From *Probability, an introduction*, Grimmett and Welsh

5.4 Some common density functions

It is fairly clear that any function f which satisfies

$$f(x) \geq 0 \qquad \text{for } x \in \mathbb{R} \tag{5.34}$$

and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{5.35}$$

is the density function of some random variable. To confirm this, simply define

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

and check that F is a distribution function by verifying (5.5)–(5.8). There are several such functions f which are especially important in practice, and we list these below.

The **uniform distribution** on the interval (a, b) has density function

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases} \tag{5.36}$$

The **exponential distribution** with parameter $\lambda > 0$ has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases} \tag{5.37}$$

The **normal (or Gaussian) distribution** with parameters μ and σ^2, sometimes written as $\mathrm{N}(\mu, \sigma^2)$, has density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \qquad \text{for } x \in \mathbb{R}. \tag{5.38}$$

The **Cauchy distribution** has density function

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for } x \in \mathbb{R}. \tag{5.39}$$

The **gamma distribution** with parameters w (> 0) and λ (> 0) has density function

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(w)} \lambda^w x^{w-1} e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \tag{5.40}$$

where $\Gamma(w)$ is the *gamma function*, defined by

$$\Gamma(w) = \int_0^\infty x^{w-1} e^{-x}\, dx. \tag{5.41}$$

Note that, for positive integers w, $\Gamma(w) = (w-1)!$ (see Exercise 5.46).

The **beta distribution** with parameters s, t (> 0) has density function

$$f(x) = \frac{1}{B(s,t)} x^{s-1}(1-x)^{t-1} \qquad \text{for } 0 \le x \le 1. \tag{5.42}$$

The *beta function*

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\, dx \tag{5.43}$$

is chosen so that f has integral equal to one. You may care to prove that

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

(see (6.44)). If $s = t = 1$, then X is uniform on $[0, 1]$.

The **chi-squared distribution with *n* degrees of freedom** (sometimes written χ_n^2) has density function

$$f(x) = \begin{cases} \dfrac{1}{2\Gamma(\frac{1}{2}n)} (\tfrac{1}{2}x)^{\frac{1}{2}n-1} e^{-\frac{1}{2}x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases} \tag{5.44}$$

A comparison of (5.44) with (5.40) shows that the χ_n^2 distribution is the same as the gamma distribution with parameters $\frac{1}{2}n$ and $\frac{1}{2}$, but we list the distribution separately here because of its common occurrence in statistics.