi)^{\tau}$$

- Or

$$\frac{\alpha\rho}{1+\alpha\rho} \frac{K}{h(\theta)} c(\varphi)^{(1-\eta\alpha)\rho} \varphi^{\rho} = x(\varphi)^{1-\alpha\rho}$$

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho} K c(\varphi)^{(1-\eta\alpha)\rho-\tau} x(\varphi)^{\alpha\rho} \varphi^{\rho} = s_0$$

Revenue

- Consider

$$x(\varphi)^{\alpha\rho} = \left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}c(\varphi)^{(1-\eta\alpha)\rho}\varphi^{\rho}\right]^{\frac{\alpha\rho}{1-\alpha\rho}}$$

$$= c(\varphi)^{\frac{(1-\eta\alpha)\rho\alpha\rho}{1-\alpha\rho}}\left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{\alpha\rho}{1-\alpha\rho}}\varphi^{\frac{\alpha\rho\rho}{1-\alpha\rho}}$$

- Thus,

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho}Kc(\varphi)^{(1-\eta\alpha)\rho-\tau}x(\varphi)^{\alpha\rho}\varphi^{\rho} = s_0$$

Revenue

- Thus,

$$x(\varphi)^{\alpha\rho} = c(\varphi)^{\frac{(1-\eta\alpha)\rho\alpha\rho}{1-\alpha\rho}} \left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{\alpha\rho}{1-\alpha\rho}} \varphi^{\frac{\alpha\rho\rho}{1-\alpha\rho}}$$

- Thus,

$$\frac{\rho(1-\eta\alpha)}{1+\alpha\rho} Kc(\varphi)^{(1-\eta\alpha)\rho-\tau} c(\varphi)^{\frac{(1-\eta\alpha)\rho\alpha\rho}{1-\alpha\rho}}$$

$$\left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{\alpha\rho}{1-\alpha\rho}} \varphi^{\frac{\alpha\rho\rho}{1-\alpha\rho}} \varphi^{\rho} = s_0$$

Revenue

- Note that

$$c(\varphi)^{\frac{1-\eta\alpha}{1-\alpha\rho}\rho-\tau} = \frac{s_0\varphi^{-\frac{\alpha\rho\rho}{1-\alpha\rho}-\rho}}{\frac{\rho(1-\eta\alpha)}{1+\alpha\rho}K\left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{\alpha\rho}{1-\alpha\rho}}} = S(\theta)\varphi^{-\frac{\alpha\rho\rho}{1-\alpha\rho}-\rho}$$

- Therefore,

$$x(\varphi) = S(\theta)^{\frac{\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}}{\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}-\tau}}\left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{1}{1-\alpha\rho}}\varphi^{\frac{\rho}{1-\alpha\rho}}\varphi^{\frac{-\frac{\alpha\rho\rho}{1-\alpha\rho}-\rho}{\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}-\tau}\times\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}}$$

$$= X(\theta)\varphi^{\frac{\rho}{1-\alpha\rho-(1-\eta\alpha)\rho/\tau}}$$

- Thus,

$$x(\varphi) = X(\theta)\varphi^{\frac{\rho}{\Gamma}}, \qquad \Gamma = 1-\alpha\rho-(1-\eta\alpha)\rho/\tau$$

Revenue

- From

$$\frac{\alpha\rho}{1+\alpha\rho}R(\varphi)=h(\theta)x(\varphi)$$

$$x(\varphi)=X(\theta)\varphi^{\frac{\rho}{\Gamma}}$$

- We obtain

$$R(\varphi)=\frac{1+\alpha\rho}{\alpha\rho}h(\theta)X(\theta)\varphi^{\frac{\rho}{\Gamma}}$$

- We finally obtain R as a function of φ.

Productivity and Revenue

- From $R(\varphi) = \frac{1+\alpha\rho}{\alpha\rho} h(\theta)X(\theta)\varphi^{\frac{\rho}{\Gamma}}$, for firms with φ_1 and φ_2, we obtain

$$\frac{R(\varphi_1)}{R(\varphi_2)} = \left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\rho}{\Gamma}}$$

The Value of a Firm

- The value of the firm is

$$J(\varphi) = \frac{1}{1+\alpha\rho} R(\varphi) - s_0 \frac{c(\varphi)^\tau}{\tau} - f - h(\theta)x(\varphi)$$

$$= \frac{1}{1+\alpha\rho} R(\varphi) - \frac{1}{\delta}\frac{\rho(1-\eta\alpha)}{1+\alpha\rho} R(\varphi) - f - \frac{\alpha\rho}{1+\alpha\rho} R(\varphi)$$

$$= \frac{1-\alpha\rho-(1-\eta\alpha)\rho/\delta}{1+\alpha\rho} R(\varphi) - f$$

$$= \frac{\Gamma}{1+\alpha\rho} R(\varphi) - f$$

$$= \frac{\Gamma}{1+\alpha\rho}\frac{1+\alpha\rho}{\alpha\rho} h(\theta)X(\theta)\varphi^{\frac{\rho}{\Gamma}} - f$$

Firm Entry

- With the sudden death probability δ, free entry of firms implies

$$-F + \int_0^\infty \delta^{-1} J(\varphi) dG(\varphi) = 0$$

- The exit cutoff φ^* satisfies

$$J(\varphi^*) = 0$$

- We can close the model as in Melitz.

Firm Entry

- Thus, free entry implies

$$\delta F = \int_0^\infty J(\varphi) dG(\varphi)$$
$$= [1 - G(\varphi^*)] \int_{\varphi^*}^\infty J(\varphi) \frac{g(\varphi)}{1 - G(\varphi^*)} d\varphi$$

$\Leftrightarrow$

$$\frac{\delta F}{1 - G(\varphi^*)} = \int_{\varphi^*}^\infty J(\varphi) \mu(\varphi) d\varphi = \bar{\pi}$$

- As in Melitz, we have the **free entry** condition:

$$\bar{\pi} = \frac{\delta F}{1 - G(\varphi^*)}$$

Firm Entry

- For a firm at the exit cutoff,

$$J(\varphi^*) = 0 \Leftrightarrow \frac{\Gamma}{1+\alpha\rho} R(\varphi^*) = f$$

- Observe

$$\frac{R(\varphi)}{R(\varphi^*)} = \left(\frac{\varphi}{\varphi^*}\right)^{\frac{\rho}{\Gamma}} \Leftrightarrow R(\varphi) = \left(\frac{\varphi}{\varphi^*}\right)^{\frac{\rho}{\Gamma}} R(\varphi^*)$$

- Thus,

$$J(\varphi) = \frac{\Gamma}{1+\alpha\rho} R(\varphi) - f = f\left[\left(\frac{\varphi}{\varphi^*}\right)^{\frac{\rho}{\Gamma}} - 1\right]$$

Firm Entry

- Thus,

$$\bar{\pi} = \int_{\varphi^*}^{\infty} J(\varphi)\mu(\varphi)d\varphi$$
$$= f\int_{\varphi^*}^{\infty} \left[\left(\frac{\varphi}{\varphi^*}\right)^{\frac{\rho}{\Gamma}} - 1\right]\mu(\varphi)d\varphi$$

- Define the average productivity as

$$\tilde{\varphi} = \left[\frac{1}{1 - G(\varphi^*)}\int_{\varphi^*}^{\infty} \varphi^{\frac{\rho}{\Gamma}} g(\varphi)d\varphi\right]^{\frac{\Gamma}{\rho}} \equiv \tilde{\varphi}(\varphi^*)$$

Firm Entry

- Therefore, the **zero cutoff profit** condition is

$$\bar{\pi} = \int_{\varphi^*}^{\infty} J(\varphi)\mu(\varphi)d\varphi$$

$$= f\left[\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\frac{\rho}{\Gamma}} - 1\right]$$

- This is nearly identical to Melitz, but the shape of $\tilde{\varphi}(\varphi^*)$ is slightly different.

Wage Distribution

- Note

$$c(\varphi)^{\frac{1-\eta\alpha}{1-\alpha\rho}\rho-\tau} = S(\theta)\varphi^{-\frac{\alpha\rho\rho}{1-\alpha\rho}-\rho}$$

$$\Leftrightarrow c(\varphi) = S(\theta)^{\frac{1}{\frac{1-\eta\alpha}{1-\alpha\rho}\rho-\tau}}\varphi^{\frac{-\frac{\alpha\rho\rho}{1-\alpha\rho}-\rho}{\frac{1-\eta\alpha}{1-\alpha\rho}\rho-\tau}}$$

$$= S(\theta)^{\frac{1-\alpha\rho}{-\tau\Gamma}}\varphi^{\frac{\rho}{\tau\Gamma}}$$

- The wage equation

$$w(\varphi) = h(\theta)(a_{\min})^{-\eta}c(\varphi)^{\eta}$$

$$= h(\theta)(a_{\min})^{-\eta}S(\theta)^{\frac{1-\alpha\rho}{-\tau\Gamma}\eta}\varphi^{\frac{\rho\eta}{\tau\Gamma}} = H(\theta)\varphi^{\frac{\rho\eta}{\tau\Gamma}}$$

Wage Distribution

- The wage equation

$$w(\varphi) = H(\theta)\varphi^{\frac{\rho\eta}{\tau\Gamma}}$$

- Note that

$$\frac{\rho\eta}{\tau\Gamma} = \frac{\rho\eta}{\tau(1-\alpha\rho)-(1-\eta\alpha)\rho}$$

$$> 0 \Leftrightarrow \tau > \frac{1-\eta\alpha}{\frac{1}{\rho}-\alpha}$$

- In this case, $w'(\varphi) > 0$.

Wage Distribution

- In a model with heterogeneous firms and heterogeneous workers, **we finally found a wage distribution**.
- Mechanism: **a more productive firm pays a higher wage rate** because it pays a higher screening cost $c(\varphi)$ to select a better group of workers to raise the average productivity of the firm $\bar{a}\big(c(\varphi)\big)$.
  - This is an example of **sorting**: better firms hire better workers.

Labor Force Participation

- For any individual, the probability that you will be invited to an interview (similar to the job-finding rate) is $m\theta^{1-\xi}$.
- Given an interview, the probability that you pass the screening process is $1 - A(c) = \ell/x$.
- From $w(\varphi) = \frac{h(\theta)x(\varphi)}{\ell(\varphi)}$, the expected to returns to labor force participation is

$$m\theta^{1-\xi}[1 - A(c)]w + \left(1 - m\theta^{1-\xi}\right) \times 0$$
$$= m\theta^{1-\xi}h(\theta)$$

Labor Force Participation

- $b > 0$: Income for non-participants.
- In equilibrium, all (ex-ante homogeneous) individuals are indifferent between participation and non-participation. Thus,

$$m\theta^{1-\xi} h(\theta) = b$$
$$\Leftrightarrow m\theta^{1-\xi}\gamma m^{-1}\theta^{\xi} = b$$
$$\Leftrightarrow \theta = b/\gamma$$

Aggregate Variables

- Each firm posts $v(\varphi) = m^{-1}\theta^{\xi}x(\varphi)$ units of vacancies. At the aggregate,

$$
\begin{aligned}
V &= \int_{\varphi^*}^{\infty} v(\varphi)\mu(\varphi)d\varphi \\
&= \int_{\varphi^*}^{\infty} m^{-1}\theta^{\xi}x(\varphi)\mu(\varphi)d\varphi \\
&= \int_{\varphi^*}^{\infty} m^{-1}\theta^{\xi}X(\theta)\varphi^{\frac{\rho}{\Gamma}}\mu(\varphi)d\varphi \\
&= m^{-1}\theta^{\xi}X(\theta)[\tilde{\varphi}(\varphi^*)]^{\frac{\rho}{\Gamma}}
\end{aligned}
$$

Aggregate Variables

- Tightness satisfies

$$\theta = \frac{V}{u} = \frac{m^{-1}\theta^{\xi}X(\theta)[\tilde{\varphi}(\varphi^{*})]^{\frac{\rho}{\Gamma}}}{u}$$

- Because $\theta = b/\gamma$, the equilibrium unemployment rate is

$$\begin{aligned} u &= m^{-1}\theta^{\xi-1}X(\theta)[\tilde{\varphi}(\varphi^{*})]^{\frac{\rho}{\Gamma}} \\ &= m^{-1}(b/\gamma)^{\xi-1}X(b/\gamma)[\tilde{\varphi}(\varphi^{*})]^{\frac{\rho}{\Gamma}} \end{aligned}$$

Aggregate Variables

- From $x(\varphi) = X(\theta)\varphi^{\frac{\rho}{\Gamma}}$ and $c(\varphi) = S(\theta)^{\frac{1-\alpha\rho}{-\tau\Gamma}}\varphi^{\frac{\rho}{\tau\Gamma}}$,
  $$y(\varphi) = \varphi\kappa x(\varphi)^{\alpha} c(\varphi)^{1-\eta\alpha}$$
  $$= \kappa X(\theta)^{\alpha} S(\theta)^{\frac{(1-\eta\alpha)(1-\alpha\rho)}{-\tau\Gamma}} \varphi^{1+\frac{\alpha\rho}{\Gamma}+\frac{(1-\eta\alpha)\rho}{\tau\Gamma}}$$
- We substitute it into
  $$Y = \left(\int_{\varphi^*}^{\infty} y(\varphi)^{\rho}\mu(\varphi)d\varphi\right)^{\frac{1}{\rho}}$$
- Now, what are $X(\theta)$ and $S(\theta)$?

Aggregate Variables

- Remember:

$$X(\theta) = S(\theta)^{\frac{\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}}{\frac{(1-\eta\alpha)\rho}{1-\alpha\rho}-\tau}} \left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{1}{1-\alpha\rho}}$$

$$S(\theta) = \frac{s_0}{\frac{\rho(1-\eta\alpha)}{1+\alpha\rho} K \left[\frac{\alpha\rho}{1+\alpha\rho}\frac{K}{h(\theta)}\right]^{\frac{\alpha\rho}{1-\alpha\rho}}}$$

$$K = PY^{1-\rho}\left[\frac{\eta}{\eta-1}\right]^{\rho}(a_{\min})^{\eta\alpha\rho}$$

Equilibrium Wage Distribution

- After finding the equilibrium level of Y, we can find the level of $S(\theta)$.
  - I leave it to you as your summer activity.
- Then we can find

$$w(\varphi) = h(\theta)(a_{\min})^{-\eta} S(\theta)^{\frac{1-\alpha\rho}{-\tau\Gamma}\eta} \varphi^{\frac{\rho\eta}{\tau\Gamma}}$$

- As you can see here, wage distribution is a challenging field of research.

Equilibrium Sorting

Partner Formation

- Consider a general environment in which players search for their partners of lifetime.
  - Marriage?
  - Labor market?
- There are two types: Good and Bad.
- λ: proportion of agents that are good.
- Payoff to finding a partner is
  - x_G if your partner is good.
  - $x_B < x_G$ if your partner is bad.

Bellman Equations

- First, consider a type-G agent.
- The value of search satisfies the following continuous-time Bellman equation:

$$rV_G = \alpha\lambda\left(\frac{x_G}{r} - V_G\right) + \alpha(1-\lambda)\left[\max\left\{\frac{x_B}{r}, V_G\right\} - V_G\right]$$

- You are happy to accept type-B agent if and only if

$$\frac{x_B}{r} > V_G$$

- Suppose this is the case. Then,

$$rV_G = \alpha\lambda\left(\frac{x_G}{r} - V_G\right) + \alpha(1-\lambda)\left[\frac{x_B}{r} - V_G\right]$$

Bellman Equations

- Thus,

$$rV_G = \frac{\alpha}{r+\alpha}[\lambda x_G + (1-\lambda)x_B]$$

- We can rewrite the condition as

$$\frac{x_B}{r} > V_G \Leftrightarrow x_B > \frac{\alpha}{r+\alpha}[\lambda x_G + (1-\lambda)x_B]$$
$$\Leftrightarrow \frac{r}{\alpha} > \lambda \frac{x_G - x_B}{x_B}$$

- LHS captures the **degree of search frictions**:
  - High r : waiting for an opportunity is painful.
  - Low α : less likely to find an opportunity.

Bellman Equations

- Consider a type-B agent.
- The value of search satisfies the following continuous-time Bellman equation:
$$rV_B = \alpha\lambda I\left(\frac{x_G}{r} - V_B\right) + \alpha(1-\lambda)\left(\frac{x_B}{r} - V_B\right)$$
- I is an indicator such that
$$I = \begin{cases} 1 \text{ if accepted} \\ 0 \text{ if rejected} \end{cases}$$
- Thus,
$$rV_B = \begin{cases} \dfrac{\alpha}{r+\alpha}\left[\lambda x_G + (1-\lambda)x_B\right] \text{ if } \dfrac{r}{\alpha} > \lambda\dfrac{x_G - x_B}{x_B} \\ \dfrac{\alpha(1-\lambda)}{r+\alpha(1-\lambda)}x_B \quad \text{if } \dfrac{r}{\alpha} < \lambda\dfrac{x_G - x_B}{x_B} \end{cases}$$

Equilibria

- If $\frac{r}{\alpha} > \lambda \frac{x_G - x_B}{x_B}$, then there is a **mixing equilibrium**.
  - Everyone is accepted.
  - The market is sufficiently frictional.
  - The quality difference $(x_G - x_B)$ is not significant.
  - Not so many type-G agents.
- If $\frac{r}{\alpha} < \lambda \frac{x_G - x_B}{x_B}$, then there is a **sorting equilibrium**.
  - Type-G accepts only type-G agents.
  - You can quickly meet others.
  - The quality difference is significant.
  - There are many type-G agents.

Further Readings

- Burdett and Coles, “Separation Cycles,” *Journal of Economic Dynamics and Control*, 1998.
- Acemoglu, “Changes in Unemployment and Wage Inequality: An Alternative Theory and Some Evidence,” *American Economic Review*, 1999.

Final Remarks

You Are Ready

- I gave you pretty much everything I know about search-matching models of the labor market.
- You are now ready to read (and replicate) the cutting-edge articles published in leading professional journals. No more textbooks.
- I guarantee that you are now **at the frontier** of macro-labor economics.
  - You can now compete with me and other professional researchers around the world!
- Good luck!