Lecture 7

Numerical Analysis

6/2

Goals

- We have two goals.
- First, given a set of parameters, we want to solve our model (often called the DMP model) using Octave/Matlab and Python.
- Second, we want to find a set of realistic parameters for the model.
- In your own research, you should start with the second task, followed by the first.

Model Equations

- Remember that the steady-state equilibrium is given by the set of (θ, w, u) satisfying

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

- Pissarides textbook assumes that the vacancy cost is proportional to labor productivity (for extensions in later Chapters).

Note on Vacancy Cost

- In many existing models, the vacancy cost is simply c rather than pc. Then, the system becomes

$$w = p - \frac{(r+\lambda)c}{q(\theta)}$$

$$w = \beta p + (1-\beta)z + \beta c\theta$$

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

- Today, we shall employ this version of the DMP model.

Specifications

- The first step is to specify all the functional forms.
- Let us suppose that the matching function is Cobb-Douglas, so that $q(\theta) = A\theta^{-\alpha}$.
- We then obtain

$$w = p - \frac{(r+\lambda)c}{A}\theta^{\alpha}$$

$$w = \beta p + (1-\beta)z + \beta c\theta$$

$$u = \frac{\lambda}{\lambda + A\theta^{1-\alpha}}$$

Parameter Values

- Finally, we need to specify the set of parameters for the analysis. But, how?
- To make the model's prediction useful, we need to find <u>a realistic set of parameters</u>.
- For now, let us <u>pretend</u> that we know a reasonable set of parameters and jump into Maxima (and Python).

Shimer (2005)

r	A	α	λ	c	β	z	p
0.012	1.355	0.72	0.1	0.213	0.72	0.4	1

- Robert Shimer, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 2005.
- This is an influential article on quantitative studies of the DMP model.
- One period = one quarter.
- We shall borrow parameters from his work.

Solving the Model

In Maxima

Parameter Values

Model

- $r = 0.012$
- $A = 1.355$
- $\alpha = 0.72$
- $\lambda = 0.1$
- $c = 0.213$
- $\beta = 0.72$
- $z = 0.4$
- $p = 1$
- Colon ":" means assignment "=".

Maxima

```
/* Parameter values */
r:0.012;
A:1.355;
alpha:0.72;
lambda:0.1;
c:0.213;
beta:0.72;
z:0.4;
p:1;
```

Job Creation Condition

- The equation is

$$w = p - \frac{(r+\lambda)c}{A}\theta^{\alpha}$$

- It is convenient to define a function of θ.
- "$:=$" defines a function.

```
/* Job-creation condition */
JC(theta):= p - (r + lambda)*(c/A)*theta^alpha;
```

Wage Equation

- The equation is
$$w = \beta p + (1 - \beta)z + \beta c\theta$$
- It is convenient to define a function of θ.

```
/* Wage equation */
Wage(theta) := beta*p + (1 - beta)*z + beta*c*theta;
```

Beveridge Curve

- The equation is

$$u = \frac{\lambda}{\lambda + A\theta^{1-\alpha}}$$

- It is convenient to define a function of θ.

```
/* Unemployment rate */
Unemp(theta) := lambda/(lambda + A*(theta^(1 - alpha)));
```

Job-Finding Rate

- The expression is

$$f_t = A\theta_t^{1-\alpha}$$

- We shall compute the equilibrium JFR in the model.

```
/* Job-Finding Rate */
JFR(theta) := A*(theta^(1-alpha));
```

Diagram

- As in the textbook, let us draw a diagram with two curves, the Job-creation condition and the wage equation.

```
/* Diagram */
plot2d([JC(theta),Wage(theta)],[theta,0.5,1.5]);
```

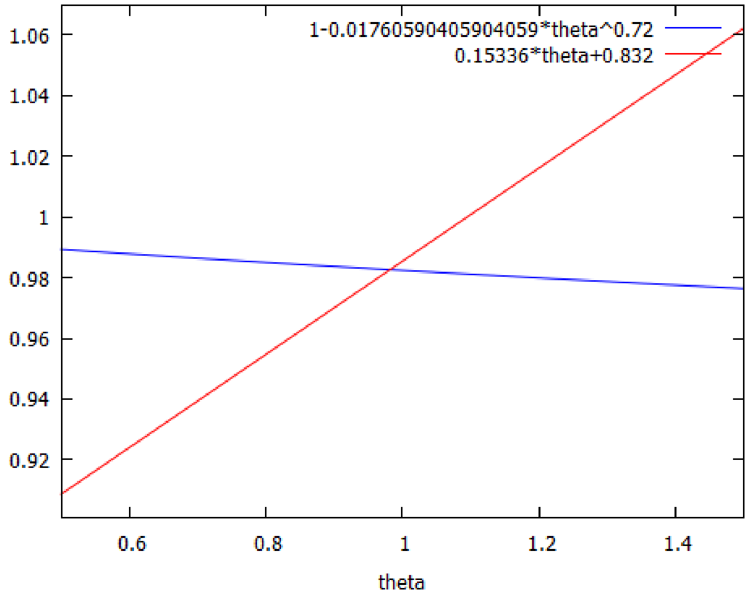
Solving the System of Equations

- Unfortunately, the $\text{solve}()$ command cannot solve our model.
  - It only solves equations that reduce to polynomials.
- We use the $\text{mnewton}()$ command:

```
/* Solving the model */
load("mnewton")$
mnewton([w=JC(theta),w=Wage(theta)],[w,theta],[1,1])[1][1];
mnewton([w=JC(theta),w=Wage(theta)],[w,theta],[1,1])[1][2];
eqtheta: rhs(mnewton([w=JC(theta),w=Wage(theta)],[w,theta],[1,1])[1][2]);
```

- We need the initial condition for computation.

Maxima output



w=0.9826210585631064
theta=0.9821404444647004
u=0.06905218641629844

Consistent with Shimer (2005)

Maxima Output

- JFR in the model is 1.348180067711758.
- The maxima code is available from NUCT.

Shimer (2005)

month, so the flow arrival rate of job offers $\mu\theta^{1-\alpha}$ should average approximately 1.35 on a quarterly basis. I do not have a long time series

Further Readings

- Robert Shimer, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 2005.
- For more information about Maxima, visit http://maxima.sourceforge.net/documentation.html
  - Maxima can be your good friend throughout your graduate study.

Solving the Model

In Python

Python Code

- We need to start with taking care of libraries.

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import root
```

Python Code

- We then define parameter values.
- To input Greek letters,
  ¥alpha
  then press Tab.
- You must not use
  lambda
  as a parameter name because it is kept as a function in Python.

```
r = 0.012
A = 1.355
α = 0.72
λ = 0.1
c = 0.213
β = 0.72
z = 0.4
p = 1
```

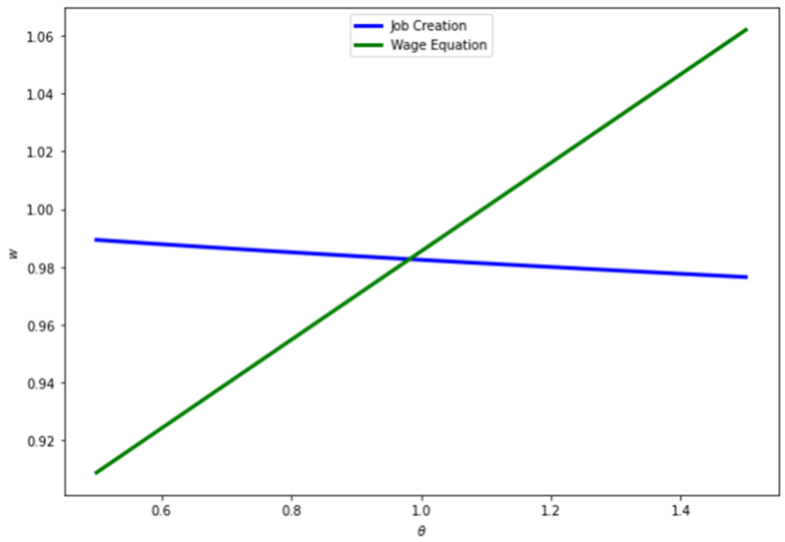
Python Code

- We then define functions:

```
# Job-creation condition
def JC(θ):
    return p - (r + λ)*(c/A)*(θ**α)
# Wage equation
def Wage(θ):
    return β*p + (1 - β)*z + β*c*θ
# Unemployment rate
def Unemp(θ):
    return λ/(λ + A*(θ**(1 - α)))
# Job-Finding rate
def JFR(θ):
    return A*(θ**(1-α))
```

Diagram

```
# Drawing a Diagram

plt.rcParams['figure.figsize'] = (10,7)

theta_grid = np.linspace(0.5, 1.5, 100)
fig, ax = plt.subplots()

JC_curve = JC(theta_grid)
Wage_curve = Wage(theta_grid)

ax.plot(theta_grid, JC_curve, 'b-', lw=3, label='Job Creation')
ax.plot(theta_grid, Wage_curve, 'g-', lw=3, label='Wage Equation')

ax.set_xlabel('$¥¥theta$')
ax.set_ylabel('$w$')
ax.legend(loc='upper center')

plt.show()
```



Solving the Model

- We then solve the system of nonlinear equations.
- The initial condition for numerical compulation is set at (1,1).

```
def func(x):
    f = [JC(x[0]) - x[1],
         Wage(x[0]) - x[1]]
    return f
sol = root(func, [1,1])
sol.x
```

Output

- We finally print the output.

```
eq_theta = sol.x[0]
eq_wage = sol.x[1]
eq_u = Unemp(eq_theta)
eq_JFR = JFR(eq_theta)

print(f'Equilibrium tightness is {eq_theta:.4f}')
print(f'Equilibrium wage rate is {eq_wage:.4f}')
print(f'Equilibrium unemployment rate is {eq_u:.4f}')
print(f'Equilibrium job-finding rate is {eq_JFR:.4f}')
```

Python Output

Equilibrium tightness is 0.9821

Equilibrium wage rate is 0.9826

Equilibrium unemployment rate is 0.0691

Equilibrium job-finding rate is 1.3482

Calibration

Calibration

- Now we have solved the steady state of the DMP model, given a set of parameters.
- How do we find the **right parameter values**?
- Modern macroeconomic research typically employs the method of **calibration**:
  - Borrow parameter values from empirical facts:
    - From the data.
    - From existing research.
  - Find parameter values using the model equations.

The Set of Parameters

- r : discount rate
- A : matching efficiency
- α : matching elasticity
- λ : separation rate
- c : vacancy cost
- β : bargaining power
- z : UI benefit
- p : labor productivity

- We have 8 parameters to determine.
- The basic idea is to introduce 8 equations to determine the parameter values.
- The facts for determining the parameter values are called **targets**.

Target Economy

- In principle, you may choose any country for your quantitative analysis.
- The advantage of studying the US economy is that you can find many articles.
  - There are many learning opportunities.
  - You can borrow parameter values.
  - You can compare your results with others.
- Many non-US economists choose their own countries in addition to US.
  - I study Japan and US for my own research.

Model Period

- Because our model has discount rate (r), we need to choose the model period.
- Typical choices are:
  - Quarter
  - Month
- Today we choose one period to be a quarter.

Calibration: r

- Collect the long run time series data on the real interest rate.
  - There is no such things as "the" real interest rate.
  - There are hopelessly many interest rates in the real world.
  - Many people, however, adopt $r = 0.01$ by arguing that the annual real interest rate is 4~5%
- If the annual real interest rate is 4%, then the quarterly real interest rate is approximately 1%.

Calibration: r

- Suppose that the quarterly interest rate is r_q. Then the annual interest rate r_a is given by

$$1 + r_a = \left(1 + r_q\right)^4$$

- Consider the nonlinear function $f(x) = (1 + x)^4$. Linearize it at $x = 0$ to obtain

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x - 0) \\ &= (1 + 0)^4 + 4(1 + 0)^3(x - 0) \\ &= 1 + 4x \end{aligned}$$

- Thus, for r_q close to 0, we can use

$$r_a \approx 4r_q$$

Calibration: r

- Suppose the target annual real interest rate is 5%. Then,
  $$1.05 = (1 + r)^4$$
- Solving it to obtain
  $$r = 0.01227$$
- The annual discount factor is
  $$\frac{1}{1 + 0.05} = 0.95238$$

Shimer (2005)

> I normalize a time period to be one quarter, and therefore set the discount rate to $r = 0.012$, equivalent to an annual discount factor of 0.953.

Calibration: λ

- To pin down the separation rate, we need worker flow data.
  - US: Current Population Survey (CPS)
  - Japan: Labor Force Survey
- Construction of worker flow data requires too much work and time.
- We typically rely on someone else's work, such as https://www.federalreserve.gov/pubs/feds/2004/200434/200434abs.html

Calibration: λ

- Shimer (2005) adopts the quarterly separation rate in the US to be 0.1, so
$$\lambda = 0.1$$
- Lin and Miyamoto (2012) find that the monthly separation rate in Japan is 0.0048, implying that the quarterly separation rate in Japan is
$$\lambda = 0.0048 \times 3 = 0.0144$$

Calibration: α

- Remember:
$$M_t = AU_t^{\alpha}V_t^{1-\alpha}$$
where $A > 0$ and $0 < \alpha < 1$.
- α is the elasticity of matching with respect to unemployment.
- If you take log of the equation, we obtain
$$\log M_t = \log A + \alpha \log U_t + (1 - \alpha) \log V_t$$
- We can estimate α by regression analysis.

Calibration: α

- There are many estimates.
- Shimer's (2005) estimate is 0.72.
- An often-cited summary of empirical results is: Petrongolo and Pissarides, "Looking into the Black Box: A Survey of the Matching Function" *Journal of Economic Literature*, 2001.
    - They conclude that the plausible range of α is 0.5 to 0.7.
- Lin and Miyamoto (2014) estimate that $\alpha = 0.6$ in Japan.

Calibration: β

- This is the worker's bargaining power in Nash wage bargaining.
  - There is no direct estimate on this value.
- In the literature, there are 2 ways:
  - Set $\beta = 0.5$ because we have no information.
  - Set $\beta = \alpha$ to ensure the Hosios condition.
    - This means job creation is efficient.
    - Shimer (2005) adopts this method.

Calibration: p

- We normalize the steady-state level to be $p = 1$.
- If we are interested in the effect of an increase in p on u, then we can compare the steady states with $p = 1$ and $p = 1.01$ (one percentage increase).

Calibration: Targets

- Sometimes we need to use <u>model equations</u> to find model parameters
- A calibration **target** is:
  - An endogenous variable (not a parameter) of the model,
  - corresponding long-run data is available,
  - Not the main variables of interest.

Calibration: A

- Job-finding rate in the model = $A\theta^{1-\alpha}$.
- Monthly job-finding rate = 0.142
- Average UV ratio in Japan = 0.78
- Matching elasticity = α = 0.5
- From these values,

$$A(0.78)^{1-0.5} = 3 \times 0.142,$$

- Thus,

$$A = \frac{3 \times 0.142}{(0.78)^{0.5}} = 0.482$$

Calibration: z and c

- Steady-state equations:

$$w = p - \frac{(r+\lambda)c}{A}\theta^{\alpha}$$
$$w = \beta p + (1-\beta)z + \beta c\theta$$

- Given a set of parameters, we can find the solution (w, θ).
- Basic idea: we set targets for w and θ and use these equations to find (z, c).
  - Reverse engineering!

Calibration: z and c

- One target is $\theta = 0.78$ from the data.
- Another target is the **unemployment benefit replacement rate**.
- In Japan, replacement rate is 0.6.
- Thus, $\frac{z}{w} = 0.6$

	Direct Rigidities		*Treatment of the Unemployed*		
	1 *Employment Protection*	*2* *Labor Standards*	*3* *Benefit Replacement Rate (%)*	*4* *Benefit Duration (years)*	*5* *Active Labor Market Policies*
Austria	16	5	50	2	8.3
Belgium	17	4	60	4	14.6
Denmark	5	2	90	2.5	10.3
Finland	10	5	63	2	16.4
France	14	6	57	3	8.8
Germany (W)	15	6	63	4	25.7
Ireland	12	4	37	4	9.1
Italy	20	7	20	0.5	10.3
Netherlands	9	5	70	2	6.9
Norway	11	5	65	1.5	14.7
Portugal	18	4	65	0.8	18.8
Spain	19	7	70	3.5	4.7
Sweden	13	7	80	1.2	59.3
Switzerland	6	3	70	1	8.2
U.K.	7	0	38	4	6.4
Canada	3	2	59	1	5.9
U.S.	1	0	50	0.5	3.0
Japan	8	1	60	0.5	4.3
Australia	4	3	36	4	3.2
New Zealand	2	3	30	4	6.8

Nickell, "Unemployment and Labor Market Rigidity: Europe versus North America," Journal of Economic Perspectives, 1997.

Calibration: z and c

- Original system (w, θ):

$$w = p - \frac{(r+\lambda)c}{A}\theta^{\alpha}$$
$$w = \beta p + (1-\beta)z + \beta c\theta$$

- Calibration equations (z, c):

$$\frac{z}{0.6} = p - \frac{(r+\lambda)c}{A}0.78^{\alpha}$$
$$\frac{z}{0.6} = \beta p + (1-\beta)z + 0.78\beta c$$

- (b, c) is chosen so that "the solution = targets".

Japanese Labor Market

- Purely exogenous parameters
  - $r = 0.01$
  - $\lambda = 0.0048 \times 3$
  - $\alpha = 0.6$
  - $\beta = 0.5$
  - $p = 1$
- Endogenous (!) parameters
  - $A = 0.482$
  - $z = ?$
  - $c = ?$

Further Readings

- Robert Shimer, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 2005.
- Lin and Miyamoto, "Gross Worker Flows and Unemployment Dynamics in Japan," *Journal of the Japanese and International Economies*, 2012.
- Lin and Miyamoto, "An Estimated Search and Matching Model of the Japanese Labor Market," *Journal of the Japanese and International Economies*, 2014.