Lecture 3

Pissarides, Equilibrium Unemployment Theory

1-2: Job Creation

4/28

Goals

- Today, we want to fully understand section 1-2 in Pissarides book.
  - As in the previous week, I will assume that you read this section as your reading assignment.
- We have two specific goals:
  - First, we want to understand the **Bellman equation (in continuous time)**, which is found almost everywhere in Pissarides book and professional articles in this field.
  - Second, we want to understand **job creation** in the labor market with search frictions.

Discounting

Discounting

- Suppose that the annual interest rate is 2%.
- Suppose also that you have 100 yen today.
- If you do not spend it, then your money will become $100 \times 1.02 = 102$ yen in the next year.
- Suppose you have a financial asset that promises you to pay 100 yen <u>after one year from today</u>.
- What is the value (or price) of this asset today?
- The answer is

$$\frac{100}{1.02} \approx 0.98$$

Discounting

- Let r denote the annual interest rate.
- A financial asset (or contract) promises you to pay X yen after one year from today.
- The **present value** of the asset (or contract) is

$$\frac{X}{1+r}$$

- If the asset pays X yen after T years from today, then the present value is

$$\frac{X}{(1+r)^T} = X(1+r)^{-T}$$

Discounting

- If interest is compounded n times (i.e., reinvested after each period) during a period, then we have nT subperiods, each bearing an interest rate of r/n.
- Then, the present value of the asset is

$$X\left(1+\frac{r}{n}\right)^{-nT}$$

- As n increases, the length of one subperiod gets smaller:
  - an year → a month → a week → a day → an hour → a minute → a second → …

Discounting

- Remember how e is defined:

$$\lim_{x \to \infty} (1 + x^{-1})^x = e$$

- Thus, the present value of the asset <u>in continuous time</u> is given by

$$\lim_{\frac{n}{r} \to \infty} X \left[\left(1 + \frac{r}{n} \right)^{\frac{n}{r}} \right]^{-rT} = Xe^{-rT}$$

Discounting

- In discrete time, if the asset pays X yen every year from today <u>forever</u>, then the value of the asset Y satisfies

$$Y = X + \frac{X}{1+r} + \frac{X}{(1+r)^2} + \frac{X}{(1+r)^3} + \cdots$$
$$= \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t X$$

- Thus, Y is an infinite sum.

Discounting

- In discrete time, Y is an infinite sum,

$$Y = \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t X$$

- In continuous time, the same infinite sum is expressed as

$$Y = \int_0^{\infty} e^{-rt} X dt$$

- The interest rate r is often referred to as the **discount rate**.

Discounting

- Consider

$$Y = \int_0^{\infty} e^{-rt} X dt$$

- We can calculate the right-hand side.
- Pick up your paper and pencil. After 3 or 4 lines of calculation, you should be able to show that

$$Y = \int_0^{\infty} e^{-rt} X dt = \frac{X}{r}$$

Bellman Equation

Bellman Equation

- Let r denote the annual interest rate.
- A financial asset (or contract) promises you to pay X yen every year from today forever. The present value of the asset Y satisfies

$$Y = X + \frac{X}{1+r} + \frac{X}{(1+r)^2} + \frac{X}{(1+r)^3} + \cdots$$

- Multiply both sides by $1/(1+r)$ to obtain

$$\frac{1}{1+r}Y = \frac{X}{1+r} + \frac{X}{(1+r)^2} + \frac{X}{(1+r)^3} + \cdots$$

Bellman Equation

- Subtract this expression from the previous one to obtain

$$Y = X + \frac{1}{1+r}Y$$

- The value Y is **recursively** defined.
  - You no longer need to consider infinite summation.
  - This is the **Bellman equation** <u>in discrete time</u>.
  - This is the simplest form. The Bellman equation is typically much bigger and more complicated.

Bellman Equation

- Suppose that the length of a period is δt, then,

$$Y = X\delta t + \frac{1}{1 + r\delta t} Y$$

- $X\delta t$ is the current payoff during an interval δt. It depends on the length the interval.
  - Consider annual wage versus hourly wage.
- $r\delta t$ is the interest rate during the interval. Again, it depends on the length of the interval.
  - Consider the long-term interest rate versus the overnight interest rate.

Bellman Equation

- Consider the Bellman equation:

$$Y = X\delta t + \frac{1}{1 + r\delta t} Y$$

- Multiply both sides by $1 + r\delta t$ to obtain

$$(1 + r\delta t)Y = (1 + r\delta t)X\delta t + Y$$

- Arrange terms to obtain

$$r\delta t Y = (1 + r\delta t)X\delta t$$

- Divide both sides by δt to obtain

$$rY = (1 + r\delta t)X$$

Bellman Equation

- Finally, take the limit as $\delta t \to 0$ to obtain the Bellman equation in continuous time as
$$rY = X$$
- Evidently, this implies
$$Y = \frac{X}{r}$$
- This verifies our previous result on page 10.

Derivation of (1.6)

- Remember that the Bellman equation takes the form:

$$Y = X\delta t + \frac{1}{1 + r\delta t} Y$$

- Substitute V for Y on the left-hand side to represent the **present value of one vacant job**.
- Substitute $-pc$ for X to represent the vacancy maintenance cost.
- Substitute $q(\theta)\delta t J + [1 - q(\theta)\delta t]V$ for Y on the right-hand side to represent the expected value.

Derivation of (1.6)

- Thus, equation (1.6) in discrete time is expressed as
$$V = -pc\delta t + \frac{1}{1 + r\delta t}\{q(\theta)\delta tJ + [1 - q(\theta)\delta t]V\}$$
- Interpretation:
  - If you open one vacancy, you must immediately pay the cost $pc\delta t$ to maintain it. For example, this is the cost of advertisement and doing preliminary screening of candidates.
  - J is the present value of a “filled job”, explained later.
  - Because the vacancy-filling probability is $q(\theta)\delta t$, with probability $q(\theta)\delta t$ this vacancy becomes filled, in which case you receive J instead of V.

Derivation of (1.6)

- Consider:
$$V = -pc\delta t + \frac{1}{1 + r\delta t}\{q(\theta)\delta tJ + [1 - q(\theta)\delta t]V\}$$
- Multiply both sides by $1 + r\delta t$ to obtain
$$(1 + r\delta t)V$$
$$= -(1 + r\delta t)pc\delta t + \{q(\theta)\delta tJ + [1 - q(\theta)\delta t]V\}$$
- Arrange terms to obtain
$$r\delta tV = -(1 + r\delta t)pc\delta t + q(\theta)\delta t(J - V)$$
- Divide both sides by δt to obtain
$$rV = -(1 + r\delta t)pc + q(\theta)(J - V)$$

Derivation of (1.6)

- Finally, take the limit as $\delta t \to 0$ to obtain the **Bellman equation in continuous time** as
$$rV = -pc + q(\theta)(J - V)$$
- This is (1.6).

Job Creation

Value of a Vacancy

- In discrete time,

$$V = -pc\delta t + \frac{1}{1 + r\delta t}\{q(\theta)\delta t J + [1 - q(\theta)\delta t]V\}$$

- In continuous time,

$$rV = -pc + q(\theta)(J - V)$$

- Interestingly, your current payoff is <u>negative</u>.
- However, you still want to create a job <u>in anticipation</u> that your state will change with probability $q(\theta)\delta t$.

Free Entry

- An important assumption in our model is that while the labor market is frictional, job creation is instantaneous.
- To be more specific, we assume that you or someone else will create an additional job as long as $V > 0$ because doing so is profitable.
- Because this process is instantaneous, the number of new jobs will be created immediately until

$$V = 0$$

- This is referred to as the **free entry condition**.

Free Entry

- We impose the free entry condition $V = 0$ to our Bellman equation

$$rV = -pc + q(\theta)(J - V)$$

- We then obtain

$$0 = -pc + q(\theta)(J - 0)$$

- Or,

$$J = \frac{pc}{q(\theta)}$$

- This is (1.7).

Value of a Filled Job

- Our maintained assumption is the only the filled job can produce output.
  - Remember that the production unit is a matched worker-firm pair.
- To simplify the model, <u>we ignore hours of work</u>. That is, an employee works for exactly one unit of each period.
- Let $p > 0$ denote the **labor productivity**.
  - Because the supply of labor is 1 unit of time, **output** is also given by p.

Value of a Filled Job

- Because output p is produced jointly with an employee, you must pay the **wage rate** w.
- After production, this pair is randomly destroyed for exogeneous reasons.
- The **separation rate** is $\lambda > 0$.
- With probability $\lambda \delta t$, this job becomes empty. In this event, you receive V instead of J.

Value of a Filled Job

- Equation (1.8) in discrete time is expressed as
$$J = (p - w)\delta t + \frac{1}{1 + r\delta t}\{\lambda \delta t V + (1 - \lambda \delta t)J\}$$
- Multiply both sides by $1 + r\delta t$ to obtain
$$(1 + r\delta t)J$$
$$= (1 + r\delta t)(p - w)\delta t + \{\lambda \delta t V + (1 - \lambda \delta t)J\}$$
- Arrange terms to obtain
$$r\delta t J = (1 + r\delta t)(p - w)\delta t + \lambda \delta t (V - J)$$
- Divide both sides by δt to obtain
$$rJ = (1 + r\delta t)(p - w) + \lambda (V - J)$$

Value of a Filled Job

- Take the limit as $\delta t \to 0$ to obtain the Bellman equation in continuous time as
$$rJ = p - w + \lambda(V - J)$$
- Because we assume free entry of vacancies, impose $V = 0$ above to finally obtain
$$rJ = p - w - \lambda J$$
- This is (1.8).
- We further rewrite it as
$$(r + \lambda)J = p - w$$

Job Creation Condition

- Under free entry, we have two equations:

$$J = \frac{pc}{q(\theta)}$$

$$(r + \lambda)J = p - w$$

- Eliminate J from these equations to obtain

$$(r + \lambda)\frac{pc}{q(\theta)} = p - w$$

- This is (1.9).
  - This is referred to as the **job creation condition**.
  - When $c = 0$, we have $w = p$ (wage rate = marginal product of labor), the competitive outcome.

Discussion

- Consider the vacancy cost pc, which is assumed to be proportional to the labor productivity.
- Why not simply c instead of pc?
- In many applications, we only need c. In many cases, you can drop p from the vacancy cost.
- However, this component will be necessary when we consider a growing economy.

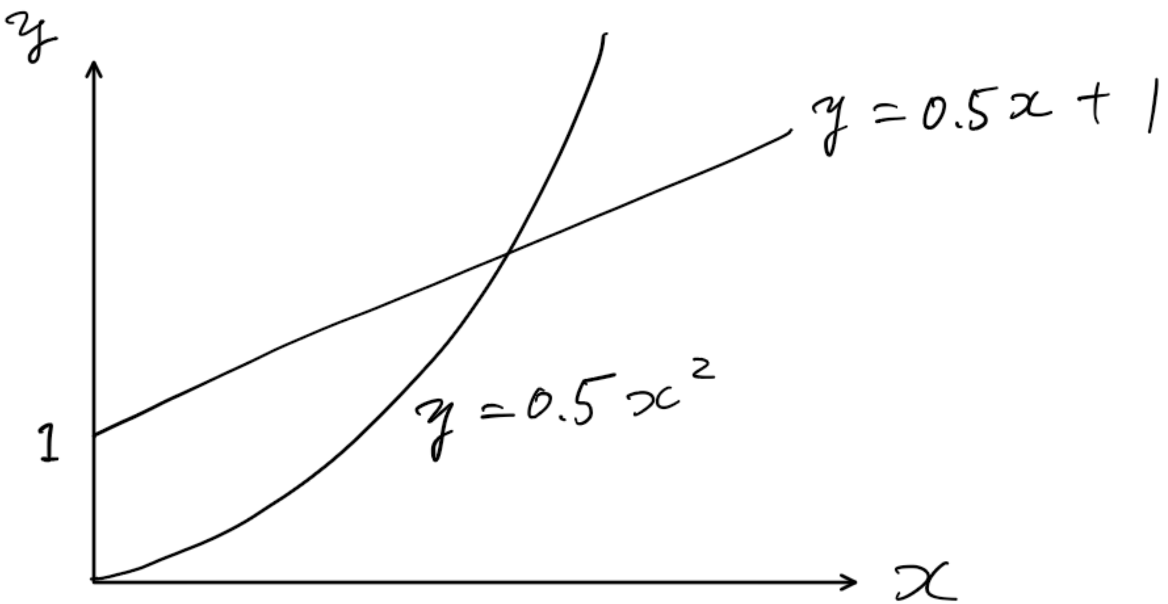
Introduction to Maxima

Example

- Consider the system of equations:

$$y = 0.5x + 1$$
$$y = 0.5x^2$$

- We focus on the positive region.
- Drawing a diagram is easy.



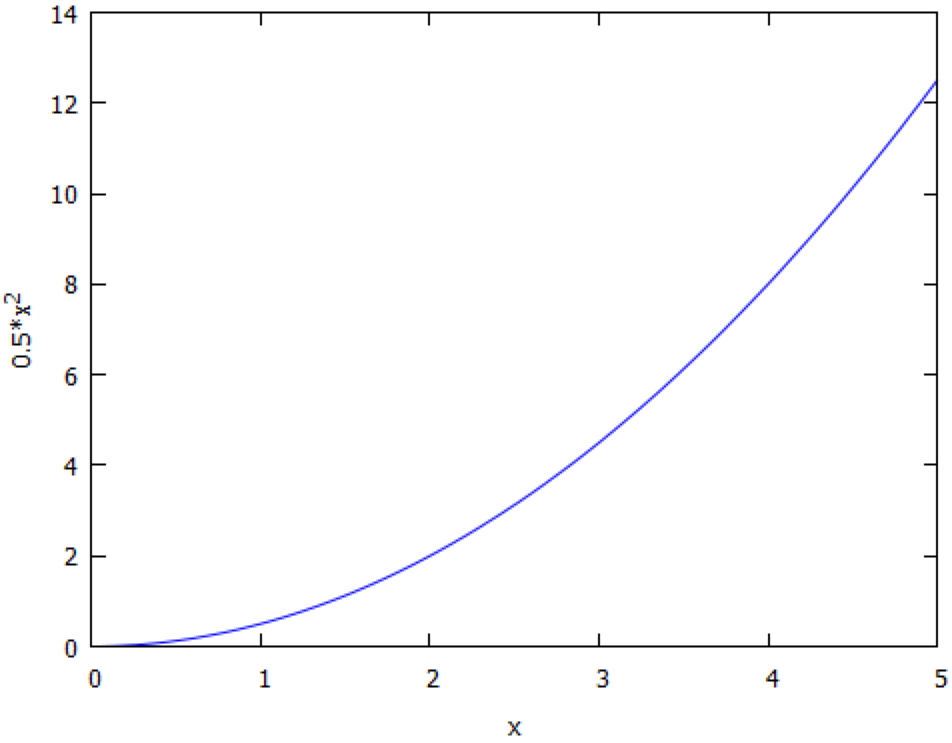
Example

- We can easily solve the equation by eliminating y:
$$0.5x^2 = 0.5x + 1$$
- Equivalently,
$$x^2 - x - 2 = 0 \Leftrightarrow (x - 2)(x + 1) = 0$$
- Thus, the positive solution is $x = 2$.
- Our first goal is to solve this problem using Maxima so that we can verify the result.

Maxima

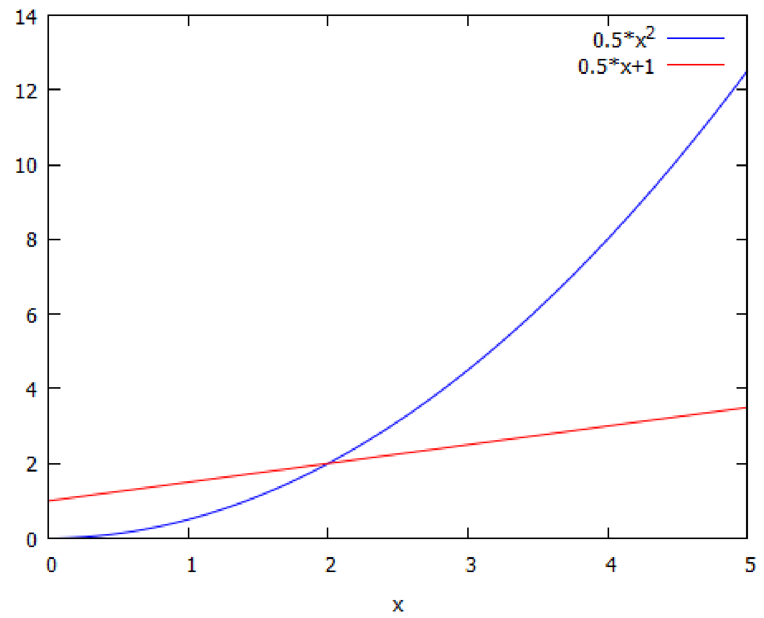
- On Maxima, execute
$$\mathrm{plot2d}(0.5 * \mathrm{x}^\wedge 2, [\mathrm{x}, 0{,}5]);$$
- $\mathrm{plot2d}()$ will make a two-dimensional diagram.
- Inside, $0.5 * \mathrm{x}^\wedge 2$ is the translation of $0.5x^2$.
- $[\mathrm{x}, 0{,}5]$ determines the plot region. It reads, x is from 0 to 5.
- Each command must end with semicolon ";".
- To execute a command, press "shift + enter".
- The output is on the next page.

Maxima



Maxima

- Execute
  $$\text{plot2d}([0.5 * x\hat{}2, 0.5 * x + 1], [x, 0,5]);$$
- This will make multiple plots.



Maxima

- To solve $x^2 - x - 2 = 0$, execute

$$\mathrm{solve(x\text{^}2 - x - 2 = 0, x);}$$

- To solve $y = 0.5x + 1, y = 0.5x^2$, execute

$$\mathrm{solve([y = 0.5 * x + 1, y = 0.5 * x\text{^}2], [x, y]);}$$

- This will give you a list of solutions.
- We can assign a new variable, soln, to the list by $\mathrm{soln\!: solve([y = 0.5 * x + 1, y = 0.5 * x\text{^}2], [x, y]);}$
- Then execute

$$\mathrm{soln;}$$

Maxima

- You can see that the variable "soln" is a **list**:
$$[[x = -1, y = 1/2], [x = 2, y = 2]]$$
- Sometimes we need to take out the numerical value such as "2" from the list for later use.
- Notice that two lists are in the list "soln". First, take out the second list from the bigger list by
$$\mathrm{soln}[2];$$
- This will take out the second element from soln and will give you another list
$$[x = 2, y = 2]$$

Maxima

- Thus, $\mathrm{soln}[2]$ is a list:
$$[x = 2, y = 2]$$
- To take out the solution $x = 2$, we need
$$\mathrm{soln}[2][1];$$
- This will take out the first element from $\mathrm{soln}[2]$. The result is "$x = 2$" not "2".
- To take out "2", we need
$$\mathrm{rhs}(\mathrm{soln}[2][1]);$$

Maxima

- Now, we generalize the system of equations:

$$y = 0.5x + b$$
$$y = 0.5x^2$$

- b is the parameter of interest.
- Suppose we want to study the impact of an increase in b on the solution.
- Let us write a Maxima code for this analysis.

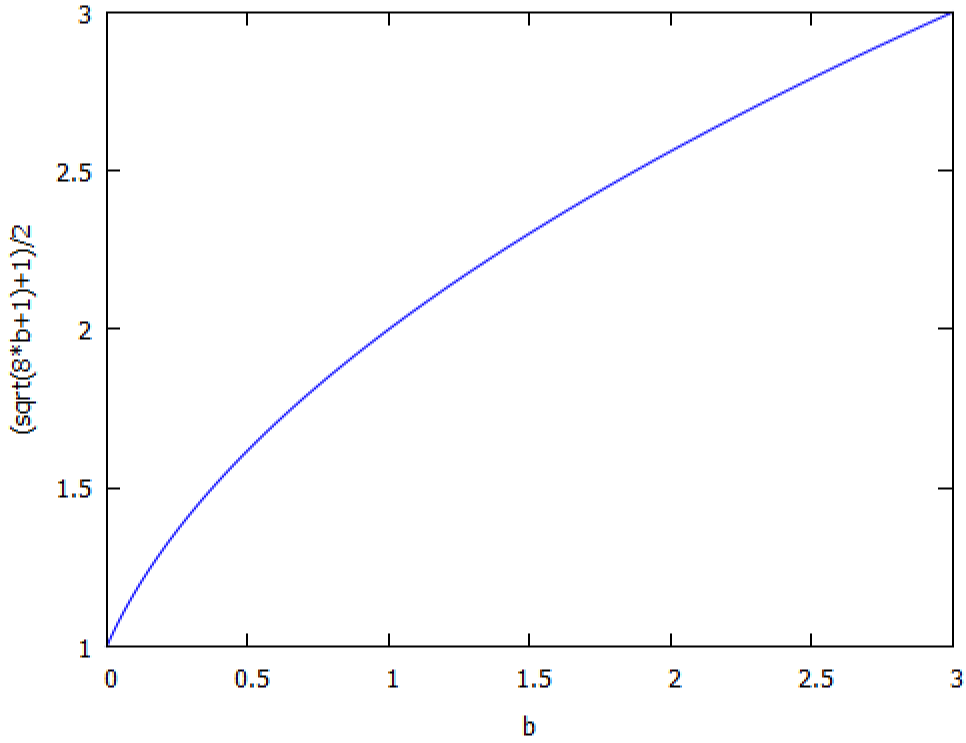
Maxima

- Execute

```
plot2d(rhs(solve([y = 0.5 * x + b, y = 0.5 * x^2], [x, y])[2][1]), [b, 0,3]);
```

- This will plot the solution x against b in the region $0 \leq b \leq 3$.
- The idea is that we plot a function of b such that the input is a value of b and the output is the solution to the equation for each value of b.
- The result is on the next page.

Maxima



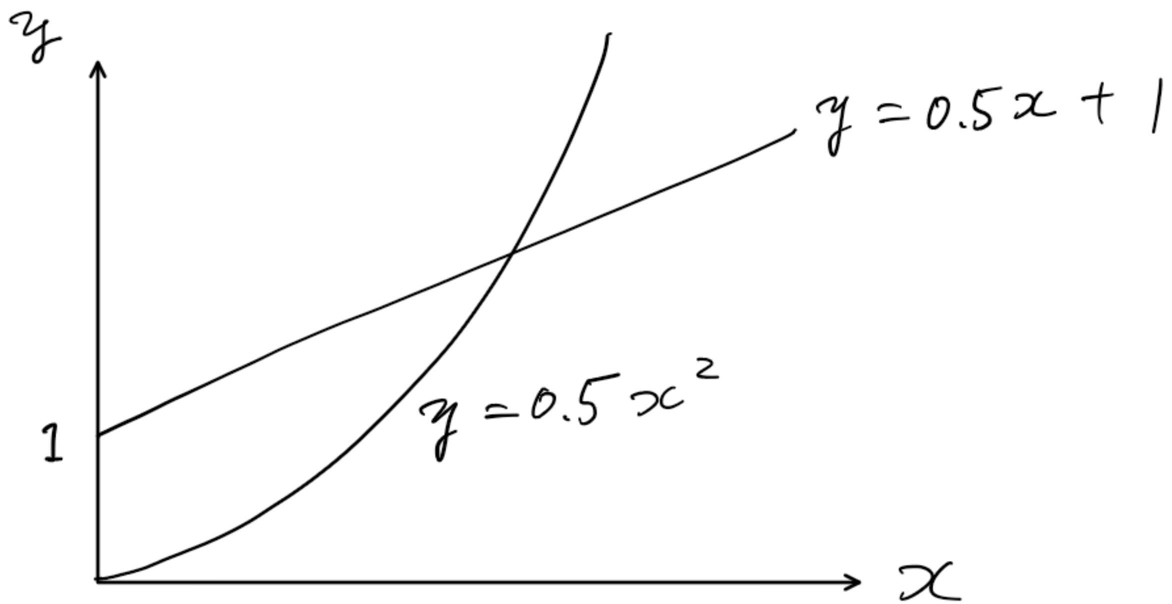
Introduction to Python

Example

- Consider:

$$y = 0.5x + 1$$
$$y = 0.5x^2$$

- We focus on the positive region.
- Drawing a diagram is easy.

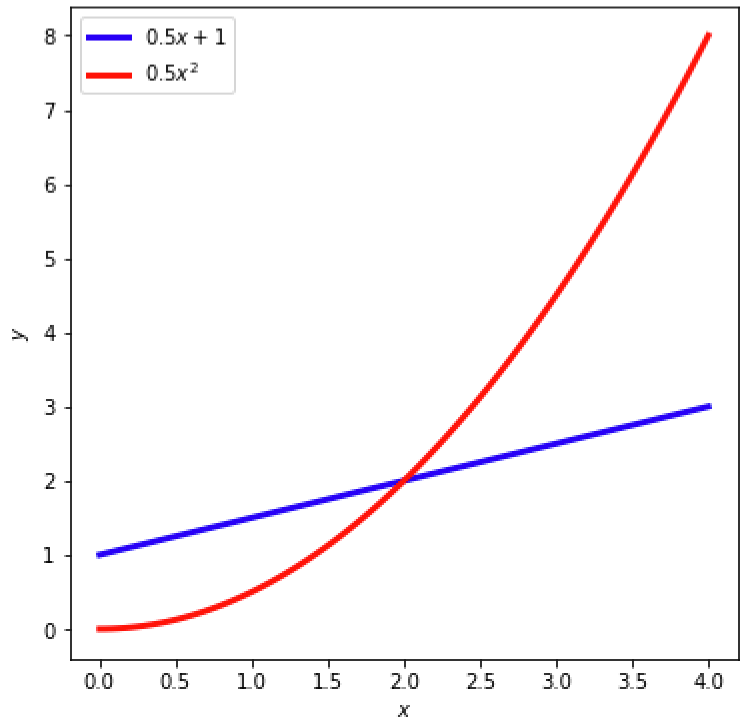


Python: Diagram

PythonCode_DiagramExample.txt

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = (6,6)

grid = np.linspace(0, 4, 100)
fig, ax = plt.subplots()

eq1 = 0.5*grid + 1
eq2 = 0.5*(grid**2)

ax.plot(grid, eq1, 'b-', lw=3, label='$0.5x+1$')
ax.plot(grid, eq2, 'r-', lw=3, label='$0.5x^2$')

ax.set_xlabel('$x$')
ax.set_ylabel('$y$')
ax.legend()

plt.show()
```

This code is available at NUCT.



Python: Solving Equations

- "f" is a list of equations.
- "x" is a list of variables.
  - x[0] = y
  - x[1] = x
  - "x[1]**2" = x^2
- root(func,[1,1]) finds the list of x and y solving the equations.
- sol.x is a list (y, x).

```
import numpy as np
from scipy.optimize import root
def func(x):
    f = [x[0] - 0.5 * x[1] - 1,
         x[0] - 0.5*(x[1]**2)]
    return f
sol = root(func, [1,1])
print(sol.x)
```

```
[2. 2.]
```

Python: Solution as a Function

PythonCode_Function_EquationSolver.txt

This code is available at NUCT.

- Let us take out the solution and define it as a function of a parameter.
- Consider
$$y = 0.5x + b$$
$$y = 0.5x^2$$
- compstat() returns the solution a function of b.

```
import numpy as np
from scipy.optimize import root
def compstat(b):

    def func(x):
        f = [x[0] - 0.5 * x[1] - b,
             x[0] - 0.5*(x[1]**2)]
        return f
    sol = root(func, [1,1])

    return sol.x[1]

compstat(1)
```

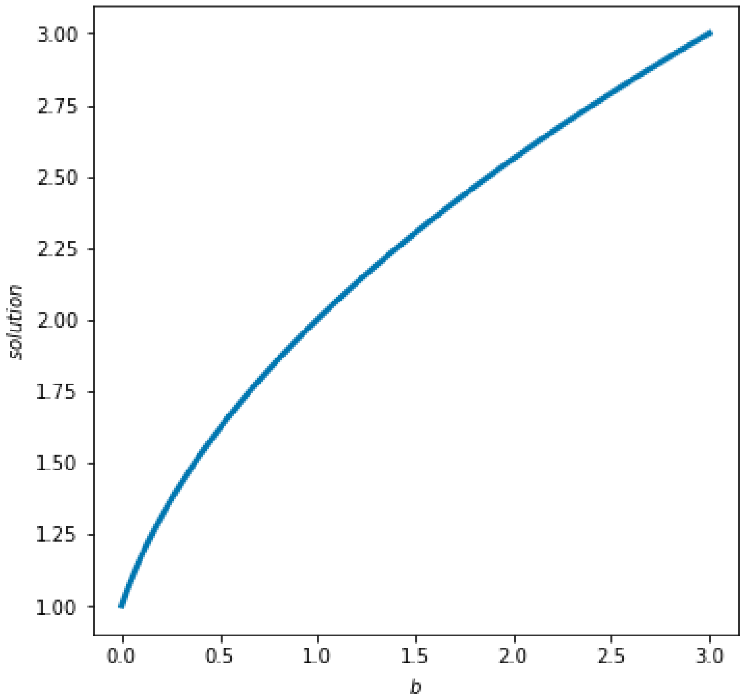
```
2.0
```

Python: Solution as a Function

PythonCode_SolutionPlot.txt

This code is available at NUCT.

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = (6,6)

grid = np.linspace(0, 3, 100)
fig, ax = plt.subplots()
values = []

for pt in grid:
    y = compstat(pt)
    values.append(y)


ax.plot(grid, values, lw=3)

ax.set_xlabel('$b$')
ax.set_ylabel('$solution$')

plt.show()
```



Reading Assignment

Reading Assignment

- Christopher A. Pissarides, *Equilibrium Unemployment Theory*, second edition, MIT Press, 2000.
- Read Section 1.3 (Workers).
- Due is 5/12.
- 5/12 Class will focus on this section.