Lecture 4

Pissarides, Equilibrium Unemployment Theory

1-3: Workers

5/12

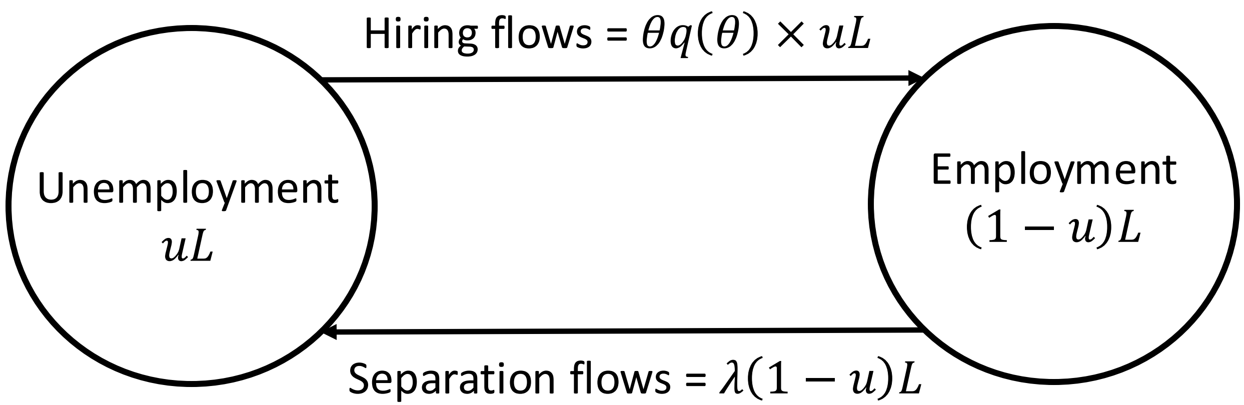
Goals

- Today, we want to fully understand section 1-3 in Pissarides book.
  - I will assume that you read this section.
  - The contents are parallel to those in section 1-2.
- There are additional issues that help you dig deeper in this section:
  - Reservation wage rate.
  - Out-of-steady-state equations in section 1.7.
  - Duration of unemployment.
  - Newton’s method.

Worker Flows and Bellman Equations

Worker Flows

- We can summarize worker flows below.



- The Bellman equations for workers must be consistent with worker flows.

Derivation of (1.10)

- Equation (1.10) in discrete time is expressed as

$$U = z\delta t + \frac{1}{1 + r\delta t}\{\theta q(\theta)\delta t W + [1 - \theta q(\theta)\delta t]U\}$$

- Interpretation:
  - If you are unemployed (= job seeker), then your current income during an interval δt is $z\delta t$. For example, this is unemployment insurance from the government. This can also include utility from leisure.
  - W is the present value of employment, explained later.
  - Because the job-finding probability is $\theta q(\theta)\delta t$, with probability $\theta q(\theta)\delta t$ you become employed, in which case you receive W instead of U.

Derivation of (1.10)

- Consider:
$$U = z\delta t + \frac{1}{1 + r\delta t}\{\theta q(\theta)\delta t W + [1 - \theta q(\theta)\delta t]U\}$$
- Multiply both sides by $1 + r\delta t$ to obtain
$(1 + r\delta t)U$
$= (1 + r\delta t)z\delta t + \{\theta q(\theta)\delta t W + [1 - \theta q(\theta)\delta t]U\}$
- Arrange terms to obtain
$$r\delta t U = (1 + r\delta t)z\delta t + \theta q(\theta)\delta t(W - U)$$
- Divide both sides by δt to obtain
$$rU = (1 + r\delta t)z + \theta q(\theta)(W - U)$$

Derivation of (1.10)

- Finally, take the limit as $\delta t \to 0$ to obtain the Bellman equation in continuous time as
$$rU = z + \theta q(\theta)(W - U)$$
- This is (1.10).

Derivation of (1.11)

- Equation (1.11) <u>in discrete time</u> is expressed as

$$W = w\delta t + \frac{1}{1 + r\delta t}\{\lambda \delta t U + (1 - \lambda \delta t)W\}$$

- Interpretation:
  - If you are employed, then your current income during an interval δt is $w\delta t$. Today we do not worry about how w is determined. That will be the topic for the next week.
  - U is the present value of unemployment.
  - Because the separation probability is $\lambda\delta t$, with probability $\lambda\delta t$ you lose your job for a purely exogenous reason and receive U instead of W.

Derivation of (1.11)

- Consider:

$$W = w\delta t + \frac{1}{1 + r\delta t}\{\lambda\delta t U + (1 - \lambda\delta t)W\}$$

- Multiply both sides by $1 + r\delta t$ to obtain

$$(1 + r\delta t)W$$
$$= (1 + r\delta t)w\delta t + \{\lambda\delta t U + (1 - \lambda\delta t)W\}$$

- Arrange terms to obtain

$$r\delta t W = (1 + r\delta t)w\delta t + \lambda\delta t(U - W)$$

- Divide both sides by δt to obtain

$$rW = (1 + r\delta t)w + \lambda(U - W)$$

Derivation of (1.11)

- Finally, take the limit as $\delta t \rightarrow 0$ to obtain the Bellman equation in continuous time as
$$rW = w + \lambda(U - W)$$
- This is (1.11).

Reservation Wage

Reservation Wage

- You want to accept any wage rate w satislying
$$W(w) \geq U$$
- There must be a number w_R that satisfies
$$W(w_R) = U$$
- At w_R, you are exactly indifferent between accepting and rejecting the offer.
- This wage rate is referred to as the **reservation wage rate**. You accept any wage rate w_R satisfying
$$w \geq w_R$$

Reservation Wage Rate

- As a function of w, $W(w)$ is given by
$$rW(w) = w + \lambda[U - W(w)]$$
- At the reservation wage rate w_R, we have
$$rW(w_R) = w_R + \lambda[U - W(w_R)]$$
- Substitute the definition $W(w_R) = U$ into the above to eliminate $W(w_R)$ to obtain
$$rU = w_R$$
- Thus, rU can be interpreted as the reservation wage rate. See page 14 in Pissarides.

Out-of-Steady-State Bellman Equations

Derivation of (1.33)

- Consider (1.6), which is
$$rV = -pc + q(\theta)(J - V)$$
- Now consider (1.33), which is
$$rV = -pc + q(\theta)(J - V) + \dot{V}$$
- The purpose here is to derive (1.33).
- Let us introduce $V(t)$ as a function of time to consider the possibility that V itself grows over time.

Derivation of (1.33)

- Consider:

$$V(t) = -pc\delta t + \frac{1}{1 + r\delta t}\{q(\theta)\delta t J(t + \delta t) + [1 - q(\theta)\delta t]V(t + \delta t)\}$$

- Multiply both sides by $1 + r\delta t$ to obtain

$$(1 + r\delta t)V(t)$$
$$= -(1 + r\delta t)pc\delta t + \{q(\theta)\delta t J(t + \delta t) + [1 - q(\theta)\delta t]V(t + \delta t)\}$$

- Arrange terms to obtain

$$r\delta t V(t) = -(1 + r\delta t)pc\delta t + q(\theta)\delta t[J(t + \delta t) - V(t + \delta t)] + V(t + \delta t) - V(t)$$

- Divide both sides by δt to obtain

$$rV(t) = -(1 + r\delta t)pc + q(\theta)[J(t + \delta t) - V(t + \delta t)] + \frac{V(t+\delta t) - V(t)}{\delta t}$$

Derivation of (1.33)

- Note that
$$\lim_{\delta t \to \infty} \frac{V(t+\delta t) - V(t)}{\delta t} = \frac{dV(t)}{dt} = \dot{V}$$
- Finally, take the limit as $\delta t \to 0$ to obtain the Bellman equation in continuous time as
$$rV = -pc + q(\theta)(J - V) + \dot{V}$$
- This is (1.33).
- In any **steady state**, we can impose $\dot{V} = 0$ and immediately obtain (1.6).

Review Questions

a) Derive (1.34).
b) Derive (1.37).
c) Derive (1.38).

Duration of Unemployment

Duration

- Suppose that the monthly job finding rate is x. Then, the average duration of unemployment is $1/x$ months.
- Proof)
- The probability that you find a job in the first month ($t = 1$) is x.
- The probability that you find a job in $t = 2$ is
$$(1-x)x$$
- The probability that you find a job in $t = 3$ is
$$[1-x-(1-x)x]x = (1-x)^2x$$

Duration

- The probability that you find a job in $t = 4$ is
$$[1 - x - (1 - x)x - (1 - x)^2 x]x$$
$$= (1 - x)[1 - x - (1 - x)x]x$$
$$= (1 - x)^3 x$$
- Thus, the probability that you find a job in $t = n$ is
$$(1 - x)^{n-1} x$$
- The average (i.e., expected) duration is
$$D = x \times 1 + (1 - x)x \times 2 + (1 - x)^2 x \times 3$$
$$+ (1 - x)^3 x \times 4 + \cdots + (1 - x)^{n-1} x \times n$$

Duration

- Consider
$$D = x \times 1 + (1-x)x \times 2 + (1-x)^2 x \times 3$$
$$+(1-x)^3 x \times 4 + \cdots + (1-x)^{n-1} x \times n$$
- Then,
$$D - (1-x)D = x + (1-x)x + (1-x)^2 x$$
$$+(1-x)^3 x + \cdots - (1-x)^n x \times n$$
- Or,
$$D$$
$$= 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^{n-1}$$
$$- (1-x)^n \times n$$

Duration

- Thus,
$$D-(1-x)D = 1-(1-x)^n$$
$$-(1-x)^n \times n + (1-x)^{n+1} \times n$$
$$= 1-(1-x)^n - (1-x)^n xn$$
- Or,
$$D = \frac{1}{x} - \frac{(1-x)^n}{x} - (1-x)^n n$$
- Take the limit as $n \to \infty$ to obtain
$$D = \frac{1}{x} - \lim_{n\to\infty}(1-x)^n n$$

Duration

- Consider

$$\lim_{n\to\infty}(1-x)^n n = \lim_{n\to\infty}\frac{n}{(1-x)^{-n}}$$

- l'Hôpital's rule implies

$$\lim_{n\to\infty}\frac{n}{(1-x)^{-n}} = \lim_{n\to\infty}\frac{1}{-(1-x)^{-n}\log(1-x)} = 0$$

- Thus,

$$D = \frac{1}{x}$$

- QED.

Newton's Method (Maxima)

- A useful command for solving equations is $\mathrm{mnewton()}$. This command finds a solution by the method of **successive approximation**, the algorithm discovered by **Newton**.
- To execute the command, we need the initial value (= our guess) for the solution.
- Unlike the $\mathrm{solve()}$ command, $\mathrm{mnewton()}$ command will give us one solution.
  - $\mathrm{solve()}$ command can be used only for linear or quadratic equations.

Newton’s Method (Maxima)

- To use mnewton() command, we need to load it first.
- “$” at the and of the command is used when you do not need to see the outcome of the line.
- [1,1] specifies the initial guess.
- As you can see, the output is a list inside a list.

```
load("mnewton")$
mnewton([y=0.5·x+1,y=0.5·x^2],[x,y],[1,1]);
```

$$[[x=2.0,y=2.0]]$$

Newton's Method

- To understand the basic idea of the Newton method, let us construct the algorithm.
- Let x^* be the solution to $f(x) = 0$.
- Suppose that we have an initial guess about the solution, x_n. $\text{mnewton}()$ command need it, too.
- Apply <u>linear approximation</u> on $f(x)$, evaluated at x_n, to obtain:
$$f(x) \approx f(x_n) + f'(x_n)(x - x_n) \equiv g(x)$$
- We can easily solve the linear equation $g(x) = 0$.

Newton’s Method

- The solution to the linear equation $g(x) = 0$ is
$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$
- This is the “updated guess” x_{n+1}. We then consider the sequence generated by:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
- Note that this is a (nonlinear) difference equation!
- As $n \to \infty$, $x_n \to x^*$. A computer needs less than a second to find x^*.

Reading Assignment

Reading Assignment

- Christopher A. Pissarides, *Equilibrium Unemployment Theory*, second edition, MIT Press, 2000.
- Read Section 1.4 (Wage Determination).
- Due is 5/19.
- 5/19 Class will focus on this section.