2023 年 3月14日　最終講義

渦運動と乱流
ー　やり残したこと

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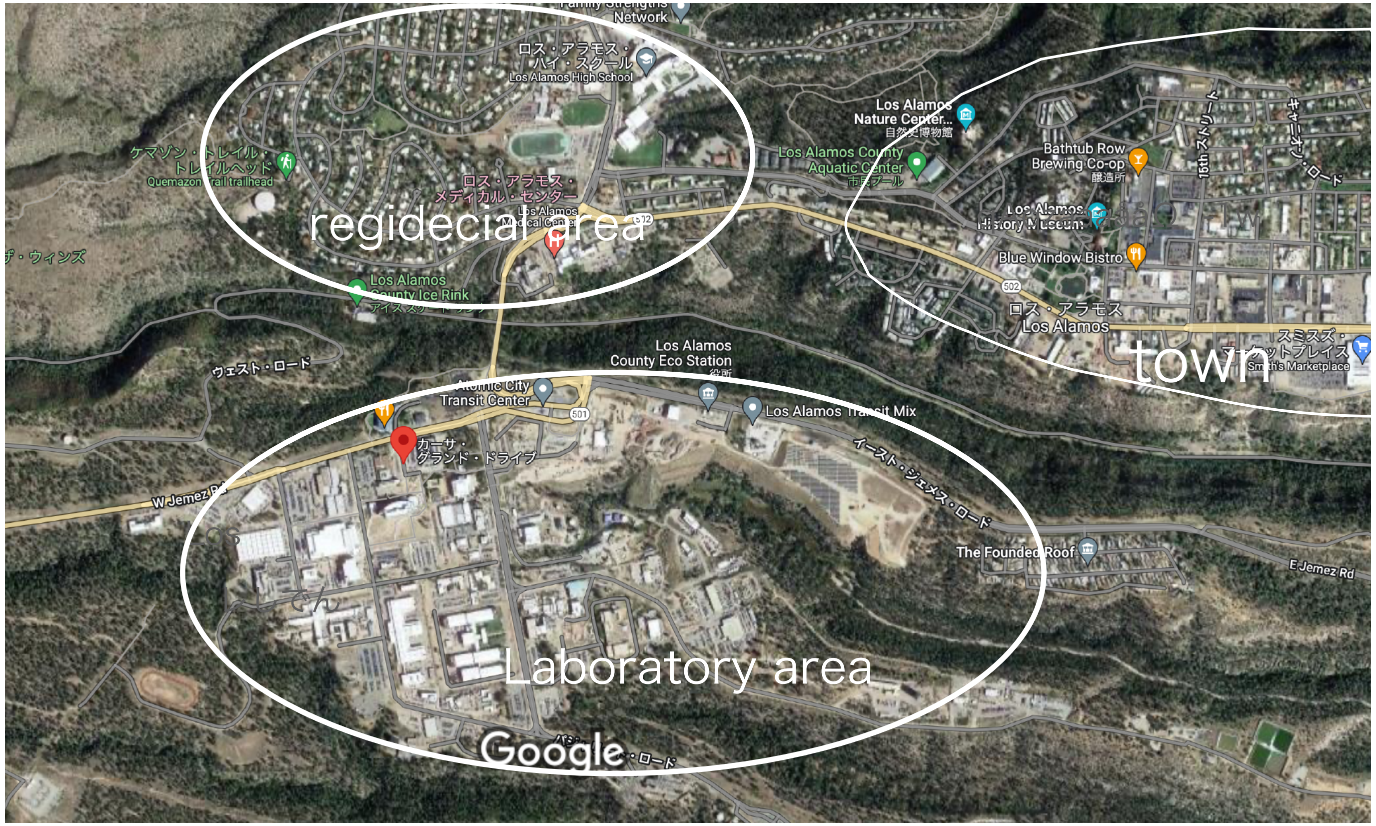
Los Alamos National Laboratory

https://cnls.lanl.gov/external/History.php　2023.3.31



Center for Nonlinear Studies (1980年設立)

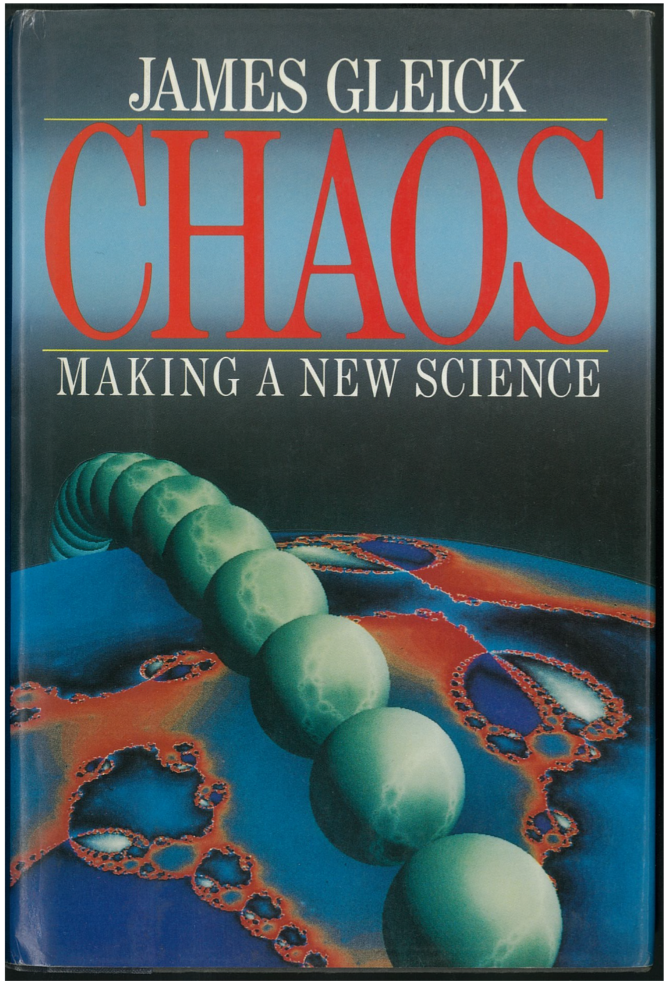
Soliton や chaos といった非線型性の特徴を踏まえ物理系の研究のあらたなパラダイムを構築することを目標としている。

soliton -> 可積分性

chaos -> 統計性

t

Well paid and free from the academic pressure to teach and publish !

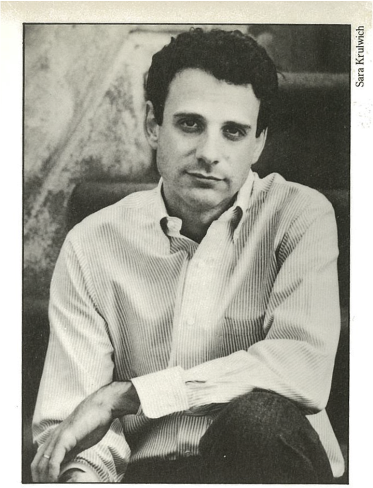


"Chaos: Making A New Science" James Gleick, London: Heinemann, 1987

1987年出版のベストセラー

Chaos に関しては今読んでも最良の　'啓蒙書' .
これとE. Ghys の「Chaos」 の movieを見れば
数式を使わずに何が本質なのか分かる.

ある人物の夜間の奇行から話は始まる…



JAMES GLEICK was born in New York City and lives there with his wife, Cynthia Crossen. Since 1978, he has been an editor and reporter at the *New York Times*.

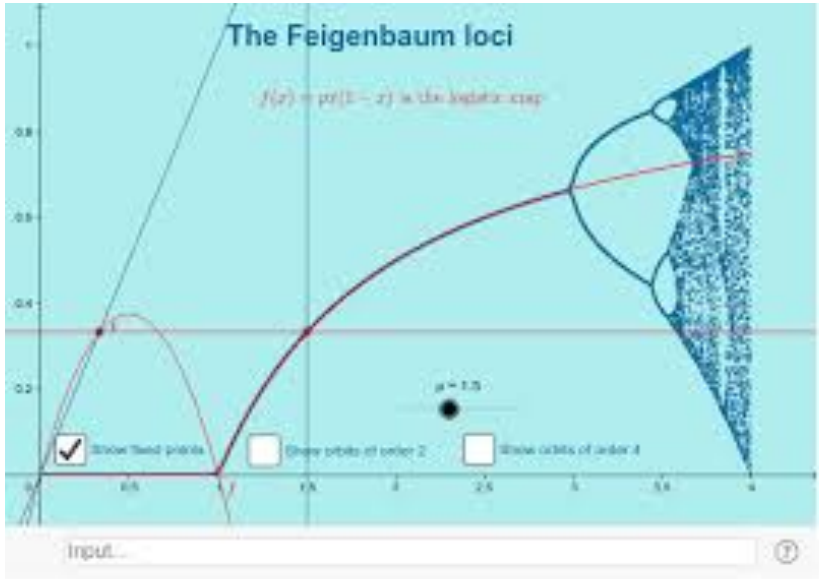
Wikipedia

James Glick (1954~)

New York Post のScience writer. "Chaos" を始めとし "Information" の将来を的確に予言している. cf. The Information: A History, a Theory, a Flood (2011).

Mitchell J. Feigenbaum

https://digitalcommons.rockefeller.edu/faculty-members/103/　2023.3.31



https://www.geogebra.org/m/FQrUDRcf　2023.3.31

$$y_{k+1} = a\ y_k(1 - y_k)$$ Logistic map

a を変化させていったときに平衡点が次々に’倍化していく.

倍化が起こる a に注目すると

$$\lim_{k\to\infty} \frac{a_{k+1} - a_k}{a_{k+2} - a_{k+1}} = \delta$$

δ の値は $\delta = 4.6692\cdots\cdots$ と確定する.（1975)

Gleick の Chaos によると

レーザー核融合に取り込むこと、
素粒子のスピン、カラー、フレーバーを解明すること、
宇宙開闢の時間を決定すること、

などは物理学者が行うべき由緒正き（legitimate) 問題である.
Feigenbaum はこれらの問題は有能な物理学者が集中して考え、計算すれば解ける容易い（obvious) 問題
あると言うであろう.

和達三樹先生

Brosl Hasslacher **Mitchell J. Feigenbaum**

Feigenbaum に

あなたは後進の研究者にlegitemate とは言えない研究を奨励しますよね？

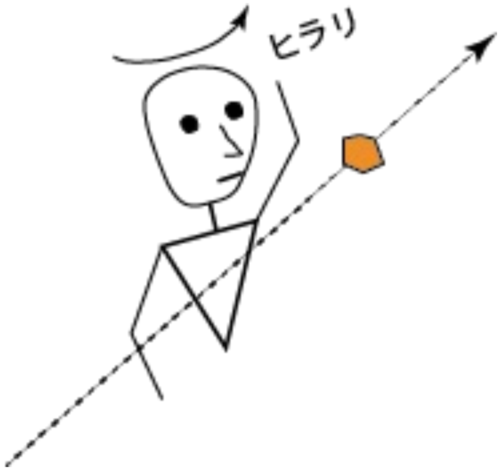
と直接聞いてみた。

He said,

“… I was lucky. “

物理屋の思考方法

相手に物を投げて反応を見る.



弾性散乱

→ 応答関数，Green核

（深部）非弾性散乱

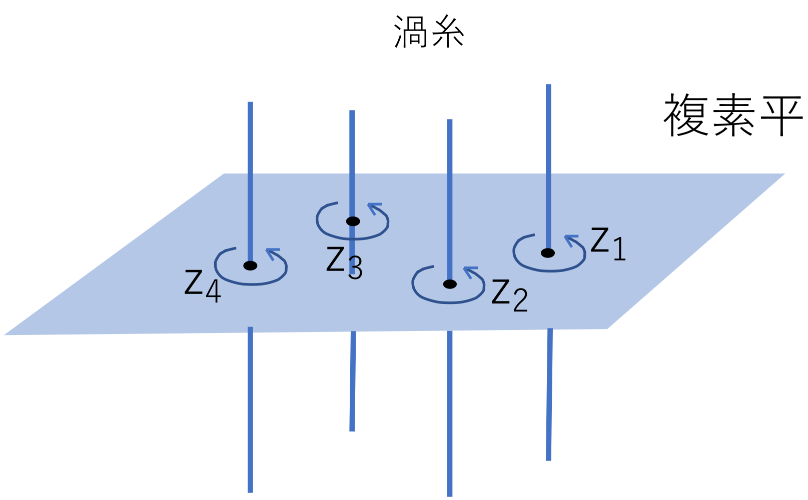
→ 衝突軌道，特異点，ブローアップ

衝突問題

三体問題の非可積分性　（吉田春夫先生）

流体における衝突

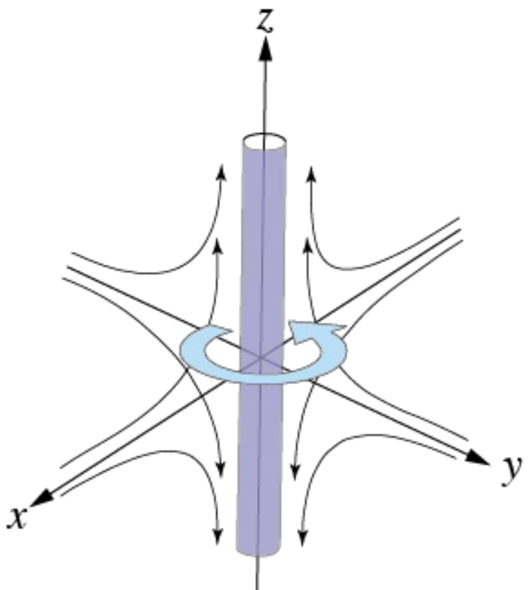
点渦



$$\frac{dz_j}{dt} = \frac{i}{2\pi}\sum_{k\neq j}^{N} \frac{\Gamma_k}{\bar{z}_j - \bar{z}_k}, \qquad (j = 1, 2, 3, \cdots)$$

Γ_k ：渦の強さ（循環）

Burgers 渦（引き伸ばされた渦）



渦管＋外部淀み点流 $\mathbf{U} = (-\alpha x/2, -\alpha y/2, \alpha z)$

$$\omega_z = \frac{\alpha\Gamma}{4\pi\nu}\exp\left[-\frac{\alpha(x^2+y^2)}{4\nu}\right]$$

$\delta = \sqrt{\frac{\nu}{\alpha}}$ （渦管のコア半径） → Navier-Stokes 方程式の定常解

点渦系の自己相似解　I

Journal of the Physical Society of Japan
Vol. 56, No. 6, June, 1987, pp. 2024–2030

Similarity Solution of Two-Dimensional Point Vortices

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Department of Physics, Faculty of Science,
Tokyo University, Hongo, Bunkyo-ku, Tokyo 113

(Received December 11, 1986)

$$\frac{dz_j}{dt} = \frac{i}{2\pi}\sum_{k\neq j}^{N} \frac{\Gamma_k}{\bar{z}_j - \bar{z}_k}, \quad (j = 1, 2, 3, \cdots)$$

$z_j = \xi_j \, f(t)$ →

$$\xi_j \frac{df}{dt}\bar{f} = \frac{i}{2\pi}\sum_{k\neq j}^{N} \frac{\Gamma_k}{\bar{\xi}_j - \bar{\xi}_k} \quad (j = 1, 2, 3, \cdots)$$

↓ 変数分離

$$\begin{cases} \dfrac{df}{dt}\bar{f} = C \equiv A + iB \\ C\,\xi_j = \dfrac{i}{2\pi}\displaystyle\sum_{k\neq j}^{N} \frac{\Gamma_k}{\bar{\xi}_j - \bar{\xi}_k} \end{cases} \quad (j = 1, 2, 3, \cdots)$$

時間関数：

$$f(t) = r(t)\ \exp[i\theta(t)]$$

$$\frac{df}{dt}\bar{f} = C \equiv A + iB$$

$$\begin{cases} \dfrac{1}{2}\dfrac{d(r^2)}{dt} = A, \quad r(0) = 1 \\ \dfrac{d\theta}{dt} = \dfrac{B}{r^2} \end{cases}$$

$$r(t) = \sqrt{2At + 1}$$

$$\theta(t) = \begin{cases} \dfrac{B}{2A}\log(2At + 1) & (A \neq 0) \\ Bt + \theta(0) & (A = 0) \end{cases}$$

$A = 0, \quad B = 0$: fixed point
$A = 0, \quad B \neq 0$: rigid rotation
$A \neq 0, \quad B = 0$: straight collision
$A \neq 0, \quad B \neq 0$: collapse, expansion

点渦系の自己相似解　II

$$C\,\xi_j = \frac{i}{2\pi}\sum_{k\neq j}^{N}\frac{\Gamma_k}{\bar{\xi}_j - \bar{\xi}_k} \quad (j = 1,2,3,\cdots)$$

N=3 の時

$$C = iB$$ （剛体回転解）

$$X \equiv \frac{\xi_1-\xi_3}{\xi_1-\xi_2}$$ について

$$(\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_3\Gamma_1)(X^2+X+1)\left[(\Gamma_1+\Gamma_2)X^3+(2\Gamma_1+\Gamma_2)X^2-(\Gamma_2+2\Gamma_3)X-(\Gamma_2+\Gamma_3)\right] = 0$$

=0 : 衝突条件　→　$$\sum_{j=1}^{N}\frac{1}{\Gamma_j} = 0$$

正三角形解　→　Lagrange 解（円分多項式）　→　正三角形の頂点におかれた3つの渦点は強さによらず重心の周りを剛体回転する.

一直線解　???

同種点渦系の平衡直線解　　－ Stieltjesの解

$$P_N(x) \;=\; \prod_{j=1}^{N}(x - x_j)$$ について

$$P_N''(x) \;-\; 2xP_N'(x) \;=\; -2NP_N(x)$$ ：Hermite 方程式

が成りたつ．これより直線上のHermite 多項式で与えられる位置にあるN個の同種の点渦は平衡解となる．（Stieltjes, 1885）

青本和彦：直交多項式入門（数学書房）

先の一直線解において $G_1 = G_2 = G_3 = G$ とおいてもHermite 方程式にならない．

曲面上の渦運動

THE ROYAL SOCIETY

Vortex motion on surfaces with constant curvature

By Yoshifumi Kimura

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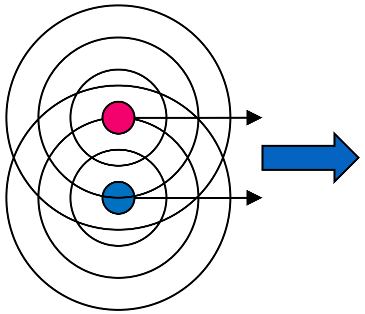
Received 22 September 1997; accepted 9 April 1998

Vortex motion on two-dimensional Riemannian surfaces with constant curvature is formulated. By way of the stereographic projection, the relation and difference between the vortex motion on a sphere (S^2) and on a hyperbolic plane (H^2) can be clearly analysed. The Hamiltonian formalism is presented for the motion of point vortices on S^2 and H^2. The set of first integrals for each Hamiltonian shows a corresponding algebraic property in terms of the Poisson bracket defined, respectively, for S^2 and H^2. As an example of analytic solutions, the motion of a vortex pair (dipole) is considered. It is shown that a dipole draws a geodesic curve as its trajectory on S^2 and H^2.

Keywords: vortex motion; hyperbolic plane; geodesic; sphere; Hamiltonian; Laplacian

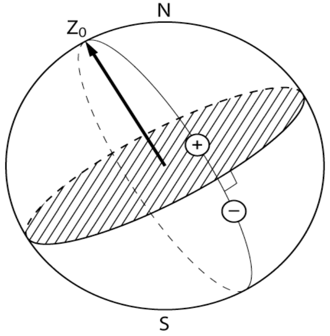
可積分な運動の例

vortex dipole



平面内では直進運動をする。球面、双曲面上ではどうか？

S^2

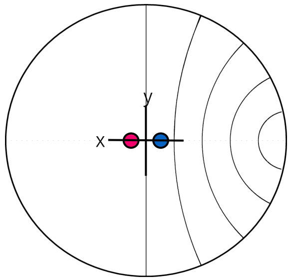


変数変換によって

$$\begin{cases} \theta_1(t) = \theta_1(0) \\ \phi_1(t) = \dfrac{1}{\cos\theta_1(0)} t + \phi_1(0) \end{cases} \longrightarrow$$

大円

H^2



$$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = \dfrac{1}{4}\epsilon(1-y^2)^2 \end{cases} \longrightarrow$$

(0,1)に漸近

conjecture: 2次元曲面上でvortex dipoleの軌跡は測地線を描く

渦対の軌跡 I

JOURNAL OF GEOMETRIC MECHANICS
doi:10.3934/jgm.2018007

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pp. **189–208**

VORTEX PAIRS ON A TRIAXIAL ELLIPSOID AND KIMURA'S CONJECTURE

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(Communicated by James Montaldi)

Abstract. We consider the problem of point vortices moving on the surface of a triaxial ellipsoid. Following Hally's approach, we obtain the equations of motion by constructing a conformal map from the ellipsoid into the sphere and composing with stereographic projection. We focus on the case of a pair of opposite vortices. Our approach is validated by testing a prediction by Kimura that a (infinitesimally close) vortex dipole follows the geodesic flow. Poincaré sections suggest that the global flow is non-integrable.

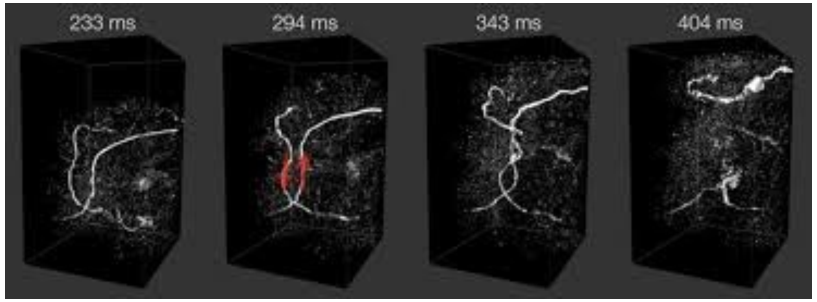
渦対の軌跡 II

Points of discussion :

(1) dipole and dipole limit (in a hyperbolic situation).

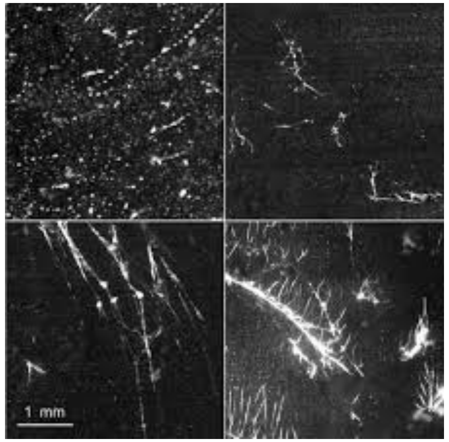
(2) local(geodesic) and global (vortex motion via Green’s fn.).

３次元の渦衝突　　ー vortex reconnection



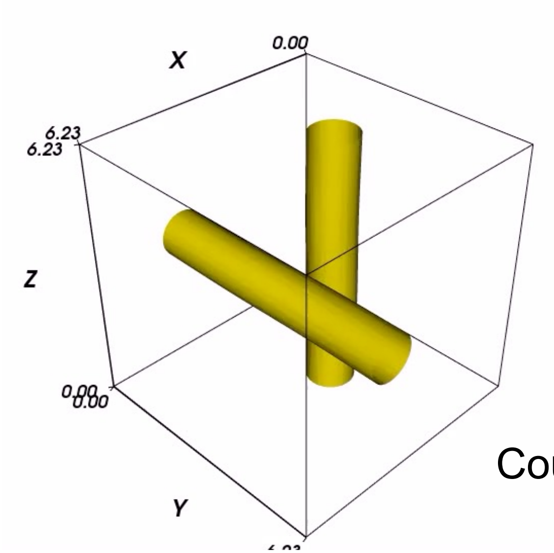
classical fluids

Kleckner, D. & Irvine, W.T.M.
Creation and dynamics of knotted vortices.
Nature Physics 9, 253258. (2013)
doi: 10.1038/nphys2560.



quantum fluids

Bewley, G.P., Paoletti, M.S., Sreenivasan, K.R., & Lathrop, D.P.: Characterization of reconnecting vortices in superfluid helium
Proc. Natl. Acad. Sci. 105, 13707 (2008).



Courtesy : T. Matsumoto

Regularization of Euler's equation

Euler's eq.

for perfect fluids

（conservation of Helicity
-> no reconnection）

regularization by sound waves

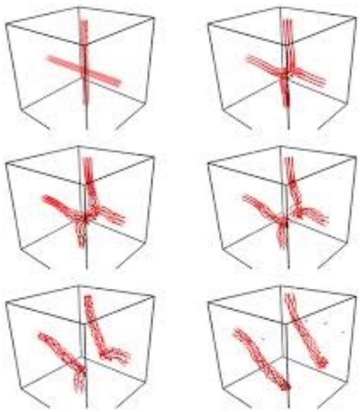
regularization by viscosity

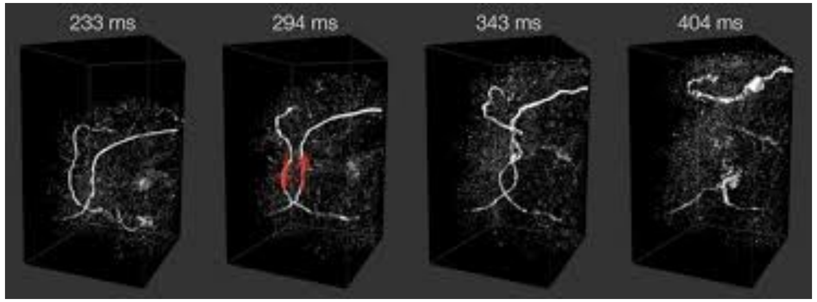
Gross-Pitaevskii eq.

for quantum fluids（BEC）

Navier-Stokes eq.

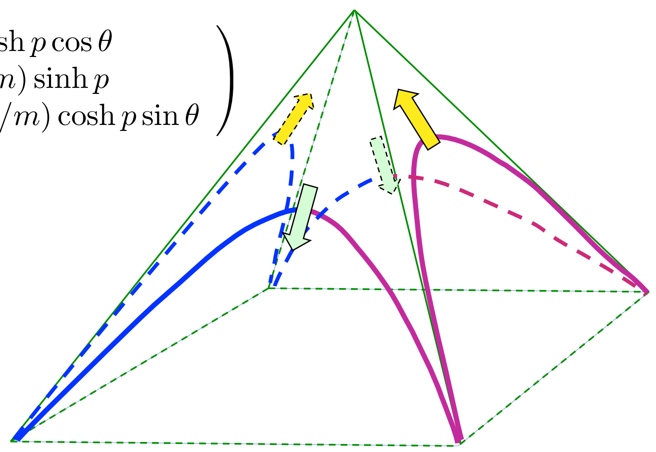
for viscous fluids





Tent model of reconnection

Y. Kimura & H.K. Moffatt, A tent model of vortex reconnection under Biot-Savart evolution. *Journal of Fluid Mechanics* **834** (2018) R1.



$$\mathbf{u}\left(\mathbf{x}(s)\right) = -\frac{\Gamma}{4\pi}\int \frac{\mathbf{t}(s') \times (\mathbf{x}(s) - \mathbf{x}(s'))}{|\mathbf{x}(s) - \mathbf{x}(s')|^3} ds'$$

Biot-Savart integral

discretization

$$\mathbf{u}(\mathbf{x}_i) = -\frac{\Gamma}{4\pi}\sum_{\substack{j=0 \\ j\neq i}}^{n-1} \frac{\mathbf{t}_j \times (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3}\, ds_j \qquad (i = 0, 1, \cdots, n-1)$$

The integral is calculated by a sum, and the singular term is removed (The cut-off method)

cf.
De Waele, A.T.A.M. & Aarts, R.G.K.M.
Route to vortex reconnection.
Phys. Rev. Lett. **72** (1994) 482-485.

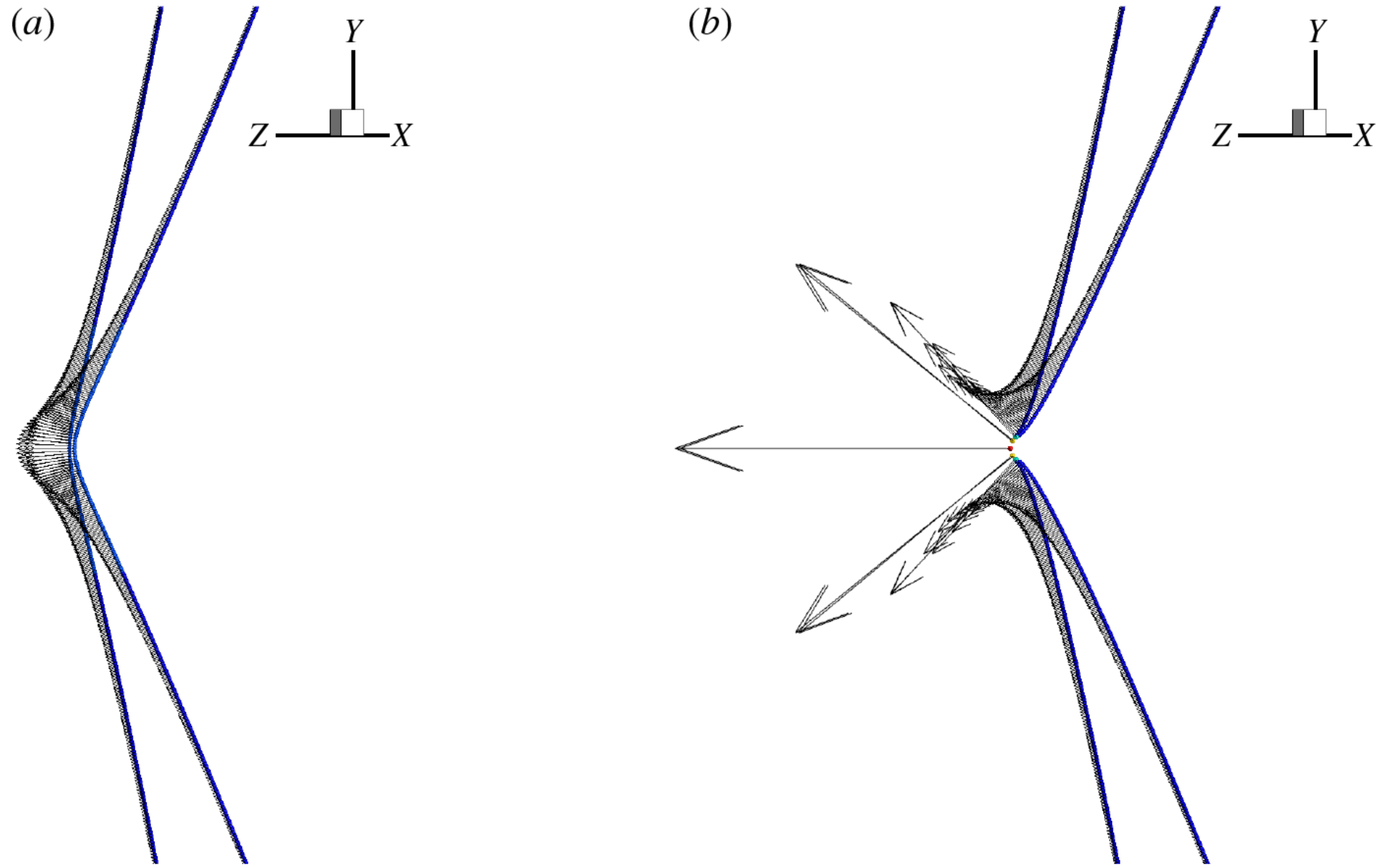


FIGURE 3. Tent-model configuration and velocity vectors at points on the vortices Γ_1 and Γ_2 ($m = 0.35$, $c = 0.1$, $\theta = \pi/4$, $n = 19\,202$); (*a*) $t = 0.449646$; (*b*) $t = 0.449716$.

Mathematical theories for a singularity of the NS eq.

(1) Beale, Kato & Majda (1984)

If a finite-time singularity occurs at $t = t_c$ (for the Euler), the vorticity $\omega = \nabla \times \boldsymbol{u}$ must be unbounded as $t \to t_c$, i.e.

$$\int_0^{t_c} \sup|\boldsymbol{\omega}(\mathbf{x}, t)|_{\mathbf{x}\in R^3} dt = \infty$$

→ *Need a vortex-stretching mechanism to sustain the emergence of a singularity.*

(2) Seregin & Šverák (2002)

If a finite-time singularity occurs at $t = t_c$ (for the NS), the pressure field $p(\boldsymbol{x},t)$ becomes unbounded below.

→ *Consistent with the situation that a singularity appears in the core of a vortex.*

(3) Caffarelli, Kohn & Nirenberg (1982)

If a singularity occurs then the space–time Hausdorff dimension of the singularity is point-like.

→ *The above mechanism should have the effect of localization as well as amplification.*

Moffatt, H.K. & Y. K.,
Towards a finite-time singularity of the Navier-Stokes equations Part 1. Derivation and analysis of dynamical system
J. Fluid Mech. (2019) **861** 930-967.

Strain around vortices

$$\mathbf{u}(q) = -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{(\mathbf{x}(q)-\mathbf{x}(p))\times\mathbf{x}'(p)\ dp}{|\mathbf{x}(q)-\mathbf{x}(p)|^3}$$

Biot-Savart integral

$$\partial_i u_j(\mathbf{x}) = \frac{\Gamma}{4\pi}\int\left[\frac{3\,(x_i - x_i(p))\,[(\mathbf{x}-\mathbf{x}(p))\times d\mathbf{x}(p)]_j}{|\mathbf{x}-\mathbf{x}(p)|^5} + \epsilon_{ijk}\frac{dx_k(p)}{|\mathbf{x}-\mathbf{x}(p)|^3}\right]$$

deformation tensor

$$\mathbf{e} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}})$$

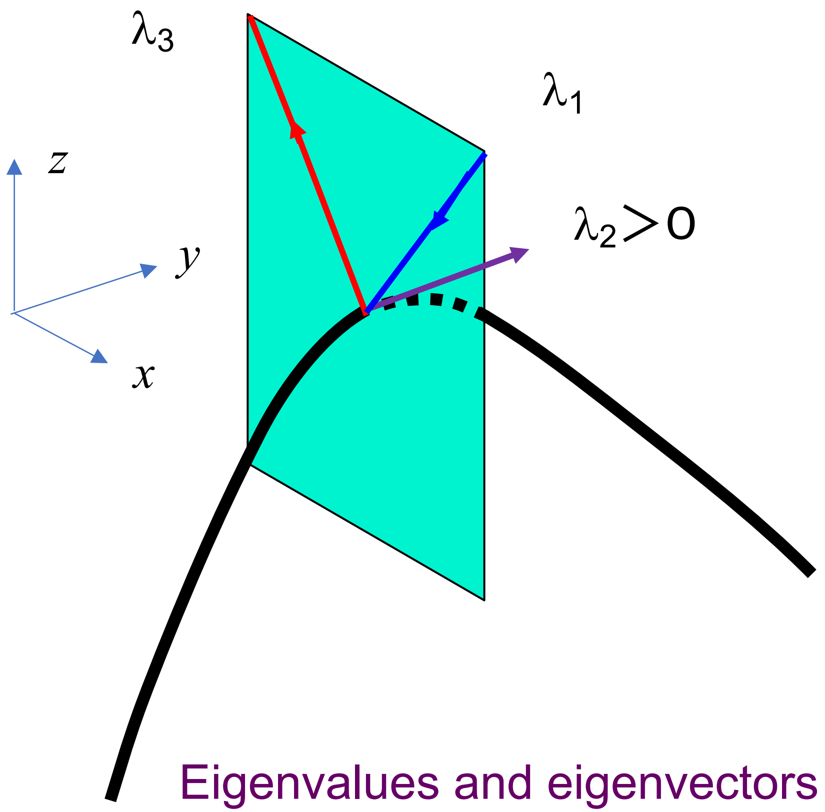
rate of strain tensor

Three eigenvalues of $\mathbf{e}$: $\lambda_1 \leq \lambda_2 \leq \lambda_3$

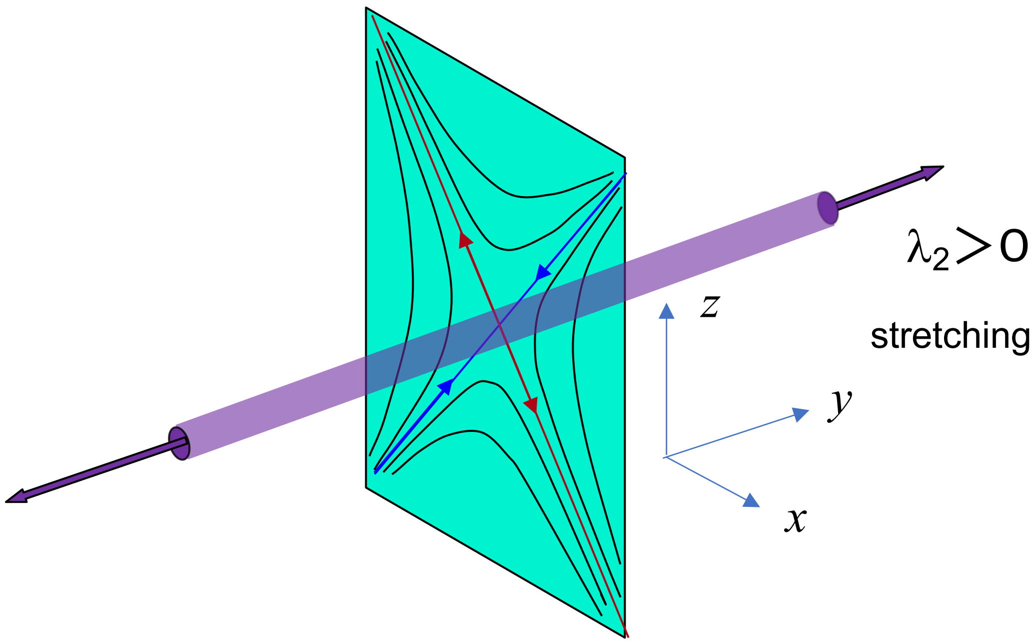
Because of incompressibility : $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 \leq 0,\quad \lambda_3 \geq 0 \quad \lambda_2\ ?$

Similarity with non-axisymmetric Burgers vortex



Eigenvalues and eigenvectors of the strain field



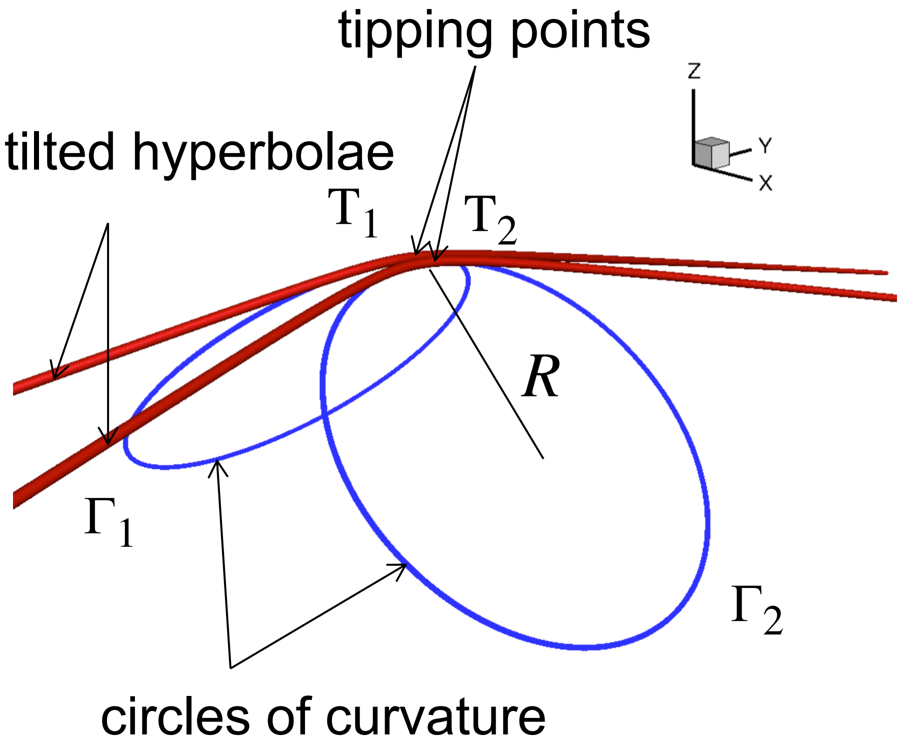
Non-axisymmetric Burgers vortex

$$\mathbf{U} = (\lambda_1 x, \lambda_2 y, \lambda_3 z) \qquad \lambda_2 > 0$$

$$\lambda_1 \leq \lambda_2 \leq \lambda_3, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

local strain field

Tilted hyperbolae → Tilted vortex rings



hen at the tipping point T_1

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

$$\mathbf{v}_1 = -\frac{\kappa_0 \Gamma}{4\pi}\left[\ \log(1/\kappa_0\delta) + \beta\ \right]\mathbf{b}$$

(self-induced velocity)

$\mathbf{b}$ → binormal unit vec.

$\kappa_0 = 1/R$: curvature

$\beta = 0.8283$ for Gaussian core

$\beta = 1.136$ for uniform vorticity core (Saffman 1970)

$$\mathbf{v}_2(\mathbf{x}_{T_1}) = \frac{\Gamma}{4\pi}\oint_{\Gamma_2}\frac{d\mathbf{x}' \wedge (\mathbf{x}_{T_1} - \mathbf{x}')}{|\mathbf{x}_{T_1} - \mathbf{x}'|^3}$$

can be evaluated in terms of elliptic integrals analytically

velocity —analytically→ rate of strain tensor

Dynamical system describing the state near the tipping point

The tipping point dynamics is determined completely by

$s(\tau)$: 1/2 of the minimum separation of vortices

$\kappa(\tau)$: curvature at the tipping points

$\delta(\tau)$: core radius

$\gamma(\tau)$: $= \Gamma_s/\Gamma$ = (surviving) circulation/ original circulation

$\tau = (\Gamma/R^2)t$ (dimensionless time)

expression of velocity ⟶ $$\frac{ds}{d\tau} = -\frac{\kappa\cos\alpha}{4\pi}\left[\log\left(\frac{s}{\delta}\right) + \beta_1\right]$$

expression of curvature ⟶ $$\frac{d\kappa}{d\tau} = \frac{\kappa\cos\alpha\sin\alpha}{4\pi s^2}$$

NS eq (Burgers type vortex) + similarity assumption ⟶ $$\frac{d\delta^2}{d\tau} = \epsilon - \frac{\kappa\cos\alpha}{4\pi s}\delta^2$$

contour integration of velocity ⟶ $$\frac{d\gamma}{d\tau} = -\epsilon\frac{s\gamma}{2\sqrt{\pi}\delta^3}\exp[-s^2/4\delta^2]$$

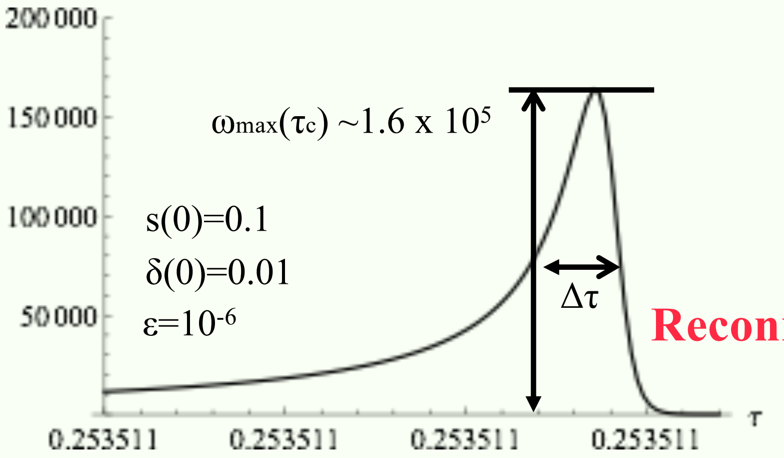
$\epsilon = 1/R_\Gamma$

$R_\Gamma = \Gamma/\nu$

Vortex Reynolds number

Our dynamical system :

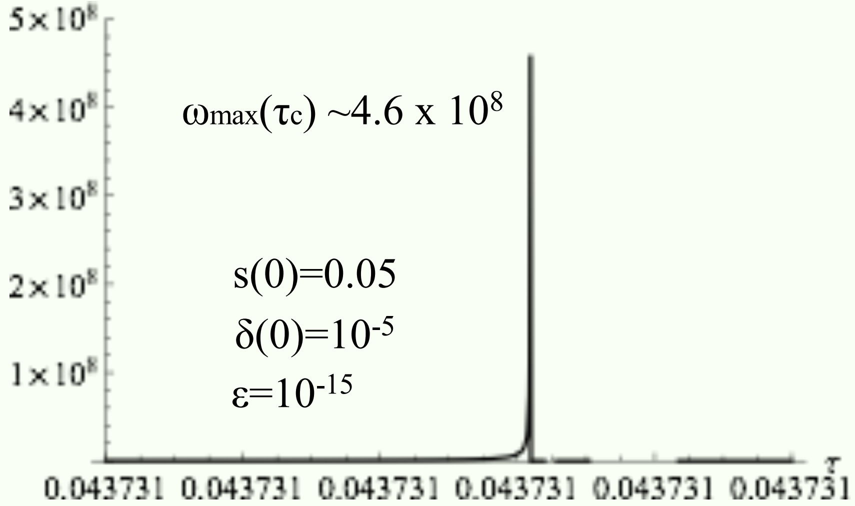
$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = -\frac{\gamma\,\kappa\cos\alpha}{4\pi}\left[\log\left(\frac{s}{\delta}\right)+\beta_1\right], \quad \frac{\mathrm{d}\kappa}{\mathrm{d}\tau} = \frac{\gamma\,\kappa\cos\alpha\sin\alpha}{4\pi s^2}, \quad \frac{\mathrm{d}\,\delta^2}{\mathrm{d}\tau} = \epsilon - \frac{\gamma\,\kappa\cos\alpha}{4\pi s}\,\delta^2, \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = -\epsilon\,\frac{s\,\gamma}{2\sqrt{\pi}\,\delta^3}\exp[-s^2/4\delta^2].$$

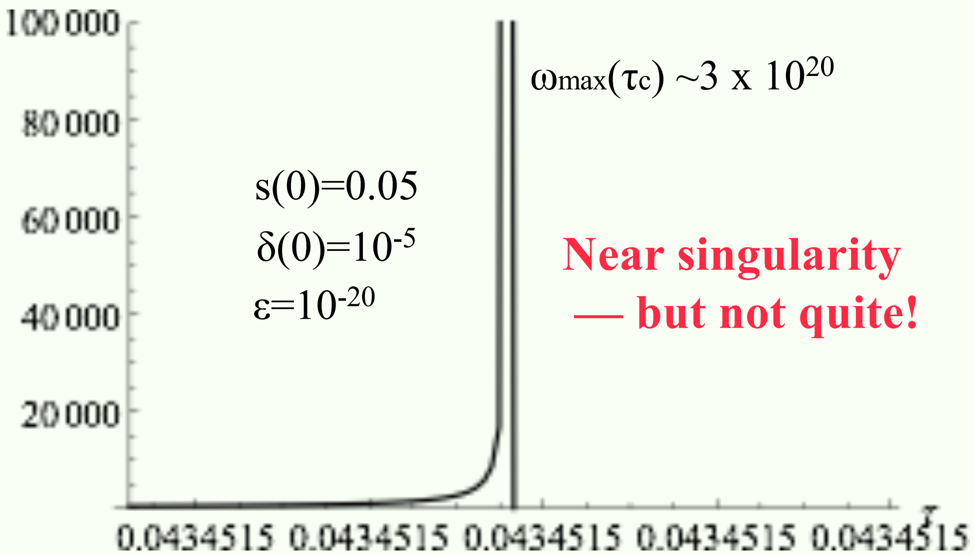


Growth of maximum vorticity $\omega_m(\tau)$ during the small time interval around the time τ_c when a maximum is attained.

Reconnection cut-off

(abscissa not resolved)





Near singularity — but not quite!

基礎的な問題

超高レイノルズ数乱流における

・特徴的な流れ構造は何か

・エネルギー散逸のメカニズムは何か

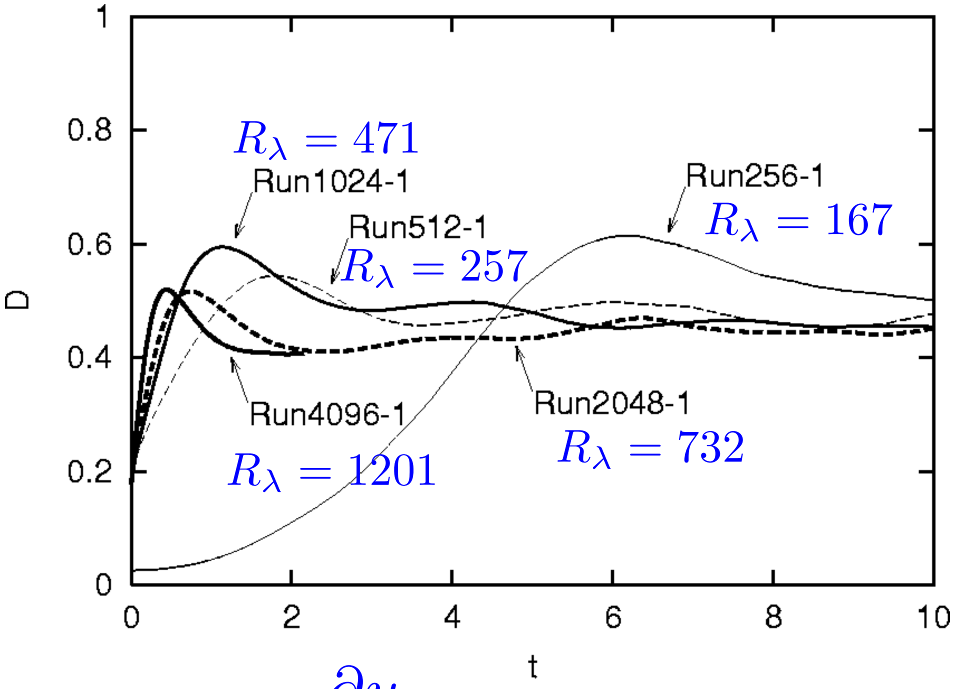
エネルギー散逸率

$$\epsilon = -\frac{dE}{dt} = -\frac{d}{dt}\left\langle \frac{1}{2}|\mathbf{u}|^2 \right\rangle = \nu \left\langle \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right\rangle$$

$\dfrac{\partial u_i}{\partial x_j} \quad (i, j = 1, 2, 3)$ ： 速度勾配，シアー，歪み

ϵ が $\nu \to 0$ の極限でも有限に留まる？

$D = \langle \epsilon \rangle L / u'^3$



Y. Kaneda *et.al.* *Phys. Fluids* (2003) **15**, L21-L24.

$\dfrac{\partial u_i}{\partial x_j} \to \infty$ ？ 速度勾配が発散する？

超高レイノルズ数乱流におけるエネルギー散逸の主体は vortex reconnection（渦の衝突）である.

やったこと

多元数理棟１階廊下
（飛田先生寄贈の絵）

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本当にどうもありがとうございました。