# Lecture 6

Pissarides, Equilibrium Unemployment Theory

1-5: Steady-State Equilibrium

5/26

# Goals

- Today and the next week, we want to fully understand section 1-5 in Pissarides book.
    - As always, I will assume that you read this section.
- We have two specific goals:
    - This week, we study how we characterize the steady state of the model. In particular, we learn the method of **comparative statics**, which is the core tool for many economic analysis. This requires paper and pencil.
    - Next week, we will study the same model using computer.

# Steady-State Equilibrium

# Beveridge curve

- The steady-state unemployment rate satisfies
$$\lambda(1-u) = \theta q(\theta) u$$
- Solve it for $u$ as
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$
- This is (1.5) and (1.21) in Pissarides.
- This equation implies a negative relationship between $u$ and $v$, the **Beveridge curve**.

# Job Creation Condition

- Under free entry, we have

$$(r+\lambda)\frac{pc}{q(\theta)} = p - w$$

- This is (1.9) and (1.22).
- This is the **job creation condition**.

# Wage Equation

- The Nash wage bargaining implies
$$w = \beta p + (1-\beta)z + \beta pc\theta$$
- This is (1.20) and (1.23).
- This is the **wage equation**.

# Steady State Conditions

- We can determine the steady-state equilibrium by the set of $(\theta, w, u)$ satisfying

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

$$(r + \lambda)\frac{pc}{q(\theta)} = p - w$$

$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

- Note that the first equation is independent of $w$. Thus, we can calculate $u$ after the equilibrium value of $\theta$ is found.

# Steady State Conditions

- Thus, we determine the steady-state equilibrium by the set of $(\theta, w)$ satisfying

$$(r+\lambda)\frac{pc}{q(\theta)} = p - w$$

$$w = \beta p + (1-\beta)z + \beta pc\theta$$

- Let us draw a diagram.
- Let $w$ be on the vertical axis:

$$w = p - (r+\lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1-\beta)z + \beta pc\theta$$

# Steady State Conditions

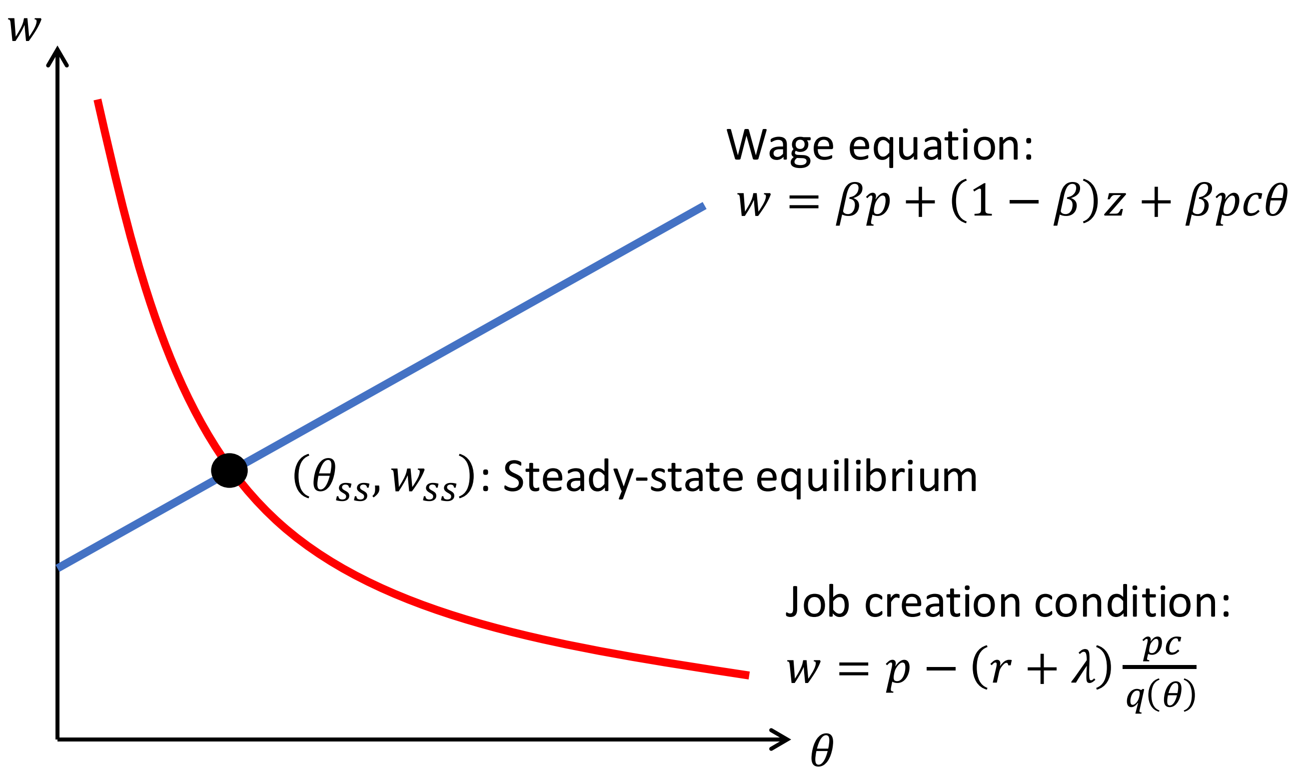
- Consider the <u>shape</u> of the job-creation condition:

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

- Because $q(\theta)$ is deceasing in $\theta$, $w$ should be deceasing in $\theta$.
- Want to verify? Let us consider the Cobb-Douglas matching function. Then, $q(\theta) = A\theta^{-\alpha}$.
- Thus,

$$w = p - (r + \lambda)pcA\theta^{\alpha}$$

# Labor Market Equilibrium



# Labor Market Equilibrium

- Let $(\theta_{ss}, w_{ss})$ be the solution to

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

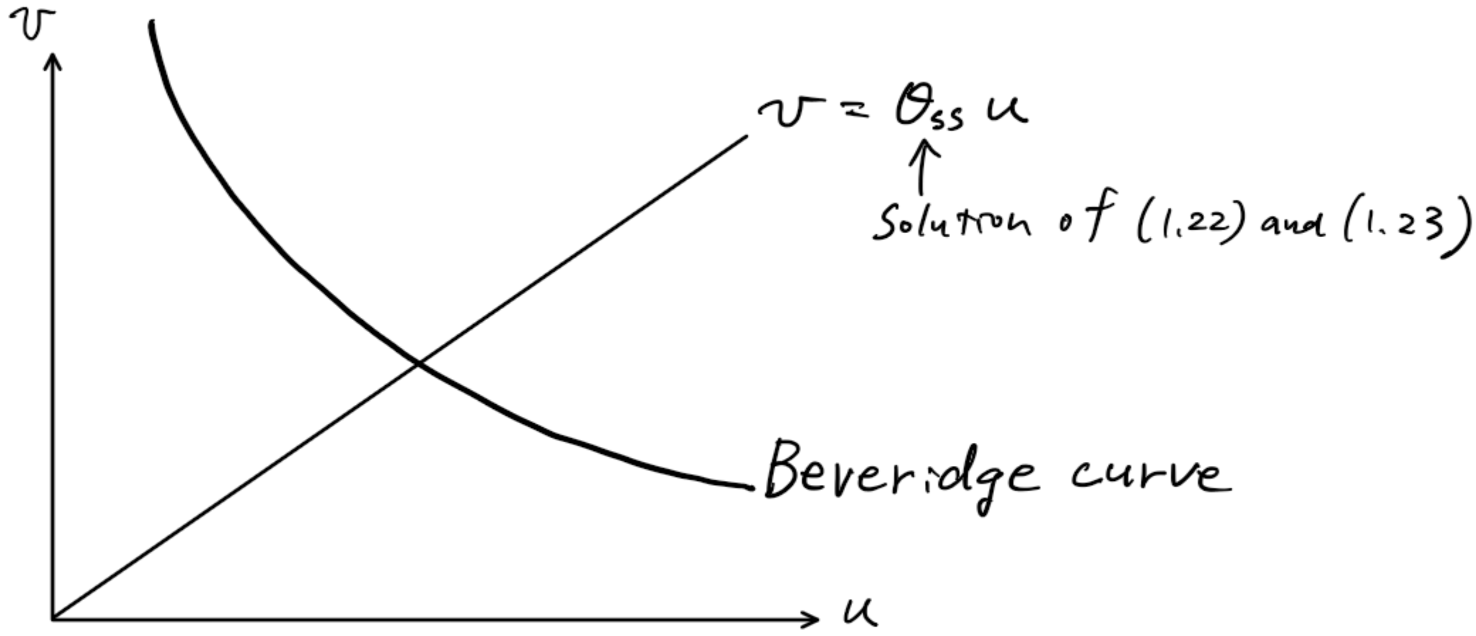
$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

- Note that $\theta_{ss}$ and $w_{ss}$ are numbers, not unknowns.
- The steady-state unemployment is therefore

$$u_{ss} = \frac{\lambda}{\lambda + \theta_{ss} q(\theta_{ss})}$$

# Labor Market Equilibrium

- By the definition of tightness ($\theta = v/u$), we obtain
$$v_{ss} = \theta_{ss} u_{ss}$$
- We can draw a diagram determining $u_{ss}$ and $v_{ss}$.



# Labor Market Equilibrium

- Once again, consider

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

- Eliminate $w$ from these equations to obtain

$$p - (r + \lambda)\frac{pc}{q(\theta)} = \beta p + (1 - \beta)z + \beta pc\theta$$

- Arrange terms to obtain

$$p - z - (r + \lambda)\frac{pc}{q(\theta)} = \beta(p - z + pc\theta)$$

# Efficiency

# Efficiency

- An increase in a vacancy has external effects:
    - Makes it easier for a worker to find a job.
    - Makes it harder for other firms to find a worker.
- Each player takes $\theta$ as given. This means that each player ignores **external effects**.
- Is job-creation efficient?
- To address this question, we ask the allocation chosen by the (hypothetical) social planner.
    - We need to make sure that the social planner faces the search-matching frictions for a fair comparison.

# Efficiency

- At the aggregate, the payoff to the society per unit of time is given by
$$(1-u)Lp + uLz - vLpc$$
  - $(1-u)L$ is the number of employed. $p$ is the productivity. $(1-u)Lp$ is total products.
  - $uL$ is the number of unemployed. $z$ is (exogenous) unemployment benefit. $uLz$ is total unemployment benefits. You can also consider it as total **home products**.
  - $vL$ is the number of vacancies. $pc$ is the unit vacancy cost. $vLpc$ is total vacancy costs.

# Efficiency

- By definition, $v = \theta u$. Thus, the payoff to the society per unit of time is given by
$$[(1-u)p + uz - \theta upc]L$$
- Now we can drop $L$. Worker flows imply
$$\dot{u} = \lambda(1-u) - \theta q(\theta)u$$
- Thus, the **social planner's problem** is to maximize
$$\int_0^\infty e^{-rt}[(1-u)p + uz - \theta upc]dt$$
subject to $\dot{u} = \lambda(1-u) - \theta q(\theta)u$.

# Efficiency

- Consider
$$\int_0^\infty e^{-rt}[(1-u)p + uz - \theta upc]dt$$
subject to $\dot{u} = \lambda(1-u) - \theta q(\theta)u$.
- This is a dynamic optimization problem in continuous time, and it requires the **optimal control theory**.
- Perhaps this is new to many of you.
  - Graduate students may notice that this is a continuous-time version of the Lagrangian method you studied in my advanced macro I course.
  - Undergrad students can jump to the result on page 26.

# Optimal Control

- Let $a$ denote the action that can be chosen instantly.
- Let $s$ denote the state that evolves over time according to the transition equation:
$$\dot{s} = G(a, s, t)$$
- The problem is to choose a sequence of $a$ to
$$\max \int_0^\infty F(a, s, t) dt$$
subject to the initial condition $s_0$ and
$$\dot{s} = G(a, s, t)$$

# Optimal Control

- The **Hamiltonian** for this problem is

$$H = F(a, s, t) + \Omega G(a, s, t)$$

  where $\Omega$ is called the **costate variable**.

- First-order conditions are:

$$\frac{\partial H}{\partial a} = 0$$

$$\dot{s} = \frac{\partial H}{\partial \Omega} = G(a, s, t)$$

$$\dot{\Omega} = -\frac{\partial H}{\partial s}$$

  with the transversality condition: $\lim_{t\to\infty} \Omega s = 0$

# Optimal Control

- Often the objective function is discounted and the transition equation is time-independent:
  $$F(a, s, t) = e^{-rt} f(a, s),$$
  where $r$ is the discount rate, and
  $$\dot{s} = G(a, s, t) = g(a, s).$$
- In that case it is convenient to construct the **current value Hamiltonian**:
  $$\widetilde{H} = e^{rt} H = f(a, s) + \omega g(a, s),$$
  where $\omega = e^{rt}\Omega$.

# Optimal Control

- The first-order conditions are

$$\frac{\partial \widetilde{H}}{\partial a} = 0,$$

$$\dot{s} = \frac{\partial \widetilde{H}}{\partial \omega} = g(a, s)$$

$$\dot{\omega} = -\frac{\partial \widetilde{H}}{\partial s} + r\omega$$

with the transversality condition:

$$\lim_{t \to \infty} e^{-rt} \omega s = 0$$

# Efficiency of Job Creation

- Let us go back to our problem (on page 16):
$$\max \int_0^\infty e^{-rt}[(1-u)p + uz - \theta upc]dt$$
subject to $\dot{u} = \lambda(1-u) - \theta q(\theta)u$.
- The current value Hamiltonian for the problem is:
$$\begin{aligned}\widetilde{H} = (1-u)p + uz - \theta upc \\ +\omega[\lambda(1-u) - \theta q(\theta)u]\end{aligned}$$
- Write down the first-order conditions with the transversality condition.

# Efficiency of Job Creation

- The first-order conditions are
  $-pcu - \omega u[q(\theta) + \theta q'(\theta)] = 0,$
  $\dot{u} = \lambda(1 - u) - \theta q(\theta) u$
  $\dot{\omega} = -\left[-p + z - \theta pc + \omega\left(-\lambda - \theta q(\theta)\right)\right] + r\omega$
  with the transversality condition:
  $$\lim_{t \to \infty} e^{-rt} \omega u = 0$$
- In any steady state, we obtain
  $$pc = -\omega[q(\theta) + \theta q'(\theta)],$$
  $$\lambda(1 - u) = \theta q(\theta) u$$
  $$\left(r + \lambda + \theta q(\theta)\right)\omega = -p + z - \theta pc$$

# Efficiency of Job Creation

- Eliminate $\omega$ to obtain

$$pc\frac{r+\lambda+\theta q(\theta)}{q(\theta)+\theta q'(\theta)} = p - z + \theta pc$$

- Remember that

$$\frac{q'(\theta)\theta}{q(\theta)} = -\eta(\theta)$$

- Thus,

$$pc\frac{r+\lambda}{q(\theta)} + pc\theta = [1-\eta(\theta)](p - z + \theta pc)$$

# Efficiency

- After some manipulations, we find that the socially optimal level of tightness satisfies

$$p - z - (r + \lambda)\frac{pc}{q(\theta)} = \eta(\theta)(p - z + pc\theta)$$

- Observe that, from page 13, the **equilibrium tightness** satisfies:

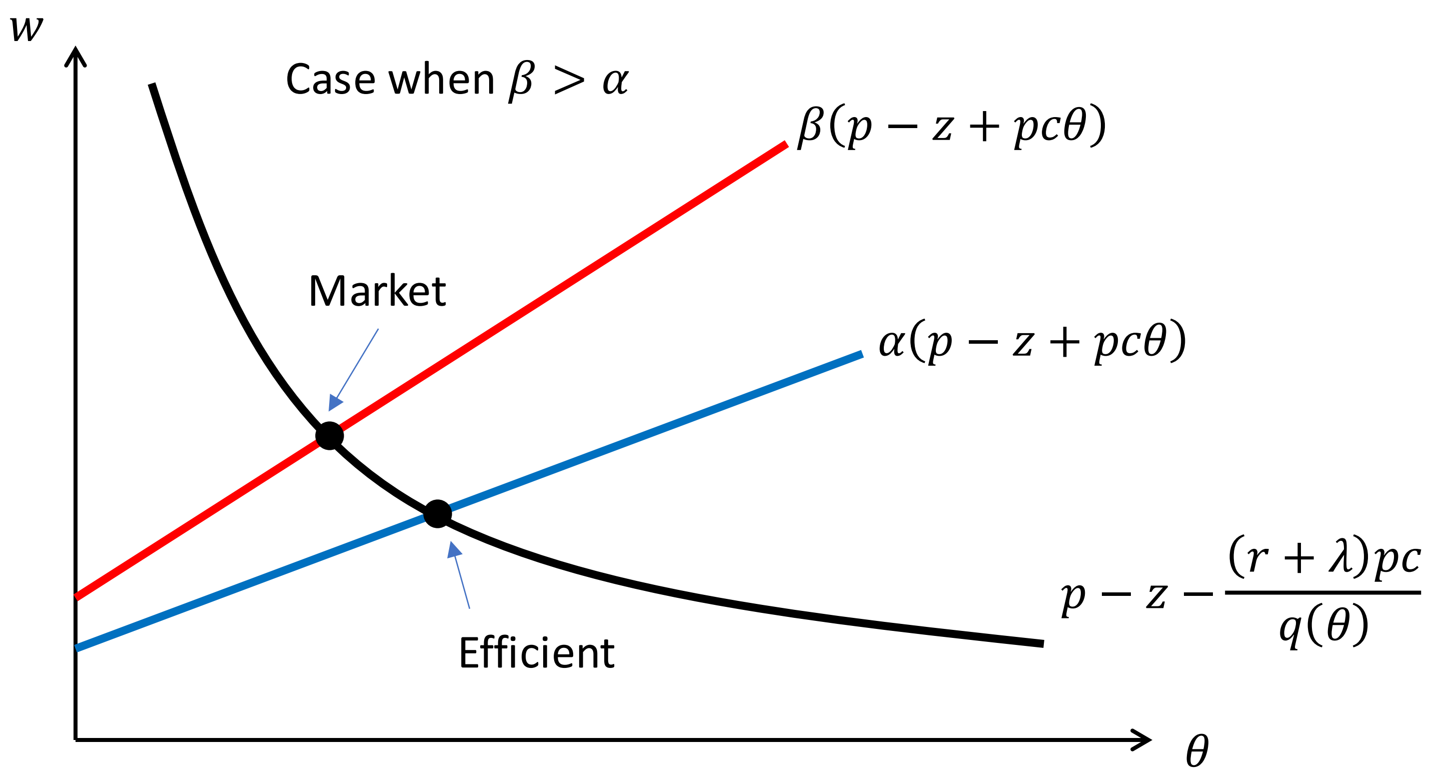
$$p - z - (r + \lambda)\frac{pc}{q(\theta)} = \beta(p - z + pc\theta)$$

- Market tightness is efficient if and only if $\eta(\theta) = \beta$.
  - This is called the **Hosios condition**.

# Efficiency

- In the Cobb-Douglas case, $\eta(\theta) = \alpha$.
- The socially optimal level of tightness satisfies
$$p - z - (r + \lambda)\frac{pc}{q(\theta)} = \alpha(p - z + pc\theta)$$
- The equilibrium tightness satisfies:
$$p - z - (r + \lambda)\frac{pc}{q(\theta)} = \beta(p - z + pc\theta)$$
- Let us draw a diagram.

# Efficiency



$w$
Case when $\beta > \alpha$
$\beta(p - z + pc\theta)$
Market
$\alpha(p - z + pc\theta)$
$p - z - \frac{(r+\lambda)pc}{q(\theta)}$
Efficient
$\theta$

# Efficiency

- From the diagram,
$$\beta > \alpha \Leftrightarrow \theta_{\text{market}} < \theta_{\text{efficient}}$$
- Market tightness is efficient if and only if
$$\beta = \alpha$$
  - Again, this is called the **Hosios condition**.

# Comparative Statics Analysis

# Comparative Statics Analysis

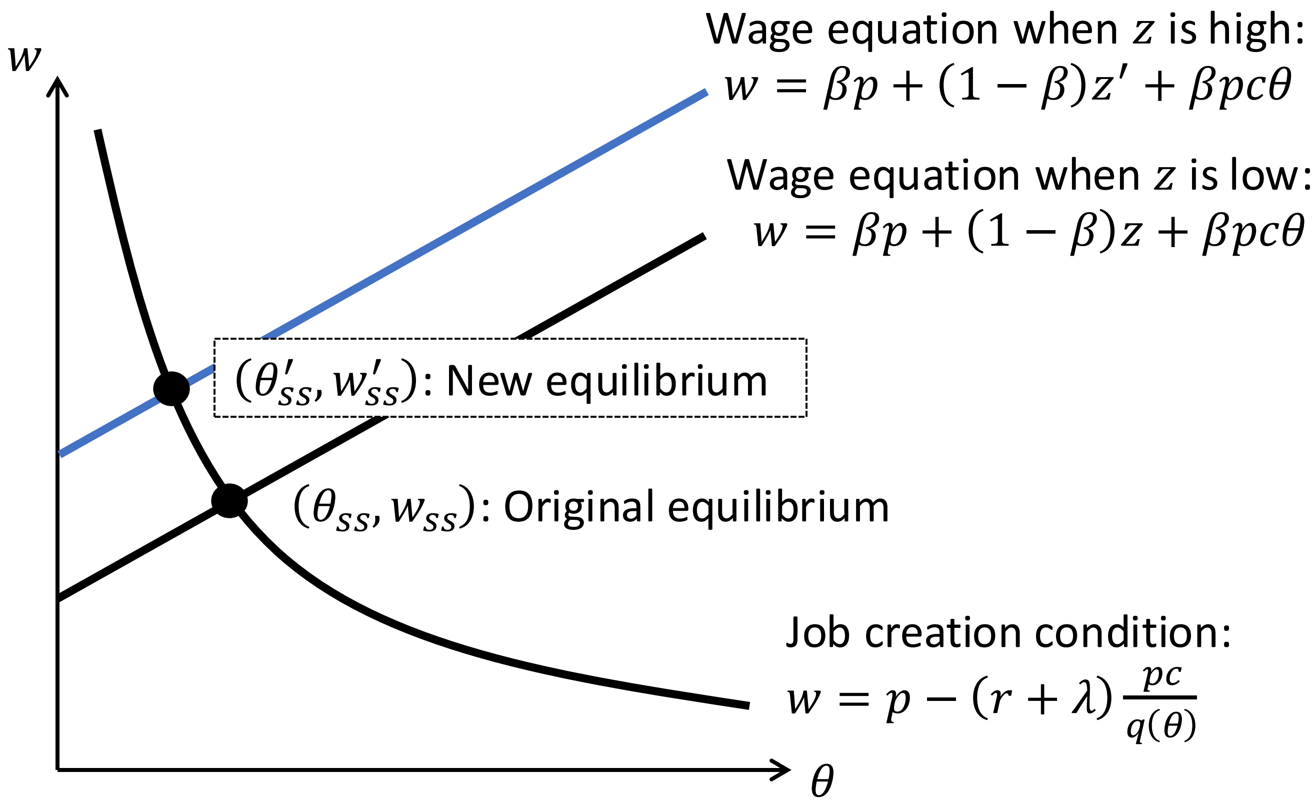
- As an example, let us consider the impact of an increase in $z$ on the steady-state tightness $\theta$.
- The system of equations we are studying are:

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

- Evidently, an increase in $z$ causes the wage curve to shift upward.
- As you can see the diagram on the next page, $\theta$ decreases and $w$ increases in the new steady-state equilibrium.

# Comparative Statics Analysis



# Comparative Statics Analysis

- Let us then consider the impact of an increase in $p$ on the steady-state tightness $\theta$.
- Drawing a diagram is not easy because $p$ is <u>everywhere</u>:

$$w = p - (r + \lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1 - \beta)z + \beta pc\theta$$

- How can we proceed?
- We need to do it analytically.

# Comparative Statics Analysis

- To analytically study the impact of an increase in $p$ on $\theta$, we totally differentiate the system of equations with respect to all endogenous variables plus the parameter you want to study:

$$w = p - (r+\lambda)\frac{pc}{q(\theta)}$$

$$w = \beta p + (1-\beta)z + \beta pc\theta$$

- Thus, we totally differentiate the system with respect to $w, \theta, p$.

# Comparative Statics Analysis

- Totally differentiate the system:

- $dw = dp - (r+\lambda)\frac{c}{q(\theta)}dp$
$$+ (r+\lambda)pcq^{-2}q'(\theta)d\theta$$

- $dw = \beta dp + \beta c\theta dp + \beta pc d\theta$

# Comparative Statics Analysis

- Eliminate $dw$ from these expressions and arrange terms to obtain
$$\frac{d\theta}{dp}=\frac{1-\beta-(r+\lambda)\dfrac{c}{q(\theta)}-\beta c\theta}{\beta pc+(r+\lambda)\dfrac{pc}{\theta q(\theta)}\eta(\theta)}$$
- The denominator is positive. Thus,
$$\frac{d\theta}{dp}>0$$
$$\Leftrightarrow 1-\beta-(r+\lambda)\frac{c}{q(\theta)}-\beta c\theta>0$$

$$(1-\beta)dp-(r+\lambda)\frac{c}{q(\theta)}dp-\beta c\theta dp$$
$$=\beta pc\, d\theta-(r+\lambda)pc\, q^{-2}q'(\theta)d\theta$$
$$\Longleftrightarrow$$
$$\left[(1-\beta)-(r+\lambda)\frac{c}{q(\theta)}-\beta c\theta\right]dp$$
$$=\left[\beta pc-\frac{(r+\lambda)pc}{q(\theta)}\frac{q'(\theta)}{q(\theta)}\right]d\theta$$
$$\frac{q'(\theta)}{q(\theta)}=-\eta(\theta)/\theta$$
$$\Longleftrightarrow$$
$$\frac{d\theta}{dp}=\frac{(1-\beta)-\dfrac{(r+\lambda)c}{q(\theta)}-\beta c\theta}{\beta pc+\dfrac{(r+\lambda)pc}{\theta q(\theta)}\eta(\theta)}$$

# Comparative Statics Analysis

- Consider the condition
$$1-\beta-(r+\lambda)\frac{c}{q(\theta)}-\beta c\theta>0$$
- Because any equilibrium must satisfy
$$w=p-(r+\lambda)\frac{pc}{q(\theta)},$$
we can rewrite the condition as
$$w>\beta p+\beta pc\theta$$
- Substitute $w=\beta p+(1-\beta)z+\beta pc\theta$ into above and arrange terms to obtain $(1-\beta)z>0$.

# Comparative Statics Analysis

- We now prove that the condition

$$1 - \beta - (r + \lambda)\frac{c}{q(\theta)} - \beta c\theta > 0$$

is satisfied in any steady-state equilibrium if and only if $(1 - \beta)z > 0$.

- Because $0 < \beta < 1, z > 0$, we know $(1 - \beta)z > 0$.
- Therefore, we now proved that

$$\frac{d\theta}{dp} = \frac{1 - \beta - (r + \lambda)\frac{c}{q(\theta)} - \beta c\theta}{\beta pc + (r + \lambda)\frac{pc}{\theta q(\theta)}\eta(\theta)} > 0$$