Answers to Practice Problems

Chapter 1

1.1 Let the Lagrange function of f_0 be

$$\begin{aligned}
&\mathscr{L}_0\left(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0\right) \\
&\quad = f_0\left(\boldsymbol{u}\right) + \mathscr{L}_{\mathrm{S}}\left(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0\right) \\
&\quad = f_0\left(\boldsymbol{u}\right) - \boldsymbol{v}_0 \cdot \left(\boldsymbol{K}\left(\boldsymbol{a}\right)\boldsymbol{u} - \boldsymbol{p}\right) \\
&\quad = \begin{pmatrix} 0 & u_2 \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} v_{01} & v_{02} \end{pmatrix} \left(\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} a_1 + a_2 & -a_2 \\ -a_2 & a_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right),
\end{aligned}$$

where $\boldsymbol{v}_0 \in \mathbb{R}^2$ is an adjoint variable (Lagrange multiplier). The stationary condition of $\mathscr{L}_0$ with respect to an arbitrary variation $\hat{\boldsymbol{v}}_0 \in U$ of $\boldsymbol{v}_0$,

$$\mathscr{L}_{0\boldsymbol{v}_0}\left(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0\right)\left[\hat{\boldsymbol{v}}_0\right] = \mathscr{L}_{\mathrm{S}}\left(\boldsymbol{a}, \boldsymbol{u}, \hat{\boldsymbol{v}}_0\right) = 0$$

holds when $\boldsymbol{u}$ satisfies the state equation. The stationary condition of $\mathscr{L}_0$ with respect to an arbitrary variation $\hat{\boldsymbol{u}} \in U$ of $\boldsymbol{u}$:

$$\begin{aligned}
\mathscr{L}_{0\boldsymbol{u}}\left(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0\right)\left[\hat{\boldsymbol{u}}\right] &= f_{0\boldsymbol{u}}\left(\boldsymbol{u}\right)\left[\hat{\boldsymbol{u}}\right] - \mathscr{L}_{\mathrm{S}\boldsymbol{u}}\left(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0\right)\left[\hat{\boldsymbol{u}}\right] \\
&= 2\begin{pmatrix} 0 & u_2 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} - \boldsymbol{v}_0 \cdot \left(\boldsymbol{K}\left(\boldsymbol{a}\right)\hat{\boldsymbol{u}}\right) \\
&= -\hat{\boldsymbol{u}} \cdot \left(\boldsymbol{K}^{\top}\left(\boldsymbol{a}\right)\boldsymbol{v}_0 - \begin{pmatrix} 0 \\ 2u_2 \end{pmatrix} \right) = 0
\end{aligned}$$

holds when $\boldsymbol{v}_0$ satisfies

$$\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} a_1 + a_2 & -a_2 \\ -a_2 & a_2 \end{pmatrix} \begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 2u_2 \end{pmatrix}. \tag{P.1.1}$$

Equation (P.1.1) is an adjoint equation with respect to f_0. Moreover, when $\boldsymbol{u}$ satisfies the state equation and $\boldsymbol{v}_0$ is the solution of Eq. (P.1.1), the following, which is the same as Eq. (1.1.36), can be obtained:

$$\begin{aligned}
\mathscr{L}_{0\boldsymbol{a}}(\boldsymbol{a},\boldsymbol{u},\boldsymbol{v}_0)[\boldsymbol{b}] &= f_0'(\boldsymbol{u}(\boldsymbol{a}))[\boldsymbol{b}] \\
&= -\left\{\boldsymbol{v}_0\cdot\begin{pmatrix}\dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_1}\boldsymbol{u} & \dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_2}\boldsymbol{u}\end{pmatrix}\right\}\boldsymbol{b} \\
&= l\begin{pmatrix}-\sigma(u_1)\,\varepsilon(v_{01}) & -\sigma(u_2-u_1)\,\varepsilon(v_{02}-v_{01})\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} \\
&= \boldsymbol{g}_0\cdot\boldsymbol{b}.
\end{aligned}$$

1.2 If $\boldsymbol{u}$ satisfies $\min_{\boldsymbol{u}\in\mathbb{R}^2}\pi(\boldsymbol{a},\boldsymbol{u})$,

$$\pi_{\boldsymbol{u}}(\boldsymbol{a},\boldsymbol{u})[\hat{\boldsymbol{u}}] = \hat{\boldsymbol{u}}\cdot(\boldsymbol{K}(\boldsymbol{a})\boldsymbol{u}-\boldsymbol{p}) = 0$$

holds with respect to an arbitrary $\hat{\boldsymbol{u}}\in\mathbb{R}^2$. In other words, it is satisfied if $\boldsymbol{u}$ is the solution of the state determination problem (Problem 1.1.3). Moreover, there exists $\alpha>0$ such that

$$\pi_{\boldsymbol{u}\boldsymbol{u}}(\boldsymbol{a},\boldsymbol{u})[\hat{\boldsymbol{u}},\hat{\boldsymbol{u}}] = \hat{\boldsymbol{u}}\cdot(\boldsymbol{K}(\boldsymbol{a})\hat{\boldsymbol{u}}) > \alpha\|\hat{\boldsymbol{u}}\|_{\mathbb{R}^2}^2 .$$

Hence, it can be confirmed that the solution $\boldsymbol{u}$ of the state determination problem (Problem 1.1.3) is a minimizer of $\pi(\boldsymbol{a},\boldsymbol{u})$. On the other hand, the maximum point of $\pi(\boldsymbol{a},\boldsymbol{u})$ with respect to $\boldsymbol{a}$ becomes the minimum point of $-\pi(\boldsymbol{a},\boldsymbol{u})$. When $\boldsymbol{u}$ is the solution to a state determination problem,

$$\begin{aligned}
-\pi_{\boldsymbol{a}}(\boldsymbol{a},\boldsymbol{u})[\boldsymbol{b}] &= -\frac{1}{2}\left\{\boldsymbol{u}\cdot\begin{pmatrix}\dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_1}\boldsymbol{u} & \dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_2}\boldsymbol{u}\end{pmatrix}\right\}\boldsymbol{b} \\
&= -\frac{1}{2}\frac{e_{\mathrm{Y}}}{l}\begin{pmatrix}u_1u_1 & (u_2-u_1)(u_2-u_1)\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} \\
&= \frac{1}{2}\boldsymbol{g}_0\cdot\boldsymbol{b}
\end{aligned}$$

holds with respect to an arbitrary $\boldsymbol{b}\in\mathbb{R}^2$. Here, $\boldsymbol{g}_0$ expresses the vector of Eq. (1.1.36).

1.3 Since $\boldsymbol{u}$ is obtained by Eq. (1.1.20),

$$f_0(\boldsymbol{u}(\boldsymbol{a})) = \left(\frac{2}{a_1}+\frac{1}{a_2}\right)^2 .$$

As per Exercise 1.1.7, let

$$\tilde{f}_0(a_1) = f_0(\boldsymbol{u}(a_1,1-a_1)) = \left(\frac{2}{a_1}+\frac{1}{1-a_1}\right)^2 .$$

Here, the values of a_1 that satisfy

$$\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}a_1} = 2\left(\frac{2}{a_1}+\frac{1}{1-a_1}\right)\left\{-\frac{2}{a_1^2}+\frac{1}{(1-a_1)^2}\right\} = 0$$

are 2, $2-\sqrt{2}$ and $2+\sqrt{2}$. The values of a_2 with respect to these are -1, $\sqrt{2}-1$ and $-\sqrt{2}-1$, respectively. Of these, the one satisfying $\boldsymbol{a} \geq \boldsymbol{0}_{\mathbb{R}^2}$ is determined when $\boldsymbol{a} = \left(2-\sqrt{2}, \sqrt{2}-1\right)^{\top}$. Moreover, due to the convexity of $\tilde{f}_0$ and f_1, this $\boldsymbol{a}$, which satisfies the KKT conditions, is the minimizer of Practice **1.1**.

1.4 The side constraint with respect to the cross-sectional area a_1 in the definition of admissible set $\mathcal{D}$ in Eq. (1.1.16) of design variable $\boldsymbol{a}$ becomes active. Hence, in addition to $f_1(\boldsymbol{a}) \leq 0$, the second inequality constraint is set to be

$$f_2(\boldsymbol{a}) = a_{01} - a_1 \leq 0.$$

Here, the cross-sectional derivative of f_2 is

$$f_{2\boldsymbol{a}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \boldsymbol{g}_2. \tag{P.1.2}$$

If the Lagrange multiplier with respect to $f_2 \leq 0$ is set to be λ_2, the KKT conditions are given by

$$\begin{aligned}
\mathscr{L}_{\boldsymbol{a}}(\boldsymbol{a}, \lambda_1, \lambda_2) &= \boldsymbol{g}_0 + \lambda_1 \boldsymbol{g}_1 + \lambda_2 \boldsymbol{g}_2 = \boldsymbol{0}_{\mathbb{R}^2}, \qquad \text{(P.1.3)}\\
\mathscr{L}_{\lambda_1}(\boldsymbol{a}, \lambda_1, \lambda_2) &= f_1(\boldsymbol{a}) = l(a_1 + a_2) - c_1 \leq 0,\\
\mathscr{L}_{\lambda_2}(\boldsymbol{a}, \lambda_1, \lambda_2) &= f_2(\boldsymbol{a}) = a_{01} - a_1 \leq 0,\\
\lambda_1 f_1(\boldsymbol{a}) &= 0,\\
\lambda_2 f_2(\boldsymbol{a}) &= 0,\\
\lambda_1 &\geq 0,\\
\lambda_2 &\geq 0.
\end{aligned}$$

With an optimal solution, $f_1 = 0$, $f_2 = 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Here, if $\boldsymbol{g}_0$, $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ of Eq. (1.1.28), Eq. (1.1.17) and Eq. (P.1.2), respectively, are substituted into Eq. (P.1.3),

$$l \begin{pmatrix} -\sigma(u_1)\,\varepsilon(u_1) \\ -\sigma(u_2 - u_1)\,\varepsilon(u_2 - u_1) \end{pmatrix} + \lambda_1 \begin{pmatrix} l \\ l \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here, if the simultaneous linear equations with respect to λ_1 and λ_2 are solved, we obtain

$$\begin{aligned}
\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} &= \begin{pmatrix} \sigma(u_2 - u_1)\,\varepsilon(u_2 - u_1) \\ -l\sigma(u_1)\,\varepsilon(u_1) + l\sigma(u_2 - u_1)\,\varepsilon(u_2 - u_1) \end{pmatrix}\\
&= \sigma(u_2 - u_1)\,\varepsilon(u_2 - u_1) \begin{pmatrix} 1 \\ l \end{pmatrix}.
\end{aligned}$$

1.5 Let us use the adjoint variable method. Equation (1.1.36) becomes

$$\mathscr{L}_{0\boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{b}]$$

$$
\begin{aligned}
&= -\left\{ \boldsymbol{v}_0 \cdot \begin{pmatrix} \dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_1}\boldsymbol{u} & \dfrac{\partial \boldsymbol{K}(\boldsymbol{a})}{\partial a_2}\boldsymbol{u} \end{pmatrix} \right\} \boldsymbol{b} \\
&= -\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} v_{01} & v_{02} \end{pmatrix} \left(\begin{pmatrix} 2a_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \begin{pmatrix} 2a_2 & -2a_2 \\ -2a_2 & 2a_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \boldsymbol{b} \\
&= -\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} 2a_1 u_1 v_{01} & 2a_2 (u_2 - u_1)(v_{02} - v_{01}) \end{pmatrix} \boldsymbol{b} \\
&= \boldsymbol{g}_0 \cdot \boldsymbol{b}.
\end{aligned}
$$

Here, if the self-adjoint relationship (Eq. (1.1.35)) is used, we get

$$
\boldsymbol{g}_0 = -\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} 2a_1 u_1^2 \\ 2a_2 (u_2 - u_1)^2 \end{pmatrix}.
$$

The Hesse matrix is calculated as shown below. The second-order derivative of the Lagrange function $\mathscr{L}_0$ with respect to arbitrary variations $(\boldsymbol{b}_1, \hat{\boldsymbol{u}}_1)$ and $(\boldsymbol{b}_2, \hat{\boldsymbol{u}}_2)$ of the design variable $(\boldsymbol{a}, \boldsymbol{u})$ becomes Eq. (1.1.38). Here, $\boldsymbol{u}$ and $\boldsymbol{v}_0$ are the solutions of the state determination problem (Problem 1.1.3) and adjoint problem (Problem 1.1.5) with respect to the design variable $\boldsymbol{a}$. Furthermore, $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ are taken to be the variations of $\boldsymbol{u}$ given that the state determination problem is satisfied with respect to arbitrary variations $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ of $\boldsymbol{a}$, respectively. That is,

$$
\begin{aligned}
\hat{\boldsymbol{u}}(\boldsymbol{a})[\boldsymbol{b}_i] = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}^{\top}} \boldsymbol{b}_i &= \begin{pmatrix} \dfrac{\partial u_1}{\partial a_1} & \dfrac{\partial u_1}{\partial a_2} \\ \dfrac{\partial u_2}{\partial a_1} & \dfrac{\partial u_2}{\partial a_2} \end{pmatrix} \begin{pmatrix} b_{i1} \\ b_{i2} \end{pmatrix} \\
&= \begin{pmatrix} -2u_1/a_1 & 0 \\ -2u_1/a_1 & -2(u_2 - u_1)/a_2 \end{pmatrix} \begin{pmatrix} b_{i1} \\ b_{i2} \end{pmatrix}.
\end{aligned}
$$

Here, the second-order derivative of Lagrange function $\mathscr{L}_0$ is

$$
\begin{aligned}
&\left(\mathscr{L}_{0\boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{b}_1] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{u}}(\boldsymbol{a})[\boldsymbol{b}_1]] \right)_{\boldsymbol{a}} [\boldsymbol{b}_2] \\
&\qquad + \left(\mathscr{L}_{0\boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{b}_1] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{u}}(\boldsymbol{a})[\boldsymbol{b}_1]] \right)_{\boldsymbol{u}} [\hat{\boldsymbol{u}}(\boldsymbol{a})[\boldsymbol{b}_2]] \\
&= \mathscr{L}_{\mathrm{S}\boldsymbol{a}\boldsymbol{a}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{u}_1, \boldsymbol{u}_2] + 2\mathscr{L}_{\mathrm{S}\boldsymbol{a}\boldsymbol{u}}(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{b}_1, \hat{\boldsymbol{u}}(\boldsymbol{a})[\boldsymbol{b}_2]] \\
&= \boldsymbol{b}_1 \cdot \left(\begin{pmatrix} \dfrac{\partial \boldsymbol{g}_0}{\partial a_1} & \dfrac{\partial \boldsymbol{g}_0}{\partial a_2} \end{pmatrix} \boldsymbol{b}_2 \right) \\
&\quad - 2\boldsymbol{b}_1 \cdot \left(\begin{pmatrix} \boldsymbol{v}_0^{\top} \boldsymbol{K}_{a_1} \\ \boldsymbol{v}_0^{\top} \boldsymbol{K}_{a_2} \end{pmatrix} \begin{pmatrix} -2u_1/a_1 & 0 \\ -2u_1/a_1 & -2(u_2 - u_1)/a_2 \end{pmatrix} \boldsymbol{b}_2 \right) \\
&= -\frac{e_{\mathrm{Y}}}{l} \boldsymbol{b}_1 \cdot \left(\begin{pmatrix} 2u_1 v_{01} & 0 \\ 0 & 2(u_2 - u_1)(v_{02} - v_{01}) \end{pmatrix} \boldsymbol{b}_2 \right) \\
&\quad - 2\boldsymbol{b}_1 \cdot \left(\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} 2a_1 v_{01} & 0 \\ -2a_2 (v_{02} - v_{01}) & 2a_2 (v_{02} - v_{01}) \end{pmatrix} \right. \\
&\qquad \left. \times \begin{pmatrix} -2u_1/a_1 & 0 \\ -2u_1/a_1 & -2(u_2 - u_1)/a_2 \end{pmatrix} \boldsymbol{b}_2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{e_{\mathrm{Y}}}{l}\boldsymbol{b}_1 \cdot \left(\begin{pmatrix} 2u_1 v_{01} & 0 \\ 0 & 2\left(u_2 - u_1\right)\left(v_{02} - v_{01}\right) \end{pmatrix} \boldsymbol{b}_2 \right) \\
&\quad - \frac{2e_{\mathrm{Y}}}{l}\boldsymbol{b}_1 \cdot \left(\begin{pmatrix} -4u_1 v_{01} & 0 \\ 0 & -4(u_2 - u_1)(v_{02} - v_{01}) \end{pmatrix} \boldsymbol{b}_2 \right) \\
&= \frac{6e_{\mathrm{Y}}}{l}\boldsymbol{b}_1 \cdot \left(\begin{pmatrix} u_1 v_{01} & 0 \\ 0 & \left(u_2 - u_1\right)\left(v_{02} - v_{01}\right) \end{pmatrix} \boldsymbol{b}_2 \right).
\end{aligned}
$$

Hence, if the self-adjoint relationship (Eq. (1.1.35)) is used, we get

$$
\boldsymbol{H}_0 = \frac{6e_{\mathrm{Y}}}{l} \begin{pmatrix} u_1^2 & 0 \\ 0 & \left(u_2 - u_1\right)^2 \end{pmatrix}.
$$

1.6 The cost function becomes

$$
f\left(\boldsymbol{a}\right) = \frac{1}{6} a_1 a_2.
$$

Hence,

$$
\boldsymbol{g}\left(\boldsymbol{a}\right) = \frac{1}{6}\begin{pmatrix} a_2 \\ a_1 \end{pmatrix}, \quad \boldsymbol{H} = \frac{1}{6}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Here, notice that the Hesse matrix $\boldsymbol{H}$ is not positive definite.

1.7 The potential energy of Problem 1.2.1 is given by extending Eq. (1.1.9) as

$$
\begin{aligned}
\pi\left(\boldsymbol{u}\right) &= \int_0^l \frac{1}{2}\sigma\left(u\right)\varepsilon\left(u\right) a_1 \,\mathrm{d}x + \cdots + \int_{(n-1)l}^{nl} \frac{1}{2}\sigma\left(u\right)\varepsilon\left(u\right) a_n \,\mathrm{d}x \\
&\quad - \boldsymbol{p}\cdot\boldsymbol{u} \\
&= \frac{1}{2}\frac{e_{\mathrm{Y}}}{l} a_1 u_1^2 + \cdots + \frac{1}{2}\frac{e_{\mathrm{Y}}}{l} a_n \left(u_n - u_{n-1}\right)^2 - p_1 u_1 - \cdots - p_n u_n.
\end{aligned}
$$

The stationary conditions of π which correspond to Eq. (1.2.1) can be written as

$$
\begin{aligned}
&\frac{e_{\mathrm{Y}}}{l} \begin{pmatrix} a_1 + a_2 & -a_2 & \cdots & 0 & 0 \\ -a_2 & a_2 + a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} + a_n & -a_{n-1} \\ 0 & 0 & \cdots & -a_{n-1} & a_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} \\
&= \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}.
\end{aligned}
$$

$\boldsymbol{K}\left(\boldsymbol{a}\right)$ is the coefficient matrix of the left-hand side of this equation.

1.8 We use $\boldsymbol{p}$ such that $\max_{\boldsymbol{p}\in\mathbb{R}^2} \pi(\boldsymbol{a},\boldsymbol{p})$ satisfies

$$-\pi_{\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{p})[\hat{\boldsymbol{p}}] = \hat{\boldsymbol{p}}\cdot(\boldsymbol{A}(\boldsymbol{a})\boldsymbol{p}+\boldsymbol{u}) = 0$$

with respect to an arbitrary $\hat{\boldsymbol{p}}\in\mathbb{R}^2$. Here, if $\boldsymbol{p}$ is the solution of the state determination problem (Problem 1.3.1), then it is satisfied. Moreover, there exists $\alpha>0$ which satisfies

$$-\pi_{\boldsymbol{p}\boldsymbol{p}}(\boldsymbol{a},\boldsymbol{p})[\hat{\boldsymbol{p}},\hat{\boldsymbol{p}}] = \hat{\boldsymbol{p}}\cdot(\boldsymbol{A}(\boldsymbol{a})\hat{\boldsymbol{p}}) > \alpha\|\hat{\boldsymbol{p}}\|_{\mathbb{R}^2}^2 .$$

Hence, $\boldsymbol{p}$ which satisfies the state determination problem can be confirmed to be the maximizer of $\pi(\boldsymbol{a},\boldsymbol{p})$. On the other hand, when $\boldsymbol{p}$ is a solution of the state determination problem,

$$\begin{aligned}
\pi_{\boldsymbol{a}}(\boldsymbol{a},\boldsymbol{p})[\boldsymbol{b}] &= -\frac{1}{2}\left\{\boldsymbol{p}\cdot\begin{pmatrix}\dfrac{\partial\boldsymbol{A}(\boldsymbol{a})}{\partial a_1}\boldsymbol{p} & \dfrac{\partial\boldsymbol{A}(\boldsymbol{a})}{\partial a_2}\boldsymbol{p}\end{pmatrix}\right\}\boldsymbol{b}\\
&= -\frac{1}{\left(a_0^2+a_1^2+a_2^2\right)^2}\begin{pmatrix} a_1\left\{a_0^2p_1+a_2^2(p_1-p_2)\right\}^2\\ a_2\left\{a_0^2p_2+a_1^2(p_2-p_1)\right\}^2\end{pmatrix}\cdot\begin{pmatrix}b_1\\ b_2\end{pmatrix}\\
&= -\begin{pmatrix}\dfrac{u_1^2}{a_1}\\ \dfrac{u_2^2}{a_2}\end{pmatrix}\cdot\begin{pmatrix}b_1\\ b_2\end{pmatrix}\\
&= \frac{1}{2}\boldsymbol{g}_0\cdot\boldsymbol{b}
\end{aligned}$$

holds with respect to an arbitrary $\boldsymbol{b}\in\mathbb{R}^2$. Here, $\boldsymbol{g}_0$ represents the vector of Eq. (1.3.19).

1.9 As in Fig. 1.5.2, let $\boldsymbol{l}=(l_0,l_1,l_2)^\top\in\mathbb{R}^3$ be the three lengths of a cylinder. Here, the value dividing the sum of the three cylinder volumes by π is given by

$$f(l_0,l_1,l_2) = r_0^2 l_0 + r_1^2 l_1 + r_2^2 l_2.$$

On the other hand, the geometric relationship leads to

$$\begin{aligned}
h_1 &= l_1\sin\theta_1-\alpha_2=0,\\
h_2 &= l_0-\alpha_1+l_1\cos\theta_1=0,\\
h_3 &= l_2\sin\theta_2-\beta_2=0,\\
h_4 &= l_0-\beta_1+l_2\cos\theta_2=0.
\end{aligned}$$

Using these relationships, we can write

$$f(l_0) = r_0^2 l_0 + r_1^2\sqrt{\alpha_2^2+(\alpha_1-l_0)^2} + r_2^2\sqrt{\beta_2^2+(\beta_1-l_0)^2}.$$

Here, the following can be obtained:

$$\frac{\mathrm{d}f}{\mathrm{d}l_0} = r_0^2 - \frac{r_1^2(\alpha_1-l_0)}{\sqrt{\alpha_2^2+(\alpha_1-l_0)^2}} - \frac{r_2^2(\alpha_2-l_0)}{\sqrt{\beta_2^2+(\beta_1-l_0)^2}}$$

$$= r_0^2 - \frac{r_1^2(\alpha_1 - l_0)}{l_1} - \frac{r_2^2(\alpha_2 - l_0)}{l_2}$$
$$= r_0^2 - r_1^2\cos\theta_1 - r_2^2\cos\theta_2 = 0.$$

Chapter 2

2.1 The eigenvalues and eigenvectors of $\boldsymbol{A}$ are written as $\lambda_1 \leq \cdots \leq \lambda_d \in \mathbb{R}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathbb{R}^d$ respectively. Here, the eigenvectors are mutually orthogonal, hence the arbitrary vector $\boldsymbol{x} \in \mathbb{R}^d$ can be written as

$$\boldsymbol{x} = \sum_{i\in\{1,\ldots,d\}} \boldsymbol{x}_i \xi_i$$

by using $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^\top \in \mathbb{R}^d$. Here, let $\|\boldsymbol{x}_1\|_{\mathbb{R}^d} = \cdots = \|\boldsymbol{x}_d\|_{\mathbb{R}^d} = 1$. Even with this, with respect to an arbitrary $\boldsymbol{\xi} \in \mathbb{R}^d$, arbitrary $\boldsymbol{x} \in \mathbb{R}^d$ can be obtained. Here, if $\boldsymbol{A}$ is positive definite, from Theorem A.2.1, $\lambda_d \geq, \ldots, \geq \lambda_1 > 0$. Hence, we get

$$\boldsymbol{x} \cdot A\boldsymbol{x} = \sum_{i\in\{1,\ldots,d\}} \lambda_i \xi_i^2 \geq \lambda_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 = \lambda_1 \|\boldsymbol{x}\|_{\mathbb{R}^d}^2 > 0$$

with respect to an arbitrary $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}_{\mathbb{R}^d}\}$. Moreover, if $\boldsymbol{A}$ is a negative definite, $\lambda_d \leq, \ldots, \leq \lambda_1 < 0$. Hence,

$$\boldsymbol{x} \cdot A\boldsymbol{x} = \sum_{i\in\{1,\ldots,d\}} \lambda_i \xi_i^2 \geq \lambda_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 = \lambda_1 \|\boldsymbol{x}\|_{\mathbb{R}^d}^2 < 0$$

is obtained with respect to an arbitrary $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}_{\mathbb{R}^d}\}$.

2.2 From Theorem 2.5.2, the required conditions for f to take a minimum value are

$$\frac{\partial f}{\partial x_1} = ax_1 + bx_2 + d = 0,$$
$$\frac{\partial f}{\partial x_2} = bx_1 + cx_2 + e = 0.$$

These equations can be written as

$$\boldsymbol{g} = \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The sufficient condition is shown by confirming that f is a convex function based on Theorem 2.5.6. In order to do so, the Hesse matrix needs to be shown to be positive semi-definite using Theorem 2.4.6. From

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} = a, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = b, \quad \frac{\partial f}{\partial x_2 \partial x_2} = c,$$

the Hesse matrix becomes

$$\boldsymbol{H} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

If the positive definiteness of this matrix is shown by Sylvester's criterion (Theorem A.2.2), we get

$$a > 0, \quad \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 > 0.$$

This relationship holds regardless of $\boldsymbol{x} \in \mathbb{R}^2$. Furthermore, if $\boldsymbol{b} = (d, c)^\top$ is used, $f(x_1, x_2)$ of this problem can be written as

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} d & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \frac{1}{2} \boldsymbol{x} \cdot (\boldsymbol{H}\boldsymbol{x}) + \boldsymbol{b} \cdot \boldsymbol{x}. \end{aligned}$$

2.3 The problem can be written as

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} \left\{ f_0(\boldsymbol{x}) = -x_1 x_2 \mid f_1(\boldsymbol{x}) = 2(x_1 + x_2) - c_1 \le 0 \right\}.$$

Let $\lambda_1 \in \mathbb{R}$ be a Lagrange multiplier with respect to the constraint of the length of the sides of the rectangle, and the Lagrange function of this problem be

$$\mathscr{L}(x_1, x_2, \lambda_1) = f_0(\boldsymbol{x}) + \lambda_1 f_1(\boldsymbol{x}) = -x_1 x_2 + \lambda_1 \{2(x_1 + x_2) - c_1\}.$$

The KKT conditions become

$$\begin{aligned} \mathscr{L}_{x_1} &= -x_2 + 2\lambda_1 = 0, \\ \mathscr{L}_{x_2} &= -x_1 + 2\lambda_1 = 0, \\ \mathscr{L}_{\lambda} &= f_1(\boldsymbol{x}) = 2(x_1 + x_2) - c_1 \le 0, \\ \lambda_1 f_1(\boldsymbol{x}) &= \lambda_1 \{2(x_1 + x_2) - c_1\} = 0, \\ \lambda_1 &\ge 0. \end{aligned}$$

From these, the KKT conditions are satisfied when

$$\lambda_1 = \frac{x_1}{2} = \frac{x_2}{2} = \frac{c_1}{8}.$$

This result indicates a square. The fact that the solution satisfying the KKT conditions is a minimizer is shown below. f_0 is not a convex function (Exercise 2.4.9). However, $\tilde{f}_0(x_1) = f_0(x_1, -x_1 + c_1/2)$ is a convex function. Here, if it is viewed as an unconstrained minimization problem of $\tilde{f}_0(x_1)$, it can be shown that (x_1, x_2) satisfying the KKT conditions is a minimizer. Figure P.1 shows the status when $c_1 = 2$.

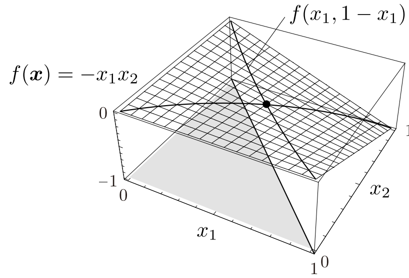

Fig. P.1: Function $f_0(\boldsymbol{x}) = -x_1 x_2$.

Chpter 3

3.1 If Eq. (3.5.7) showing the Newton–Raphson method is rewritten for f,

$$x_{k+1} = x_k - \frac{f(x_k)}{g(x_k)}.$$

Moreover, if $g(x_k)$ is replaced by the difference,

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

is obtained.

3.2 Let $f\left(\boldsymbol{x}_k + \bar{\epsilon}_g \bar{\boldsymbol{y}}_g\right)$ be $\bar{f}(\bar{\epsilon}_g)$, and furthermore

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}\bar{\epsilon}_g}(\bar{\epsilon}_g) = \bar{g}(\bar{\epsilon}_g) = \boldsymbol{g}\left(\boldsymbol{x}_k + \bar{\epsilon}_g \bar{\boldsymbol{y}}_g\right) \cdot \bar{\boldsymbol{y}}_g.$$

In the strict line search method (Problem 3.4.1), $\bar{\epsilon}_g$ is determined so that

$$\bar{g}(\bar{\epsilon}_g) = 0$$

is satisfied. When obtaining the solution of this non-linear equation using the Newton–Raphson method, $\bar{\epsilon}_{g\,l+1} = \bar{\epsilon}_{gl} - \bar{g}(\bar{\epsilon}_{gl}) / h(\bar{\epsilon}_{gl})$ should be sought so that

$$\bar{g}(\bar{\epsilon}_{g\,l+1}) = \bar{g}(\bar{\epsilon}_{gl}) + h(\bar{\epsilon}_{gl})(\bar{\epsilon}_{g\,l+1} - \bar{\epsilon}_{gl}) = 0$$

is satisfied. Here, $h(\bar{\epsilon}_{gl})$ is a second-order derivative function of $\bar{f}$. When using the secant method, we would set

$$h(\bar{\epsilon}_{gl}) = \frac{\bar{g}(\bar{\epsilon}_{gl}) - \bar{g}(\bar{\epsilon}_{g\,l-1})}{\bar{\epsilon}_{gl} - \bar{\epsilon}_{g\,l-1}}$$

and use

$$\bar{\epsilon}_{g\,l+1} = \bar{\epsilon}_{gl} - \frac{\bar{\epsilon}_{gl} - \bar{\epsilon}_{g\,l-1}}{\bar{g}(\bar{\epsilon}_{gl}) - \bar{g}(\bar{\epsilon}_{g\,l-1})} \bar{g}(\bar{\epsilon}_{gl})$$

in order to obtain $\bar{\epsilon}_{g\,l+1}$.

3.3 In the conjugate gradient method, set $\boldsymbol{x}_0 = \boldsymbol{0}_X$ and $\bar{\boldsymbol{y}}_{g0} = -\boldsymbol{g}_0 = -\boldsymbol{g}(\boldsymbol{x}_0) = -\boldsymbol{b}$, and calculate $\bar{\epsilon}_{gk}$ using Eq. (3.4.8) with respect to $k \in \mathbb{N} \cup \{0\}$, and $\boldsymbol{x}_k$, $\boldsymbol{g}_k$, β_k and $\bar{\boldsymbol{y}}_{gk}$ using from Eq. (3.4.9) to Eq. (3.4.12) with respect to $k \in \mathbb{N}$. Therefore, the following holds:

$$\begin{aligned}
&\bar{\boldsymbol{y}}_{k+1} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\quad = \left(-\boldsymbol{g}_{k+1} + \beta_{k+1}\bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\quad = \left(-\boldsymbol{g}_{k+1} + \frac{\boldsymbol{g}_{k+1} \cdot \boldsymbol{g}_{k+1}}{\boldsymbol{g}_k \cdot \boldsymbol{g}_k}\bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\quad = \left\{-\boldsymbol{g}_k - \bar{\epsilon}_{gk}\boldsymbol{B}\bar{\boldsymbol{y}}_{gk} + \frac{\left(\boldsymbol{g}_k + \bar{\epsilon}_{gk}\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{g}_k + \bar{\epsilon}_{gk}\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)}{\boldsymbol{g}_k \cdot \boldsymbol{g}_k}\bar{\boldsymbol{y}}_{gk}\right\} \\
&\qquad \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\quad = \left\{-\boldsymbol{g}_k - \bar{\epsilon}_{gk}\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right. \\
&\qquad \left. + \frac{\boldsymbol{g}_k \cdot \boldsymbol{g}_k + 2\bar{\epsilon}_{gk}\boldsymbol{g}_k \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) + \bar{\epsilon}_{gk}^2\left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)}{\boldsymbol{g}_k \cdot \boldsymbol{g}_k}\bar{\boldsymbol{y}}_{gk}\right\} \\
&\qquad \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\quad = \left\{-\boldsymbol{g}_k - \frac{\boldsymbol{g}_k \cdot \boldsymbol{g}_k}{\bar{\boldsymbol{y}}_{gk} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)}\left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)\right\} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) + \bar{\boldsymbol{y}}_{gk} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \\
&\qquad + 2\boldsymbol{g}_k \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) + \frac{\left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)}{\bar{\boldsymbol{y}}_{gk} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right)}\boldsymbol{g}_k \cdot \boldsymbol{g}_k \\
&\quad = \left(\boldsymbol{g}_k + \bar{\boldsymbol{y}}_{gk}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) = \beta_k\bar{\boldsymbol{y}}_{k-1} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right).
\end{aligned}$$

Here, $\bar{\boldsymbol{y}}_{gk-1} \cdot \boldsymbol{g}_k = 0$ was used because we use the strict line search. When $k = 0$, from $\bar{\boldsymbol{y}}_{g0} = -\boldsymbol{g}_0$, $\left(\boldsymbol{g}_0 + \bar{\boldsymbol{y}}_{g0}\right) \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{g0}\right) = 0$ is established. Therefore, with respect to $k \in \mathbb{N}$, $\bar{\boldsymbol{y}}_{g\,k+1} \cdot \left(\boldsymbol{B}\bar{\boldsymbol{y}}_{gk}\right) = 0$ holds.

3.4 The gradient $\boldsymbol{g}(\boldsymbol{a})$ and Hessian $\boldsymbol{H}$ of $f(\boldsymbol{a})$ with respect to a variation of $\boldsymbol{a}$ obtained in Practice **1.6** are used. The Newton method uses $\boldsymbol{H}\boldsymbol{b} = -\boldsymbol{g}(\boldsymbol{a}_0)$, that is,

$$\frac{1}{6}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} a_{02} \\ a_{01} \end{pmatrix}$$

to obtain the search vector $\boldsymbol{b}$. When solving this equation, we get

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = -\begin{pmatrix} a_{01} \\ a_{02} \end{pmatrix}.$$

Here, the point updated using the first Newton method:

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} a_{01} \\ a_{02} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{01} - a_{01} \\ a_{02} - a_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

becomes the minimum point of f. The reason that the minimum point could be obtained after using Newton method just once is because the Taylor expansion of f is fully described by the gradient and the Hesse matrix. In that case, it can be confirmed that the positive definiteness of the Hesse matrix is not required.

Chapter 4

4.1 Let U and V be

$$\begin{aligned} U = \{ & u \in H^1 ((0,l) \times (0,t_{\mathrm{T}}) ; \mathbb{R}) \mid \\ & u(0,t) = 0 \text{ for } t \in (0,t_{\mathrm{T}}), \; u(x,0) = \alpha(x) \text{ for } x \in (0,l) \}, \\ V = \{ & v \in H^1 ((0,l) \times (0,t_{\mathrm{T}}) ; \mathbb{R}) \mid \\ & v(0,t) = 0 \text{ for } t \in (0,t_{\mathrm{T}}), \; v(x,0) = 0 \text{ for } x \in (0,l) \}. \end{aligned}$$

Select and fix an element u_0 of $H^1((0,l) \times (0,t_{\mathrm{T}}); \mathbb{R})$ satisfying $u_0(x,0) = \alpha(x)$. The first variation of $f(u)$ with respect to an arbitrary $v \in V$ becomes

$$\begin{aligned} & f'(u)[v] \\ & \quad = \int_0^{t_{\mathrm{T}}} \left\{ \int_0^l (\rho \dot{u} \dot{v} - e \nabla u \nabla v + bv) a_{\mathrm{S}} \,\mathrm{d}x + p_{\mathrm{N}} v(l,t) a_{\mathrm{S}}(l,t) \right\} \mathrm{d}t \\ & \qquad - \int_0^l \rho \beta v(x,t_{\mathrm{T}}) a_{\mathrm{S}} \,\mathrm{d}x \\ & \quad = \int_0^{t_{\mathrm{T}}} \left\{ \int_0^l (-\rho \ddot{u} + \nabla(e \nabla u) + b) v a_{\mathrm{S}} \,\mathrm{d}x \right. \\ & \qquad \left. - (e \nabla u(l,t) - p_{\mathrm{N}}) v(l,t) a_{\mathrm{S}}(l) \right\} \mathrm{d}t \\ & \qquad + \int_0^l \rho (\dot{u}(x,t_{\mathrm{T}}) - \beta) v(x,t_{\mathrm{T}}) a_{\mathrm{S}} \,\mathrm{d}x. \end{aligned}$$

Hence, the stationary condition of $f(u)$ with respect to an arbitrary $v \in V$ is given by the condition such that $f'(u)[v] = 0$ with respect to $u - u_0 \in V$. In other words, we get

$$\begin{aligned} \rho \ddot{u} - \nabla(e \nabla u) = \rho \ddot{u} - \nabla \sigma(u) = b & \text{ for } (x,t) \in (0,l) \times (0,t_{\mathrm{T}}), \\ e \nabla u(l,t) = \sigma(u(l,t)) = p_{\mathrm{N}} & \text{ for } t \in (0,t_{\mathrm{T}}), \\ \dot{u}(x,t_{\mathrm{T}}) = \beta & \text{ for } x \in (0,l). \end{aligned}$$

At that time, for $f(u)$ and $f'(u)[v]$ to have meaning, we need the following to hold:

$$\begin{aligned} & \rho \in L^\infty((0,l);\mathbb{R}), \quad \alpha \in H^1((0,l);\mathbb{R}), \quad \beta \in L^2((0,l);\mathbb{R}), \\ & b \in L^2((0,l) \times (0,t_{\mathrm{T}});\mathbb{R}), \quad p_{\mathrm{N}} \in L^2((0,t_{\mathrm{T}});\mathbb{R}). \end{aligned}$$

4.2 The first variation of the action integral $f(\boldsymbol{u})$ with respect to an arbitrary variation $\boldsymbol{v} \in V$ of $\boldsymbol{u} \in U$ becomes

$$
\begin{aligned}
& f'(\boldsymbol{u}, \dot{\boldsymbol{u}})[\boldsymbol{v}, \dot{\boldsymbol{v}}] \\
&= \int_0^{t_\mathrm{T}} \left(\frac{\partial l}{\partial \boldsymbol{u}} \cdot \boldsymbol{v} + \frac{\partial l}{\partial \dot{\boldsymbol{u}}} \cdot \dot{\boldsymbol{v}} \right) \mathrm{d}t \\
&= \int_0^{t_\mathrm{T}} \left(\frac{\partial l}{\partial \boldsymbol{u}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial l}{\partial \dot{\boldsymbol{u}}} \right) \cdot \boldsymbol{v} \, \mathrm{d}t + \frac{\partial l}{\partial \dot{\boldsymbol{u}}}(t_\mathrm{T}) \cdot \boldsymbol{v}(t_\mathrm{T}) - \frac{\partial l}{\partial \dot{\boldsymbol{u}}}(0) \cdot \boldsymbol{v}(0) \\
&= \int_0^{t_\mathrm{T}} \left(\frac{\partial l}{\partial \boldsymbol{u}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial l}{\partial \dot{\boldsymbol{u}}} \right) \cdot \boldsymbol{v} \, \mathrm{d}t.
\end{aligned}
$$

With respect to an arbitrary $\boldsymbol{v} \in V$, for $f'(\boldsymbol{u}, \dot{\boldsymbol{u}})[\boldsymbol{v}, \dot{\boldsymbol{v}}] = 0$ to hold, the Lagrange equation of motion needs to hold.

4.3 The first variation of the action integral $f(\boldsymbol{u}, \boldsymbol{q})$ with respect to an arbitrary variation $\boldsymbol{v} \in V$ of $\boldsymbol{u} \in U$ and an arbitrary variation $\boldsymbol{r} \in Q$ of $\boldsymbol{q} \in Q$ becomes

$$
\begin{aligned}
& f'(\boldsymbol{u}, \boldsymbol{q})[\boldsymbol{v}, \boldsymbol{r}] \\
&= \int_0^{t_\mathrm{T}} \left(-\dot{\boldsymbol{q}} \cdot \boldsymbol{v} - \frac{\partial \mathscr{H}}{\partial \boldsymbol{u}} \cdot \boldsymbol{v} - \dot{\boldsymbol{r}} \cdot \boldsymbol{u} - \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}} \cdot \boldsymbol{r} \right) \mathrm{d}t \\
&= \int_0^{t_\mathrm{T}} \left\{ -\left(\dot{\boldsymbol{q}} + \frac{\partial \mathscr{H}}{\partial \boldsymbol{u}} \right) \cdot \boldsymbol{v} + \left(\dot{\boldsymbol{u}} - \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}} \right) \cdot \boldsymbol{r} \right\} \mathrm{d}t.
\end{aligned}
$$

With respect to an arbitrary $\boldsymbol{v} \in V$ and an arbitrary $\boldsymbol{r} \in Q$, for $f'(\boldsymbol{u}, \boldsymbol{q})[\boldsymbol{v}, \boldsymbol{r}] = 0$ to hold, the Hamilton equation of motion needs to hold. Moreover, when the Hamilton equation of motion holds,

$$
\dot{\mathscr{H}}(\boldsymbol{u}, \boldsymbol{q}) = \frac{\partial \mathscr{H}}{\partial \boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = \frac{\partial \mathscr{H}}{\partial \boldsymbol{u}} \cdot \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}} - \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}} \cdot \frac{\partial \mathscr{H}}{\partial \boldsymbol{u}} = 0
$$

holds. Furthermore, with respect to a spring mass system of Fig. 4.1.1, when the external force $p = 0$, since the momentum is given by $q = m\dot{u}$, we get

$$
\mathscr{H}(u, q) = -l(u, q) + q\dot{u} = -\frac{1}{2}m\dot{u}^2 + \frac{1}{2}ku^2 + q\dot{u} = \frac{1}{2}m\dot{u}^2 + \frac{1}{2}ku^2.
$$

In other words, it shows that when there are no external forces in play, the sum of kinetic energy and potential energy becomes a Hamilton function and that it is conserved.

4.4 If $Y \Subset Z$, there exists some positive constant c and with respect to an arbitrary $\boldsymbol{x} \in Y$,

$$
\|\boldsymbol{x}\|_Z \leq c \|\boldsymbol{x}\|_Y
$$

holds. Here, if the definitions of norms (Definition 4.4.5) with respect to Y' and Z' are used,

$$\frac{1}{c}\|\boldsymbol{\phi}\|_{Y'} = \sup_{\boldsymbol{x}\in Y\setminus\{\boldsymbol{0}_Y\}} \frac{|\langle\boldsymbol{\phi},\boldsymbol{x}\rangle|}{c\|\boldsymbol{x}\|_Y} \le \sup_{\boldsymbol{x}\in Z\setminus\{\boldsymbol{0}_Z\}} \frac{|\langle\boldsymbol{\phi},\boldsymbol{x}\rangle|}{\|\boldsymbol{x}\|_Z} = \|\boldsymbol{\phi}\|_{Z'}$$

is established with respect to an arbitrary $\boldsymbol{\phi} \in Z'$. Therefore, from $\|\boldsymbol{\phi}\|_{Y'} \le c\|\boldsymbol{\phi}\|_{Z'}$, $Z' \Subset Y'$ is obtained.

Chapter 5

5.1 From the fact that Dirichlet condition is given over the whole boundary, $U = H_0^1(\Omega;\mathbb{R})$. In this case,

$$\begin{aligned}\int_\Omega (-\Delta u + u)\, v \,\mathrm{d}x &= \int_\Omega (-\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}u + u)\, v\,\mathrm{d}x \\ &= -\int_{\partial\Omega} v\boldsymbol{\nabla}u\cdot\boldsymbol{\nu}\,\mathrm{d}\gamma + \int_\Omega (\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v + uv)\,\mathrm{d}x \\ &= \int_\Omega (\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v + uv)\,\mathrm{d}x = \int_\Omega bv\,\mathrm{d}x\end{aligned}$$

holds with respect to an arbitrary $v \in U$. Here, the weak form of this problem becomes a problem seeking $\tilde{u} = u - u_{\mathrm{D}} \in U$ satisfying

$$a(u,v) = l(v)$$

with respect to an arbitrary $v \in U$, where

$$a(u,v) = \int_\Omega (\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v + uv)\,\mathrm{d}x, \quad l(v) = \int_\Omega bv\,\mathrm{d}x.$$

For this weak-form solution to exist uniquely, the assumptions for the Lax–Milgram theorem need to hold. $U = H_0^1(\Omega;\mathbb{R})$ is a Hilbert space. Moreover, from the fact that

$$a(v,v) = \|v\|^2_{H^1(\Omega;\mathbb{R})}$$

holds with respect to an arbitrary $v \in H_0^1(\Omega;\mathbb{R})$, a is coercive and bounded. Hence, just $\hat{l} \in U'$ needs to hold. With respect to $\hat{l}$,

$$\begin{aligned}\left|\hat{l}(v)\right| &\le \int_\Omega |bv|\,\mathrm{d}x + \int_\Omega (|\boldsymbol{\nabla}u_{\mathrm{D}}\cdot\boldsymbol{\nabla}v| + |u_{\mathrm{D}}v|)\,\mathrm{d}x \\ &\le \|b\|_{L^2(\Omega;\mathbb{R})}\|v\|_{L^2(\Omega;\mathbb{R})} + \|\boldsymbol{\nabla}u_{\mathrm{D}}\|_{L^2(\Omega;\mathbb{R}^d)}\|\boldsymbol{\nabla}v\|_{L^2(\Omega;\mathbb{R}^d)} \\ &\quad + \|u_{\mathrm{D}}\|_{L^2(\Omega;\mathbb{R})}\|v\|_{L^2(\Omega;\mathbb{R})} \\ &\le \left(\|b\|_{L^2(\Omega;\mathbb{R})} + \|u_{\mathrm{D}}\|_{H^1(\Omega;\mathbb{R})}\right)\|v\|_{H^1(\Omega;\mathbb{R})}\end{aligned}$$

holds. Therefore, we need $b \in L^2(\Omega;\mathbb{R})$ and $u_{\mathrm{D}} \in H^1(\Omega;\mathbb{R})$.

5.2 The point $\boldsymbol{x}_{\mathrm{A}}$ is a boundary between a homogeneous Dirichlet and homogeneous Neumann boundaries at which the opening angle is $\alpha = \pi/2$. From Theorem 5.3.2 (2), getting $\boldsymbol{u} \in H^2\left(B_{\mathrm{A}}; \mathbb{R}^2\right)$ around the neighborhood B_{A} of the point $\boldsymbol{x}_{\mathrm{A}}$, the point $\boldsymbol{x}_{\mathrm{A}}$ is not a singular point. On the other hand, the point $\boldsymbol{x}_{\mathrm{B}}$ is a boundary between homogeneous Neumann and non-homogeneous Neumann boundaries at which the opening angle α is $\pi/2$. There is no singularity in the solution at this angle from Theorem 5.3.2 (1). However, $\boldsymbol{p}_{\mathrm{N}}$ changes as a step function around the neighborhood B_{B} of $\boldsymbol{x}_{\mathrm{B}}$ as $(0,0)^{\top}$ and $(0,-1)^{\top}$ across the boundary Γ_p. From this, if we view it as $\boldsymbol{p}_{\mathrm{N}} \in L^{\infty}\left(B_{\mathrm{B}}; \mathbb{R}^2\right)$, we have $\boldsymbol{u} \in C^{0,1}\left(B_{\mathrm{B}}; \mathbb{R}^2\right)$ which is not included in $H^2\left(B_{\mathrm{B}}; \mathbb{R}^2\right)$.

5.3 The function space with respect to this problem is set as

$$U = \left\{ \boldsymbol{u} \in H^1\left((0, t_{\mathrm{T}}); H^1\left(\Omega; \mathbb{R}^d\right)\right) \;\middle|\; \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}} \times (0, t_{\mathrm{T}}), \right.$$
$$\left. \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Omega \times \{0, t_{\mathrm{T}}\} \right\}.$$

Assume $\boldsymbol{u}_{\mathrm{D}0}$, $\boldsymbol{u}_{\mathrm{D}\top} \in H^1\left(\Omega; \mathbb{R}^d\right)$ and $\boldsymbol{u}_{\mathrm{D}} \in H^1\left((0, t_{\mathrm{T}}); H^1\left(\Omega; \mathbb{R}^d\right)\right)$. Furthermore, assume $\boldsymbol{b} \in L^2\left((0, t_{\mathrm{T}}); L^2\left(\Omega; \mathbb{R}^d\right)\right)$, $\boldsymbol{p}_{\mathrm{N}} \in L^2\left((0, t_{\mathrm{T}}); L^2\left(\Gamma_{\mathrm{N}}; \mathbb{R}^d\right)\right)$. Here, the weak form of this problem can be obtained by multiplying an arbitrary $\boldsymbol{v} \in U$ to the first equation, integrating with $\Omega \times (0, t_{\mathrm{T}})$ and using the fundamental boundary conditions as follows. "Obtain $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U$ which satisfy

$$\int_0^{t_{\mathrm{T}}} \left(b\left(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}\right) - a\left(\boldsymbol{u}, \boldsymbol{v}\right) + l\left(\boldsymbol{v}\right)\right) \mathrm{d}t = 0$$

with respect to an arbitrary $\boldsymbol{v} \in U$, where let

$$b\left(\boldsymbol{u}, \boldsymbol{v}\right) = \int_{\Omega} \rho \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x,$$
$$a\left(\boldsymbol{u}, \boldsymbol{v}\right) = \int_{\Omega} \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}) \, \mathrm{d}x,$$
$$l\left(\boldsymbol{v}\right) = \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v} \, \mathrm{d}\gamma."$$

5.4 Let the function space with respect to $\boldsymbol{\phi}$ be

$$U = \left\{ \boldsymbol{\phi} \in H^1\left(\Omega; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{\phi} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}} \right\}.$$

In this case, substituting $\boldsymbol{u}\left(\boldsymbol{x}, t\right) = \boldsymbol{\phi}\left(\boldsymbol{x}\right) \mathrm{e}^{\lambda t}$ with respect to $\boldsymbol{\phi} \in U$ into $\rho \ddot{\boldsymbol{u}}^{\top} - \boldsymbol{\nabla}^{\top} \boldsymbol{S}\left(\boldsymbol{u}\right) = \boldsymbol{0}_{\mathbb{R}^d}^{\top}$, integrating this equation over Ω after having an arbitrary $\boldsymbol{v} \in U$ multiplied by it, and considering the fundamental boundary condition $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}$ on $\Gamma_{\mathrm{D}} \times (0, t_{\mathrm{T}})$, the weak form of the natural frequency problem can be obtained as below. "Obtain $(\boldsymbol{\phi}, \lambda) \in U \times \mathbb{C}$ satisfying

$$\lambda^2 b\left(\boldsymbol{\phi}, \boldsymbol{v}\right) + a\left(\boldsymbol{\phi}, \boldsymbol{v}\right) = 0$$

with respect to an arbitrary $\boldsymbol{v} \in U$."

Commentary This problem is an eigenvalue problem (the equation is an eigen equation) on a function space U. In this problem, if a non-negative definiteness (coerciveness including 0) of $a(\cdot,\cdot)$ and positive definiteness (coverciveness) of $b(\cdot,\cdot)$ are considered, eigenpairs $(\boldsymbol{\phi}_i, \lambda_i)_{i\in\mathbb{N}}$ of the number of dimensions of U, which is the same as a countably infinite number, exist. In this case, $\lambda_i^2 \le 0$, in other words, $\lambda_i = \pm \mathrm{i}\omega_i$ (i is the imaginary unit) is derived. From this result, $\boldsymbol{\phi}_i(\boldsymbol{x})\left(\mathrm{e}^{\mathrm{i}\omega_i t} + \mathrm{e}^{-\mathrm{i}\omega_i t}\right) = \boldsymbol{\phi}_i \cos\omega_i t$ becomes a solution of the eigen value problem and ω_i and $\boldsymbol{\phi}_i$ are called eigenfrequencies and eigenmodes.

5.5 Let the function space with respect to $\boldsymbol{u}$ and p be as follows respectively:

$$
\begin{aligned}
U = \{ & \boldsymbol{u} \in H^1\left((0,t_{\mathrm{T}}); H^1\left(\Omega;\mathbb{R}^d\right)\right) \mid \\
& \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \partial\Omega \times (0,t_{\mathrm{T}}) \cup \Omega \times \{0\}\},\\
V = \{ & \boldsymbol{u} \in H^1\left((0,t_{\mathrm{T}}); H^1\left(\Omega;\mathbb{R}^d\right)\right) \mid \\
& \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \partial\Omega \times (0,t_{\mathrm{T}}) \cup \Omega \times \{t_{\mathrm{T}}\}\},\\
P = & \left\{ p \in L^2\left((0,t_{\mathrm{T}}); L^2(\Omega;\mathbb{R})\right) \,\middle|\, \int_\Omega p \,\mathrm{d}x = 0 \right\}.
\end{aligned}
$$

Here, if an arbitrary $\boldsymbol{v} \in V$ is used to multiply the Navier–Stokes equation and integrate it over $(0,t_{\mathrm{T}}) \times \Omega$, and a basic boundary condition $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}$ on $\partial\Omega \times (0,t_{\mathrm{T}}) \cup \Omega \times \{0\}$ is considered, a weak-form equation with respect to the Navier–Stokes equation can be obtained. On the other hand, if an arbitrary $q \in P$ is used to multiply through the equation of continuity and integrate it over $(0,t_{\mathrm{T}}) \times \Omega$, the weak form with respect to the equation of continuity can be obtained. This can be written as below. "Obtain $(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}, p) \in U \times Q$ which satisfies

$$
\int_0^{t_{\mathrm{T}}} \left(b(\dot{\boldsymbol{u}}, \boldsymbol{v}) + c(\boldsymbol{u})(\boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, p)\right) \mathrm{d}t = \int_0^{t_{\mathrm{T}}} l(\boldsymbol{v})\,\mathrm{d}t,
$$

$$
\int_0^{t_{\mathrm{T}}} d(\boldsymbol{u}, q)\,\mathrm{d}t = 0
$$

with respect to an arbitrary $(\boldsymbol{v}, q) \in U \times Q$, where we let

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \mu \left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{v}^\top\right) \mathrm{d}x,\\
b(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x,\\
c(\boldsymbol{u})(\boldsymbol{w}, \boldsymbol{v}) &= \int_\Omega \rho \left(\left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{w}\right) \cdot \boldsymbol{v}\,\mathrm{d}x,\\
d(\boldsymbol{v}, q) &= -\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{v}\,\mathrm{d}x,\\
l(\boldsymbol{v}) &= \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v}\,\mathrm{d}\gamma."
\end{aligned}
$$

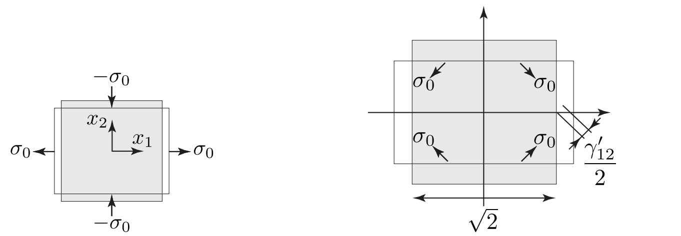

Fig. P.1: Deformation with shearing stress

5.6 When a stress such as that in Fig. P.1 (a) occurs, the linear strain becomes

$$\varepsilon_{11} = -\varepsilon_{22} = \frac{1+\nu_{\mathrm{P}}}{e_{\mathrm{Y}}}\sigma_0. \tag{P.5.1}$$

On the other hand, in a coordinate system which is just one $\pi/4$ rotation in the anti-clockwise direction such as in Fig. P.1 (b),

$$\varepsilon_{11} = \frac{\gamma'_{12}/\sqrt{2}}{\sqrt{2}} = \frac{\gamma'_{12}}{2} = \varepsilon'_{12} = \frac{\sigma_0}{2\mu_{\mathrm{L}}} \tag{P.5.2}$$

holds. From Eq. (P.5.1) and Eq. (P.5.2), $e_{\mathrm{Y}} = 2\mu_{\mathrm{L}}\left(1+\nu_{\mathrm{P}}\right)$ holds.

Chapter 6

6.1 The weak form of this problem can be written as

$$a\left(u,v\right) + c\left(u,v\right) = l_1\left(v\right) \tag{P.6.1}$$

with respect to an arbitrary $v : (0,1) \to \mathbb{R}$ satisfying $v\left(0\right) = v\left(1\right) = 0$, where $a\left(\,\cdot\,,\,\cdot\,\right)$ and $l_1\left(\,\cdot\,\right)$ use the definitions in Exercise 6.1.5. Moreover, let

$$c\left(u,v\right) = \int_0^1 uv \,\mathrm{d}x.$$

The result when approximate functions u_h and v_h are substituted in $a\left(u,v\right)$ and $l_1\left(v\right)$ is as per Exercise 6.1.5. Here, if u_h and v_h are substituted in $c\left(u,v\right)$, we get

$$\begin{aligned} c\left(u_h,v_h\right) &= \int_0^1 \left\{\sum_{i\in\{1,\ldots,m\}} \alpha_i \sin(i\pi x)\right\}\left\{\sum_{j\in\{1,\ldots,m\}} \beta_j \sin(j\pi x)\right\} \mathrm{d}x \\ &= \boldsymbol{\beta}^{\top}\boldsymbol{C}\boldsymbol{\alpha}. \end{aligned}$$

Here, $\boldsymbol{C} = (c(\sin(i\pi x), \sin(j\pi x)))_{ij}$ and

$$\begin{aligned} & c(\sin(i\pi x), \sin(j\pi x)) \\ & \quad = \int_0^1 \sin(i\pi x)\sin(j\pi x)\ \mathrm{d}x \\ & \quad = -\frac{1}{2}\int_0^1 \left[\cos\{(i+j)\pi x\} - \cos\{(i-j)\pi x\}\right]\ \mathrm{d}x = \frac{1}{2}\delta_{ij}. \end{aligned}$$

From the answer to Exercise 6.1.5 and the result above, Eq. (P.6.1) becomes

$$(\boldsymbol{A} + \boldsymbol{C})\,\boldsymbol{\alpha} = \boldsymbol{f}.$$

In other words,

$$\begin{aligned} & \left(\frac{\pi^2}{2}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 9 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\right)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix} \\ & \quad = \frac{1}{\pi}\begin{pmatrix} 2 \\ 0 \\ 2/3 \\ \vdots \\ \left\{(-1)^{m+1} + 1\right\}/m \end{pmatrix}, \end{aligned}$$

or

$$\frac{i^2\pi^2 + 1}{2}\alpha_i = \frac{(-1)^{i+1} + 1}{i\pi}.$$

If this simultaneous linear equation is solved,

$$\alpha_i = \frac{2\left\{(-1)^{i+1} + 1\right\}}{i\pi\left(i^2\pi^2 + 1\right)}$$

is obtained. Therefore, the approximate function becomes

$$u_h = \sum_{i \in \{1,\ldots,m\}} \frac{2\left\{(-1)^{i+1} + 1\right\}}{i\pi\left(i^2\pi^2 + 1\right)} \sin(i\pi x).$$

6.2 The weak form of this problem is given by Eq. (P.6.1). $a(u, v)$ and $l_1(v)$ with approximate functions u_h and v_h substituted in are as shown in Exercise 6.2.1. Here, if u_h and v_h are substituted in $c(u, v)$, we get

$$c(u_h, v_h) = \sum_{i \in \{1,\ldots,m\}} \int_{x_{i-1}}^{x_i} u_h v_h\ \mathrm{d}x = \sum_{i \in \{1,\ldots,m\}} c_i(u_h, v_h),$$

$$\begin{aligned}
&c_i\left(u_h, v_h\right)\\
&\quad = \begin{pmatrix} v_{i(1)} & v_{i(2)} \end{pmatrix} \begin{pmatrix} \displaystyle\int_{x_{i-1}}^{x_i} \varphi_{i(1)}\varphi_{i(1)} \,\mathrm{d}x & \displaystyle\int_{x_{i-1}}^{x_i} \varphi_{i(1)}\varphi_{i(2)} \,\mathrm{d}x \\ \displaystyle\int_{x_{i-1}}^{x_i} \varphi_{i(2)}\varphi_{i(1)} \,\mathrm{d}x & \displaystyle\int_{x_{i-1}}^{x_i} \varphi_{i(2)}\varphi_{i(2)} \,\mathrm{d}x \end{pmatrix} \begin{pmatrix} u_{i(1)} \\ u_{i(2)} \end{pmatrix}\\
&\quad = \bar{\boldsymbol{v}}_i \cdot \bar{\boldsymbol{C}}_i \bar{\boldsymbol{u}}_i = \bar{\boldsymbol{v}} \cdot \boldsymbol{Z}_i^\top \bar{\boldsymbol{C}}_i \boldsymbol{Z}_i \bar{\boldsymbol{u}} = \bar{\boldsymbol{v}} \cdot \tilde{\boldsymbol{C}}_i \bar{\boldsymbol{u}}.
\end{aligned}$$

Here, $\bar{\boldsymbol{C}}_i = \left(\bar{c}_{i\alpha\beta}\right)_{\alpha,\beta} \in \mathbb{R}^2$ becomes

$$\begin{aligned}
\bar{c}_{i11} &= \int_{x_{i-1}}^{x_i} \varphi_{i(1)}\varphi_{i(1)} \,\mathrm{d}x = \frac{1}{\left(x_i - x_{i-1}\right)^2} \int_{x_{i-1}}^{x_i} \left(x_i - x\right)^2 \,\mathrm{d}x\\
&= \frac{x_i - x_{i-1}}{3},\\
\bar{c}_{i12} = \bar{c}_{i21} &= \int_{x_{i-1}}^{x_i} \varphi_{i(1)}\varphi_{i(2)} \,\mathrm{d}x\\
&= \frac{1}{\left(x_i - x_{i-1}\right)^2} \int_{x_{i-1}}^{x_i} \left(x_i - x\right)\left(x - x_{i-1}\right) \,\mathrm{d}x = \frac{x_i - x_{i-1}}{6},\\
\bar{c}_{i22} &= \int_{x_{i-1}}^{x_i} \varphi_{i(2)}\varphi_{i(2)} \,\mathrm{d}x = \frac{1}{\left(x_i - x_{i-1}\right)^2} \int_{x_{i-1}}^{x_i} \left(x - x_{i-1}\right)^2 \,\mathrm{d}x\\
&= \frac{x_i - x_{i-1}}{3}.
\end{aligned}$$

In other words,

$$\bar{\boldsymbol{C}}_i = \frac{x_i - x_{i-1}}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Matrix $\bar{\boldsymbol{C}}$, which is the sum of all elements, becomes

$$\bar{\boldsymbol{C}} = \frac{h}{6} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Therefore, the approximate equation becomes

$$\left(\frac{1}{h} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} + \frac{h}{6} \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = h \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Supplementary The integrals on the finite element are simplified if the domain is changed to a standard domain. Let the mapping $\xi:\ (x_{i-1}, x_i) \to (0,1)$ be

$$\xi = \frac{x - x_{i-1}}{h},$$

where $h = x_i - x_{i-1}$. Here, the Jacobian becomes

$$\frac{d\xi}{dx} = h.$$

The base function becomes

$$\varphi_{i(1)}(x) = \frac{x_i - x}{h} = 1 - \xi = \hat{\varphi}_{i(1)}(\xi),$$
$$\varphi_{i(2)}(x) = \frac{x - x_{i-1}}{h} = \xi = \hat{\varphi}_{i(2)}(\xi).$$

This time, $\bar{\boldsymbol{C}}_i$ can be calculated as

$$\bar{c}_{i11} = \int_0^1 \hat{\varphi}_{i(1)}\hat{\varphi}_{i(1)} h \; d\xi = h \int_0^1 (1-\xi)^2 \; d\xi = h \int_0^1 \eta^2 \; d\eta = \frac{h}{3},$$
$$\bar{c}_{i12} = \bar{c}_{i21} = \int_0^1 \hat{\varphi}_{i(1)}\hat{\varphi}_{i(2)} h \; d\xi = h \int_0^1 (1-\xi)\xi \; d\xi = \frac{h}{6},$$
$$\bar{c}_{i22} = \int_0^1 \hat{\varphi}_{i(2)}\hat{\varphi}_{i(2)} h \; d\xi = h \int_0^1 \xi^2 \; d\xi = \frac{h}{3}.$$

6.3 Let us think about a domain Ω_i of a triangular finite element such as in Fig. P.1. Here, with respect to the cross product of two vectors $\boldsymbol{x}_{i(2)} - \boldsymbol{x}_{i(1)}$ and $\boldsymbol{x}_{i(3)} - \boldsymbol{x}_{i(1)}$,

$$\begin{aligned}
&2\,|\Omega_i|\,\boldsymbol{e}_3 \\
&= \begin{pmatrix} x_{i(2)1} - x_{i(1)1} \\ x_{i(2)2} - x_{i(1)2} \\ 0 \end{pmatrix} \times \begin{pmatrix} x_{i(3)1} - x_{i(1)1} \\ x_{i(3)2} - x_{i(1)2} \\ 0 \end{pmatrix} \\
&= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_{i(2)1} - x_{i(1)1} & x_{i(2)2} - x_{i(1)2} & 0 \\ x_{i(3)1} - x_{i(1)1} & x_{i(3)2} - x_{i(1)2} & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 & 1 \\ x_{i(2)1} - x_{i(1)1} & x_{i(2)2} - x_{i(1)2} & 0 \\ x_{i(3)1} - x_{i(1)1} & x_{i(3)2} - x_{i(1)2} & 0 \end{vmatrix} \boldsymbol{e}_3 \\
&= \left(\begin{vmatrix} 0 & 0 & 1 \\ x_{i(2)1} - x_{i(1)1} & x_{i(2)2} - x_{i(1)2} & 0 \\ x_{i(3)1} - x_{i(1)1} & x_{i(3)2} - x_{i(1)2} & 0 \end{vmatrix} + \begin{vmatrix} x_{i(1)1} & x_{i(1)2} & 0 \\ x_{i(1)1} & x_{i(1)2} & 1 \\ x_{i(1)1} & x_{i(1)2} & 1 \end{vmatrix} \right) \boldsymbol{e}_3 \\
&= \begin{vmatrix} x_{i(1)1} & x_{i(1)2} & 1 \\ x_{i(2)1} & x_{i(2)2} & 1 \\ x_{i(3)1} & x_{i(3)2} & 1 \end{vmatrix} \boldsymbol{e}_3 = \gamma \boldsymbol{e}_3
\end{aligned}$$

holds, where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are unit orthogonal vectors of x_1, x_2 and x_3 coordinate systems. Hence, $\gamma = 2\,|\Omega_i|$ is obtained.

6.4 Let the finite elements with finite element numbers $\{3,5\}$, $\{4,6\}$, $\{1,7\}$ and $\{2,8\}$ be called Type 1, Type 2, Type 3 and Type 4, respectively.

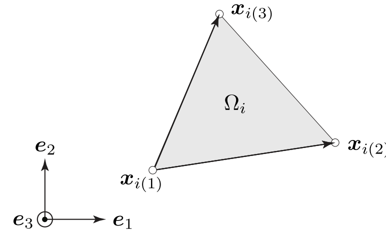

Fig. P.1: Triangular Ω_i and points $\boldsymbol{x}_{i(1)}$, $\boldsymbol{x}_{i(2)}$ and $\boldsymbol{x}_{i(3)}$.

The result from Exercise 6.3.2 is used with respect to Type 1 and Type 2. With respect to Type 3, $\gamma = h^2$, $|\Omega_i| = h^2/2$ and

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix}x_{i(2)2} - x_{i(3)2}\\ x_{i(3)2} - x_{i(1)2}\\ x_{i(1)2} - x_{i(2)2}\end{pmatrix} = \frac{1}{h^2}\begin{pmatrix}-h\\ h\\ 0\end{pmatrix},$$
$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix}x_{i(3)1} - x_{i(2)1}\\ x_{i(1)1} - x_{i(3)1}\\ x_{i(2)1} - x_{i(1)1}\end{pmatrix} = \frac{1}{h^2}\begin{pmatrix}-h\\ 0\\ h\end{pmatrix}.$$

Therefore,

$$\bar{\boldsymbol{A}}_1 = \frac{1}{2}\begin{pmatrix}2 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}, \quad \bar{\boldsymbol{b}}_1 = \frac{h^2}{6}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

is obtained. With respect to Type 4 too, in a similar way, $\gamma = h^2$, $|\Omega_i| = h^2/2$ and

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \frac{1}{h^2}\begin{pmatrix}0\\ h\\ -h\end{pmatrix}, \begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \frac{1}{h^2}\begin{pmatrix}-h\\ h\\ 0\end{pmatrix},$$
$$\bar{\boldsymbol{A}}_2 = \frac{1}{2}\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}, \quad \bar{\boldsymbol{b}}_2 = \frac{h^2}{6}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

can be obtained. On the other hand, the local node number and total node numbers can be made correspondent in the way shown in Table P.1.

Table P.1: The relationship between the local nodes $\boldsymbol{x}_{i(1)}$, $\boldsymbol{x}_{i(2)}$, $\boldsymbol{x}_{i(3)}$ and total nodes $\boldsymbol{x}_j$.

$i \in \mathcal{E}$	1	2	3	4	5	6	7	8
$\boldsymbol{x}_{i(1)}$	$\boldsymbol{x}_1$	$\boldsymbol{x}_4$	$\boldsymbol{x}_2$	$\boldsymbol{x}_2$	$\boldsymbol{x}_4$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_8$
$\boldsymbol{x}_{i(2)}$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_7$	$\boldsymbol{x}_8$	$\boldsymbol{x}_8$	$\boldsymbol{x}_9$
$\boldsymbol{x}_{i(3)}$	$\boldsymbol{x}_2$	$\boldsymbol{x}_2$	$\boldsymbol{x}_6$	$\boldsymbol{x}_3$	$\boldsymbol{x}_8$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$	$\boldsymbol{x}_6$
Type	3	4	1	2	1	2	3	4

If a sum of all elements is taken, $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{l}}$ become

$$\bar{\boldsymbol{A}} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -2 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 8 & -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix},$$

$$\bar{\boldsymbol{l}} = \frac{h^2}{6}\begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \\ 4 \\ 4 \\ 1 \\ 4 \\ 1 \end{pmatrix}.$$

Here, the fundamental boundary conditions $u_1 = u_2 = u_3 = u_4 = u_7 = 0$ and $v_1 = v_2 = v_3 = v_4 = v_7 = 0$ and $h = 1/2$ can be used to obtain

$$\begin{pmatrix} 8 & -2 & -2 & 0 \\ -2 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_5 \\ u_6 \\ u_8 \\ u_9 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 4 \\ 4 \\ 4 \\ 1 \end{pmatrix}.$$

Solving this, we get

$$\begin{pmatrix} u_5 \\ u_6 \\ u_8 \\ u_9 \end{pmatrix} = \frac{1}{16 \times 12}\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 6 & 2 & 4 \\ 2 & 2 & 6 & 4 \\ 2 & 4 & 4 & 12 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \\ 4 \\ 1 \end{pmatrix} = \frac{1}{96}\begin{pmatrix} 15 \\ 22 \\ 22 \\ 26 \end{pmatrix}.$$

6.5 With respect to a finite element $i \in \mathcal{E}$ in Fig. 6.4.7, a standard domain is set to be $\Xi_i = (0,1)^2$. The isoparametric representations of the approximate functions and coordimates become

$$\hat{u}_h(\boldsymbol{\xi}) = \sum_{\alpha\in\{1,\ldots,4\}} \hat{\varphi}_\alpha(\boldsymbol{\xi})\, u_{i\alpha} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\xi}) \cdot \bar{\boldsymbol{u}}_i,$$
$$\hat{v}_h(\boldsymbol{\xi}) = \sum_{\alpha\in\{1,\ldots,4\}} \hat{\varphi}_\alpha(\boldsymbol{\xi})\, v_{i\alpha} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\xi}) \cdot \bar{\boldsymbol{v}}_i,$$
$$\hat{x}_{h1}(\boldsymbol{\xi}) = \sum_{\alpha\in\{1,\ldots,4\}} \hat{\varphi}_\alpha(\boldsymbol{\xi})\, x_{i1\alpha} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\xi}) \cdot \bar{\boldsymbol{x}}_{i1},$$
$$\hat{x}_{h2}(\boldsymbol{\xi}) = \sum_{\alpha\in\{1,\ldots,4\}} \hat{\varphi}_\alpha(\boldsymbol{\xi})\, x_{i2\alpha} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\xi}) \cdot \bar{\boldsymbol{x}}_{i2}.$$

Here, let $x_{i1(2)} - x_{i1(1)} = h_1$ and $x_{i2(2)} - x_{i2(1)} = h_2$ and

$$\begin{pmatrix}\lambda_{11}\\ \lambda_{12}\\ \lambda_{21}\\ \lambda_{22}\end{pmatrix} = \begin{pmatrix}\left(x_{i1(2)} - x_1\right)/h_1\\ \left(x_1 - x_{i1(1)}\right)/h_2\\ \left(x_{i2(2)} - x_2\right)/h_1\\ \left(x_2 - x_{i2(1)}\right)/h_2\end{pmatrix} = \begin{pmatrix}(1-\xi_1)\\ \xi_1\\ (1-\xi_2)\\ \xi_2\end{pmatrix},$$
$$\hat{\boldsymbol{\varphi}} = \begin{pmatrix}\hat{\varphi}_1(\boldsymbol{\xi})\\ \hat{\varphi}_2(\boldsymbol{\xi})\\ \hat{\varphi}_3(\boldsymbol{\xi})\\ \hat{\varphi}_4(\boldsymbol{\xi})\end{pmatrix} = \begin{pmatrix}(1-\xi_1)(1-\xi_2)\\ \xi_1(1-\xi_2)\\ \xi_1\xi_2\\ (1-\xi_1)\xi_2\end{pmatrix}.$$

In this case,

$$\begin{aligned}\partial_{\boldsymbol{\xi}}\hat{\varphi}_\alpha(\boldsymbol{\xi}) &= \begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial\xi_1\\ \partial\hat{\varphi}_\alpha/\partial\xi_2\end{pmatrix} = \begin{pmatrix}\partial\hat{x}_1/\partial\xi_1 & \partial\hat{x}_2/\partial\xi_1\\ \partial\hat{x}_1/\partial\xi_2 & \partial\hat{x}_2/\partial\xi_2\end{pmatrix}\begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial x_1\\ \partial\hat{\varphi}_\alpha/\partial x_2\end{pmatrix}\\ &= \begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}\begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial x_1\\ \partial\hat{\varphi}_\alpha/\partial x_2\end{pmatrix}\end{aligned}$$

holds. Hence,

$$\begin{aligned}\begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial x_1\\ \partial\hat{\varphi}_\alpha/\partial x_2\end{pmatrix} &= \frac{1}{\omega(\boldsymbol{\xi})}\begin{pmatrix}\partial\hat{x}_2/\partial\xi_2 & -\partial\hat{x}_2/\partial\xi_1\\ -\partial\hat{x}_1/\partial\xi_2 & \partial\hat{x}_1/\partial\xi_1\end{pmatrix}\begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial\xi_1\\ \partial\hat{\varphi}_\alpha/\partial\xi_2\end{pmatrix}\\ &= \frac{1}{h_1h_2}\begin{pmatrix}h_2 & 0\\ 0 & h_1\end{pmatrix}\begin{pmatrix}\partial\hat{\varphi}_\alpha/\partial\xi_1\\ \partial\hat{\varphi}_\alpha/\partial\xi_2\end{pmatrix}\end{aligned}$$

can be obtained, where

$$\begin{aligned}&\begin{pmatrix}\partial\hat{\varphi}_1/\partial\xi_1 & \partial\hat{\varphi}_2/\partial\xi_1 & \partial\hat{\varphi}_3/\partial\xi_1 & \partial\hat{\varphi}_4/\partial\xi_1\\ \partial\hat{\varphi}_1/\partial\xi_2 & \partial\hat{\varphi}_2/\partial\xi_2 & \partial\hat{\varphi}_3/\partial\xi_2 & \partial\hat{\varphi}_4/\partial\xi_2\end{pmatrix}\\ &\quad= \begin{pmatrix}-(1-\xi_2) & (1-\xi_2) & \xi_2 & -\xi_2\\ -(1-\xi_1) & -\xi_1 & \xi_1 & (1-\xi_1)\end{pmatrix}.\end{aligned}$$

Using this result, the element coefficient matrix $\bar{\boldsymbol{A}}_i = \left(\bar{a}_{i\alpha\beta}\right)_{\alpha\beta} \in \mathbb{R}^{4\times 4}$ becomes

$$\bar{a}_{i\alpha\beta} = \int_{\Omega_i}\begin{pmatrix}\partial\varphi_\alpha/\partial x_1\\ \partial\varphi_\alpha/\partial x_2\end{pmatrix}\cdot\begin{pmatrix}\partial\varphi_\beta/\partial x_1\\ \partial\varphi_\beta/\partial x_2\end{pmatrix}\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\Xi_i} \begin{pmatrix} \partial\hat{\varphi}_\alpha/\partial x_1 \\ \partial\hat{\varphi}_\alpha/\partial x_2 \end{pmatrix} \cdot \begin{pmatrix} \partial\hat{\varphi}_\beta/\partial x_1 \\ \partial\hat{\varphi}_\beta/\partial x_2 \end{pmatrix} \omega(\boldsymbol{\xi})\,\mathrm{d}\xi \\
&= \frac{1}{h_1 h_2} \int_{\Xi_i} \begin{pmatrix} \dfrac{\partial\hat{\varphi}_\alpha}{\partial\xi_1} & \dfrac{\partial\hat{\varphi}_\alpha}{\partial\xi_2} \end{pmatrix} \begin{pmatrix} h_2 & 0 \\ 0 & h_1 \end{pmatrix} \begin{pmatrix} h_2 & 0 \\ 0 & h_1 \end{pmatrix} \begin{pmatrix} \partial\hat{\varphi}_\beta/\partial\xi_1 \\ \partial\hat{\varphi}_\beta/\partial\xi_2 \end{pmatrix} \mathrm{d}\xi \\
&= \int_{\Xi_i} \left(\frac{h_2}{h_1} \frac{\partial\hat{\varphi}_\alpha}{\partial\xi_1} \frac{\partial\hat{\varphi}_\beta}{\partial\xi_1} + \frac{h_1}{h_2} \frac{\partial\hat{\varphi}_\alpha}{\partial\xi_2} \frac{\partial\hat{\varphi}_\beta}{\partial\xi_2} \right) \mathrm{d}\xi.
\end{aligned}
$$

Letting $\sigma = h_2/h_1$, we get

$$
\bar{a}_{i11} = \int_{\Xi_i} \left[\sigma \{-(1-\xi_2)\}^2 + \sigma^{-1} \{-(1-\xi_1)\}^2 \right] \mathrm{d}\xi = \frac{1}{3}\left(\sigma + \sigma^{-1}\right).
$$

From these calculations we get

$$
\bar{\boldsymbol{A}}_i = \frac{1}{6} \begin{pmatrix} 2\sigma + 2\sigma^{-1} & -2\sigma + \sigma^{-1} & -\sigma - \sigma^{-1} & \sigma - 2\sigma^{-1} \\ -2\sigma + \sigma^{-1} & 2\sigma + 2\sigma^{-1} & \sigma - 2\sigma^{-1} & -\sigma - \sigma^{-1} \\ -\sigma - \sigma^{-1} & \sigma - 2\sigma^{-1} & 2\sigma + 2\sigma^{-1} & -2\sigma + \sigma^{-1} \\ \sigma - 2\sigma^{-1} & -\sigma - \sigma^{-1} & -2\sigma + \sigma^{-1} & 2\sigma + 2\sigma^{-1} \end{pmatrix}.
$$

The known term vector $\bar{\boldsymbol{l}}_i = \left(\bar{l}_{i\alpha}\right)_\alpha \in \mathbb{R}^4$ becomes

$$
\bar{l}_{i\alpha} = \int_{\Omega_i} b\hat{\varphi}_\alpha\,\mathrm{d}x = b_0 \int_{\Xi_i} \hat{\varphi}_\alpha(\boldsymbol{\xi})\,\omega(\boldsymbol{\xi})\,\mathrm{d}\xi.
$$

Therefore,

$$
\bar{\boldsymbol{l}}_i = b_0 h_1 h_2 \begin{pmatrix} \displaystyle\int_{\Xi_i} (1-\xi_1)(1-\xi_2)\,\mathrm{d}\xi \\ \displaystyle\int_{\Xi_i} \xi_1(1-\xi_2)\,\mathrm{d}\xi \\ \displaystyle\int_{\Xi_i} \xi_1\xi_2\,\mathrm{d}\xi \\ \displaystyle\int_{\Xi_i} (1-\xi_1)\xi_2\,\mathrm{d}\xi \end{pmatrix} = \frac{b_0 h_1 h_2}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
$$

6.6 Let $\Xi = (0,1)^2$ be a standard domain. With respect to $\alpha \in \{1,\dots,4\}$, let $\hat{\varphi}_{(\alpha)}(\boldsymbol{\xi})$ are basis functions on Ξ. Here, the following holds:

$$
\boldsymbol{\varepsilon}(\boldsymbol{\xi}) = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u_{h1}}{\partial x_1} \\ \dfrac{\partial u_{h2}}{\partial x_2} \\ \dfrac{\partial u_{h2}}{\partial x_1} + \dfrac{\partial u_{h1}}{\partial x_2} \end{pmatrix}
$$

$$
= \begin{pmatrix}
\frac{\partial \hat{\varphi}_1}{\partial x_1} & \frac{\partial \hat{\varphi}_2}{\partial x_1} & \frac{\partial \hat{\varphi}_3}{\partial x_1} & \frac{\partial \hat{\varphi}_4}{\partial x_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial \hat{\varphi}_1}{\partial x_2} & \frac{\partial \hat{\varphi}_2}{\partial x_2} & \frac{\partial \hat{\varphi}_3}{\partial x_2} & \frac{\partial \hat{\varphi}_4}{\partial x_2} \\
\frac{\partial \hat{\varphi}_1}{\partial x_2} & \frac{\partial \hat{\varphi}_2}{\partial x_2} & \frac{\partial \hat{\varphi}_3}{\partial x_2} & \frac{\partial \hat{\varphi}_4}{\partial x_2} & \frac{\partial \hat{\varphi}_1}{\partial x_1} & \frac{\partial \hat{\varphi}_2}{\partial x_1} & \frac{\partial \hat{\varphi}_3}{\partial x_1} & \frac{\partial \hat{\varphi}_4}{\partial x_1}
\end{pmatrix}
\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \end{pmatrix}
$$

$$
= \frac{1}{\omega(\boldsymbol{\xi})}
\begin{pmatrix}
\frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_1}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_1}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_2}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_2}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_3}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_3}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_4}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_4}{\partial \xi_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_1}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_1}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_2}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_2}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_3}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_3}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_4}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_4}{\partial \xi_2} \\
-\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_1}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_1}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_2}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_2}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_3}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_3}{\partial \xi_2} & -\frac{\partial \hat{x}_1}{\partial \xi_2}\frac{\partial \hat{\varphi}_4}{\partial \xi_1} + \frac{\partial \hat{x}_1}{\partial \xi_1}\frac{\partial \hat{\varphi}_4}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_1}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_1}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_2}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_2}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_3}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_3}{\partial \xi_2} & \frac{\partial \hat{x}_2}{\partial \xi_2}\frac{\partial \hat{\varphi}_4}{\partial \xi_1} - \frac{\partial \hat{x}_2}{\partial \xi_1}\frac{\partial \hat{\varphi}_4}{\partial \xi_2}
\end{pmatrix}
\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \end{pmatrix}
$$

$$
= \boldsymbol{B}\,(\boldsymbol{\xi})\,\bar{\boldsymbol{u}}_i,
$$

where $\omega(\boldsymbol{\xi}) = \det\left(\partial_{\boldsymbol{\xi}} \boldsymbol{x}^{\top}\right)$. The element coefficient matrix becomes

$$
\boldsymbol{K}_i = \int_{\Omega_i} \boldsymbol{B}^{\top}(\boldsymbol{x}) \boldsymbol{D} \boldsymbol{B}(\boldsymbol{x})\, \mathrm{d}x = \int_{\Xi} \boldsymbol{B}^{\top}(\boldsymbol{\xi}) \boldsymbol{D} \boldsymbol{B}(\boldsymbol{\xi}) \omega(\boldsymbol{\xi})\ \mathrm{d}\xi.
$$

Here, the integral of the right-hand side can be obtained by the Gaussian quadrature.

Chapter 8

8.1 When the θ-type elastic problem (Problem 8.9.2) was made into a state determination problem, a self-adjoint relationship was obtained with respect to the mean compliance f_0 defined by Eq. (8.9.6). Similarly, when the θ-type Poisson problem (Problem 8.2.3) is made into a state determination problem, if

$$f_0(u) = \int_D b(\theta)\, u \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} p_\mathrm{N} u \,\mathrm{d}\gamma - \int_{\Gamma_\mathrm{D}} \phi^\alpha(\theta)\, u_\mathrm{D} \partial_\nu u \,\mathrm{d}\gamma$$

is taken to be an objective function, the self-adjoint relationship is obtained. Moreover, the θ-derivative of f_0 becomes

$$\tilde{f}_0'(\theta)[\vartheta] = \langle g_0, \vartheta \rangle = \int_D \left(2b_\theta u - \alpha\phi^{\alpha-1}\phi_\theta \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u\right) \vartheta \,\mathrm{d}x.$$

8.2 The θ-type expanded Poisson problem becomes as below.

Problem P.8.1 (θ-type expanded Poisson problem) Let D be a $d \in \{2,3\}$-dimensional Lipschitz domain. With respect to $\theta \in \mathcal{D}$, $b \in C^1\left(\mathcal{D}; L^{2q_\mathrm{R}}(D;\mathbb{R})\right)$, $c_\Omega \in L^\infty(D;\mathbb{R})$, $p_\mathrm{B} \in L^{2q_\mathrm{R}}(\partial D;\mathbb{R})$, $c_{\partial\Omega} \in L^\infty(\partial D;\mathbb{R})$ are assumed to be given, where let $q_\mathrm{R} > d$. Here, obtain $u : D \to \mathbb{R}$ that satisfies

$$-\boldsymbol{\nabla} \cdot (\phi^\alpha(\theta) \boldsymbol{\nabla} u) + c_\Omega u = b(\theta) \quad \text{in } D,$$
$$\phi^\alpha(\theta)\, \partial_\nu u + c_{\partial\Omega} u = p_\mathrm{B} \quad \text{on } \partial D.$$

□

Let the Lagrange function with respect to Problem P.8.1 be

$$\mathscr{L}_\mathrm{S}(\theta, u, v) = \int_D \left(-\phi^\alpha(\theta) \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v - c_\Omega uv + b(\theta)\, v\right) \mathrm{d}x + \int_{\partial\Omega} \left(-c_{\partial\Omega} uv + p_\mathrm{B} v\right) \mathrm{d}\gamma$$

by applying Problem 5.1.4. As an analogy with the mean compliance with respect to the θ-type linear elastic problem, let an objective function be

$$f_0(u) = \int_D b(\theta)\, u \,\mathrm{d}x + \int_{\partial D} p_\mathrm{B} u \,\mathrm{d}\gamma, \tag{P.8.1}$$

and a constraint function with respect to the domain measure be Eq. (8.9.7). Here, the θ-type topology optimization problem becomes as follows.

Problem P.8.2 (θ-type topology optimization problem) Let $\mathcal{D}$ be Eq. (8.1.4), and $\mathcal{S} = W^{1,2q_\mathrm{R}}(D;\mathbb{R})$. Let f_0 and f_1 be Eq. (P.8.1) and Eq. (8.9.7), respectively. In this case, obtain θ satisfying

$$\min_{(\theta,u)\in\mathcal{D}\times\mathcal{S}} \left\{ f_0(\theta, u) \mid f_1(\theta) \le 0, \text{ Problem P.8.1} \right\}.$$

□

In order to obtain the θ-derivative of f_0, let the Lagrange function with respect to f_0 be

$$\begin{aligned}\mathscr{L}_0(\theta, u, v_0) &= f_0(\theta, u) + \mathscr{L}_\mathrm{S}(\theta, u, v_0)\\ &= \int_D \{-\phi^\alpha(\theta)\,\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_0 + b(\theta)(u + v_0)\}\,\mathrm{d}x\\ &\quad + \int_{\partial\Omega} p_\mathrm{B}(u + v_0)\ \mathrm{d}\gamma.\end{aligned}$$

Let the Fréchet derivative of $\mathscr{L}_0$ with respect to an arbitrary variation $(\vartheta, \hat{u}, \hat{v}_0) \in X \times U \times U$ (where $U = H^1(D;\mathbb{R})$) of (θ, u, v_0) be

$$\begin{aligned}\mathscr{L}_0'(\theta, u, v_0)[\vartheta, \hat{u}, \hat{v}_0] &= \mathscr{L}_{0\theta}(\theta, u, v_0)[\vartheta] + \mathscr{L}_{0u}(\theta, u, v_0)[\hat{u}]\\ &\quad + \mathscr{L}_{0v_0}(\theta, u, v_0)[\hat{v}_0].\end{aligned} \tag{P.8.2}$$

The third term on the right-hand side of Eq. (P.8.2) becomes

$$\mathscr{L}_{0v_0}(\theta, u, v_0)[\hat{v}_0] = \mathscr{L}_{\mathrm{S}v_0}(\theta, u, v_0)[\hat{v}_0] = \mathscr{L}_\mathrm{S}(\theta, u, \hat{v}_0).$$

Moreover, the second term on the right-hand side of Eq. (P.8.2) becomes

$$\mathscr{L}_{0u}(\theta, u, v_0)[\hat{u}] = \mathscr{L}_\mathrm{S}(\theta, \hat{u}, v_0).$$

Here, the self-adjoint relationship:

$$u = v_0$$

holds. Furthermore, the first term on the right-hand side of Eq. (P.8.2) becomes

$$\mathscr{L}_{0\theta}(\theta, u, v_0)[\vartheta] = \int_D \left\{b_\theta \cdot (u + v_0) - \alpha\phi^{\alpha-1}\phi_\theta \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_0\right\}\vartheta\ \mathrm{d}x.$$

Hence, we get

$$\begin{aligned}\tilde{f}_0'(\theta)[\vartheta] &= \mathscr{L}_{0\theta}(\theta, u, v_0)[\vartheta] = \langle g_0, \vartheta\rangle\\ &= \int_D \left(2b_\theta \cdot u - \alpha\phi^{\alpha-1}\phi_\theta \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u\right)\vartheta\ \mathrm{d}x.\end{aligned}$$

On the other hand, the θ-derivative of $f_1(\theta)$ becomes

$$f_1'(\theta)[\vartheta] = \langle g_1, \vartheta\rangle = \int_D \phi_\theta \vartheta\ \mathrm{d}x.$$

Here, the KKT conditions with respect to Problem P.8.2 are given as the conditions for which

$$\begin{aligned}\langle g_0 + \lambda_1 g_1, \vartheta\rangle &= \left\langle 2b_\theta \cdot u + \left(-\alpha\phi^{\alpha-1}\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + \lambda_1\right)\phi_\theta, \vartheta\right\rangle = 0,\\ f_1(\theta) &\le 0,\end{aligned}$$

$$\begin{aligned}\lambda_1 f_1(\theta) &= 0,\\ \lambda_1 &\geq 0\end{aligned}$$

hold with respect to an arbitrary $\vartheta \in X$. Here, λ_1 is the Lagrange multiplier with respect to the domain measure constraint.

8.3 Let the Lagrange function with respect to Problem 8.12.1 be

$$\mathscr{L}(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m) = \beta + \sum_{i \in \{1,\ldots,m\}} \lambda_i \mathscr{L}_i(\theta, \beta, u, v_i),$$

where $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_m\}^\top$ is a Lagrange multiplier with respect to $f_1 - \beta \leq 0$, $\ldots$, $f_m - \beta \leq 0$, and

$$\mathscr{L}_i(\theta, \beta, u, v_i) = f_i(\theta, u) - \beta + \mathscr{L}_{\mathrm{S}}(\theta, u, v_i).$$

Here, let $\mathscr{L}_{\mathrm{S}}$ be defined in Eq. (8.2.4). The Fréchet derivative of $\mathscr{L}$ with respect to an arbitrary variation $\left(\vartheta, \hat{\beta}, \hat{u}, \hat{v}_1, \ldots, \hat{v}_m\right) \in X \times \mathbb{R} \times U^{m+1}$ of $(\theta, \beta, u, v_1, \ldots, v_m)$ is written as

$$\begin{aligned}&\mathscr{L}'(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m)\left[\vartheta, \hat{\beta}, \hat{u}, \hat{v}_1, \ldots, \hat{v}_m\right]\\ &\quad= \mathscr{L}_\theta(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m)[\vartheta]\\ &\qquad+ \mathscr{L}_\beta(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m)\left[\hat{\beta}\right]\\ &\qquad+ \sum_{i \in \{1,\ldots,m\}} \lambda_i \mathscr{L}_{iu}(\theta, \beta, u, v_i)[\hat{u}]\\ &\qquad+ \sum_{i \in \{1,\ldots,m\}} \lambda_i \mathscr{L}_{iv_i}(\theta, \beta, u, v_i)[v_i'].\end{aligned} \tag{P.8.3}$$

The fourth term on the right-hand side of Eq. (P.8.3) becomes 0 when u is the weak solution of the state determination problem. The third term on the right-hand side of Eq. (P.8.3) becomes

$$\begin{aligned}&\sum_{i \in \{1,\ldots,m\}} \lambda_i \mathscr{L}_{iu}(\theta, \beta, u, v_i)[\hat{u}]\\ &\quad= \sum_{i \in \{1,\ldots,m\}} \lambda_i \left(f_{iu}(\theta, u)[\hat{u}] + \mathscr{L}_{\mathrm{S}u}(\theta, u, v_i)[\hat{u}]\right).\end{aligned}$$

When v_1, $\ldots$, v_m are the weak solutions of adjoint problem (Problem 8.5.1) with respect to f_1, $\ldots$ and f_m, respectively, it becomes 0. The second term on the right-hand side of Eq. (P.8.3) becomes

$$\mathscr{L}_\beta(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m)\left[\hat{\beta}\right] = (1 - \lambda_1 - \ldots - \lambda_m)\hat{\beta}.$$

Furthermore, the first term on the right-hand side of Eq. (P.8.3) can be written as

$$\begin{aligned} &\mathscr{L}_{\theta}\left(\theta, \beta, u, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m\right)[\vartheta] \\ &\quad = \sum_{i \in \{1,\ldots,m\}} \lambda_i \mathscr{L}_{i\theta}\left(\theta, \beta, u, v_i\right)[\vartheta] = \sum_{i \in \{1,\ldots,m\}} \lambda_i \left\langle g_i, \vartheta \right\rangle . \end{aligned}$$

Here g_i is given by Eq. (8.5.6).

Hence, the KKT conditions with respect to Problem 8.12.1 are given as the conditions under which

$$\begin{aligned} \lambda_1 + \cdots + \lambda_m &= 1, \qquad (\text{P.8.4}) \\ \left\langle \sum_{i \in \{1,\ldots,m\}} \lambda_i g_i, \vartheta \right\rangle &= 0, \\ f_i(\theta) \le 0 &\quad \text{for } i \in \{1, \ldots, m\}, \\ \lambda_i f_i(\theta) = 0 &\quad \text{for } i \in \{1, \ldots, m\}, \\ \lambda_i \ge 0 &\quad \text{for } i \in \{1, \ldots, m\} \end{aligned}$$

holds with respect to an arbitrary $\vartheta \in X$.

Moreover, the solution to this problem using the gradient method with respect to constrained problems becomes as seen below. Imagine a situation with a simple algorithm (Algorithm 3.7.2) shown in Section 3.7.1, and suppose the replacements such as those shown in Section 8.7 are conducted. In this problem, g_0 ($\boldsymbol{g}_0$ in Problem 3.7.1) becomes 0. Therefore $\vartheta_{g0} = 0$. Moreover, set $\beta = \max_{i \in \{1,\ldots,m\}} f_i - \epsilon$ with ϵ as a positive constant. Here, Eq. (8.7.3) for obtaining the Lagrange multiplier becomes

$$\left(\left\langle g_i, \vartheta_{gj} \right\rangle\right)_{(i,j) \in I_{\mathrm{A}}^2} \left(\lambda_j\right)_{j \in I_{\mathrm{A}}} = -\left(f_i\right)_{i \in I_{\mathrm{A}}} . \qquad (\text{P.8.5})$$

If $\left(g_i\right)_{i \in I_{\mathrm{A}}}$ is linearly independent, $\left(\lambda_j\right)_{j \in I_{\mathrm{A}}}$ satisfying Eq. (P.8.5) is uniquely determined. Here, if $c = \sum_{j \in I_{\mathrm{A}}} \lambda_j$ is used to replace $\left(\lambda_j / c\right)_{j \in I_{\mathrm{A}}}$ with $\left(\lambda_j\right)_{j \in I_{\mathrm{A}}}$ and $\left(c \vartheta_{gj}\right)_{j \in I_{\mathrm{A}}}$ with $\left(\vartheta_{gj}\right)_{j \in I_{\mathrm{A}}}$, Eq. (P.8.4) and Eq. (P.8.5) are simultaneously satisfied. However, if Eq. (8.7.2) is used to seek ϑ_g, these replacements become unnecessary.

8.4 If $\boldsymbol{u}$ is the solution of the state determination problem (Problem 8.9.2), it satisfies $\min_{\boldsymbol{u} \in U} \pi$ (Theorem 5.2.9). On the other hand, the maximum point with respect to θ of $\pi(\theta, \boldsymbol{u})$ becomes the minimum point of $-\pi(\theta, \boldsymbol{u})$. When $\boldsymbol{u}$ is a solution of the state determination problem,

$$-\pi_{\theta}(\theta, \boldsymbol{u})[\vartheta] = \frac{1}{2} \left\langle g_0, \vartheta \right\rangle$$

holds with respect to an arbitrary $\vartheta \in X$. Here, g_0 represents a vector of Eq. (8.9.14).

8.5 If $(\boldsymbol{u}, p)$ is the solution of a state determination problem (Problem 8.10.2), it satisfies $\min_{\boldsymbol{u}\in U} \max_{p\in P} \pi$ (Theorem 5.6.6). On the other hand, when $(\boldsymbol{u}, p)$ is the solution of the state determination problem,

$$\pi_\theta (\theta, \boldsymbol{u}, p) [\vartheta] = \frac{1}{2} \langle g_0, \vartheta \rangle$$

holds with respect to an arbitrary $\vartheta \in X$. Here, g_0 represents a vector of Eq. (8.10.17).

Chapter 9

9.1 With respect to the second term on the right-hand side of Eq. (9.8.9),

$$\begin{aligned} &\left\| \left(\sum_{j\in\{1,\ldots,d-1\}} \{\boldsymbol{\tau}_j \cdot \boldsymbol{\nabla} (p_{\mathrm{N}} v_i)\} \boldsymbol{\tau}_j \right) \cdot \boldsymbol{\varphi} \right\|_{L^1(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\ &\quad \le (d-1) \max_{j\in\{1,\ldots,d-1\}} \Big(\|\boldsymbol{\tau}_j\|^2_{L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\ &\qquad \times \|\boldsymbol{\nabla} (p_{\mathrm{N}} v_i)\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \Big) \|\boldsymbol{\varphi}\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \end{aligned} \tag{P.9.1}$$

holds. Here,

$$\begin{aligned} \|\boldsymbol{\nabla} (p_{\mathrm{N}} v_i)\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} &\le \|p_{\mathrm{N}} v_i\|_{H^1(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\ &\le \|p_{\mathrm{N}}\|_{W^{1,4}(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \|v_i\|_{W^{1,4}(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\ &\le \|\gamma_{\partial\Omega}\|^2 \|p_{\mathrm{N}}\|_{C^{1,1}(D;\mathbb{R})} \|v_i\|_{W^{2,4}(D;\mathbb{R})} \end{aligned}$$

holds. Hence,

$$\begin{aligned} &(\text{Eq. (P.9.1) の右辺}) \\ &\quad \le \|\gamma_{\partial\Omega}\|^3 (d-1) \max_{j\in\{1,\ldots,d-1\}} \|\boldsymbol{\tau}_j\|^2_{H^{3/2}\cap C^{0,1}(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\ &\qquad \times \|p_{\mathrm{N}}\|_{C^{1,1}(D;\mathbb{R})} \|v_i\|_{W^{2,4}(D;\mathbb{R})} \|\boldsymbol{\varphi}\|_X \end{aligned}$$

holds. If Hypothesis 9.5.1 is satisfied, the right-hand side of the equation above becomes bounded, and the second term on the right-hand side of Eq. (9.8.9) becomes an element of X'. Furthermore, from the fact that $\boldsymbol{\nabla} (p_{\mathrm{N}} v_i) = v_i \boldsymbol{\nabla} p_{\mathrm{N}} + p_{\mathrm{N}} \boldsymbol{\nabla} v_i \in W^{1,4} (D;\mathbb{R})$ and $\boldsymbol{\tau}_j \in H^{3/2} \cap C^{0,1} (\Gamma_p (\boldsymbol{\phi}); \mathbb{R})$, the second term on the right-hand side of Eq. (9.8.9) is included in $H^{1/2} \cap L^\infty \left(\Gamma_p (\boldsymbol{\phi}); \mathbb{R}^d\right)$.

9.2 Let the Lagrange function of Problem 9.15.1 be

$$\mathscr{L}_{\mathrm{S}} (\boldsymbol{\phi}, u, v) = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} (p_{\mathrm{R}} v - c_{\partial\Omega} u v) \, \mathrm{d}\gamma.$$

Moreover, the Lagrange function with respect to f_i is set to be

$$\begin{aligned}\mathscr{L}_i(\boldsymbol{\phi}, u, v_i) &= f_i(\boldsymbol{\phi}, u) + \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v_i)\\ &= -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i \,\mathrm{d}x\\ &\quad + \int_{\partial\Omega(\boldsymbol{\phi})} \left(\eta_{\mathrm{R}i}(\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i\right) \mathrm{d}\gamma.\end{aligned}$$

Applying the formulae using the shape derivative of a function, the shape derivative of $\mathscr{L}_i$ can be written as

$$\begin{aligned}&\mathscr{L}_i'(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}, \hat{u}, \hat{v}_i]\\ &\quad = \mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}] + \mathscr{L}_{iu}(\boldsymbol{\phi}, u, v_i)[\hat{u}] + \mathscr{L}_{iv_i}(\boldsymbol{\phi}, u, v_i)[\hat{v}_i].\end{aligned} \tag{P.9.2}$$

The third term on the right-hand side of Eq. (P.9.2) becomes

$$\mathscr{L}_{iv_i}(\boldsymbol{\phi}, u, v_i)[\hat{v}_i] = \mathscr{L}_{\mathrm{S}v_i}(\boldsymbol{\phi}, u, v_i)[\hat{v}_i] = \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, \hat{v}_i).$$

If u is a weak solution of the state determination problem (Problem 9.15.1), it becomes 0. Moreover, the second term on the right-hand side of Eq. (P.9.2) becomes

$$\begin{aligned}&\mathscr{L}_{iu}(\boldsymbol{\phi}, u, v_i)[\hat{u}]\\ &\quad = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}\hat{u} \cdot \boldsymbol{\nabla} v_i \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \left(\eta_{\mathrm{R}iu}(\boldsymbol{\phi}, u)[\hat{u}] - c_{\partial\Omega} v_i \hat{u}\right) \mathrm{d}\gamma.\end{aligned}$$

When v_i is a weak solution of an adjoint problem with respect to f_i such as the following, the second term on the right-hand side of Eq. (P.9.2) becomes 0 too.

Problem P.9.1 (Adjoint problem with respect to f_i) When a solution u of Problem 9.15.1 with respect to $\boldsymbol{\phi} \in \mathcal{D}$ is given, obtain $v_i : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$\begin{aligned}-\Delta v_i &= 0 \quad \text{in } \Omega(\boldsymbol{\phi}),\\ \partial_\nu v_i + c_{\partial\Omega}(\boldsymbol{\phi}) v_i &= \eta_{\mathrm{R}iu}(\boldsymbol{\phi}, u) \quad \text{on } \partial\Omega(\boldsymbol{\phi}).\end{aligned}$$

□

Furthermore, the first term on the right-hand side of Eq. (P.9.2) becomes

$$\begin{aligned}&\mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}]\\ &\quad = \int_{\Omega(\boldsymbol{\phi})} \Big\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top \boldsymbol{\nabla} v_i\right) + \boldsymbol{\nabla} v_i \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top \boldsymbol{\nabla} u\right)\\ &\qquad - \left(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i\right) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \Big\} \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\partial\Omega(\boldsymbol{\phi})} \Big\{ \kappa \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \\
&\quad - \boldsymbol{\nabla}_\tau \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \cdot \boldsymbol{\varphi}_\tau \Big\} \mathrm{d}\gamma \\
&+ \int_{\Theta(\boldsymbol{\phi})} \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma.
\end{aligned}
$$

In order to obtain this integral, the fact that $\partial\Omega(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$ was used. Moreover, the known function was assumed to be fixed with the material.

With the above results in mind, if u and v_i are assumed to be the weak solutions of Problems 9.15.1 and P.9.1,

$$
\begin{aligned}
\tilde{f}_i' (\boldsymbol{\phi}) [\boldsymbol{\varphi}] &= \mathscr{L}_{i\boldsymbol{\phi}'} (\boldsymbol{\phi}, u, v_i) [\boldsymbol{\varphi}] = \langle \boldsymbol{g}_i, \boldsymbol{\varphi} \rangle \\
&= \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{G}_{\Omega i} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}^\top + g_{\Omega i} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \right) \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{g}_{\partial\Omega i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\Theta(\boldsymbol{\phi})} \boldsymbol{g}_{\Theta i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma
\end{aligned}
$$

can be written. Here, we get

$$
\begin{aligned}
\boldsymbol{G}_{\Omega i} &= \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top + \boldsymbol{\nabla} v_i \left(\boldsymbol{\nabla} u \right)^\top, \\
g_{\Omega i} &= -\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i, \\
\boldsymbol{g}_{\partial\Omega i} &= \kappa \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \boldsymbol{\nu} \\
&\quad - \sum_{j \in \{1, \ldots, d-1\}} \left\{ \boldsymbol{\tau}_j \cdot \boldsymbol{\nabla} \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \right\} \boldsymbol{\tau}_j, \\
\boldsymbol{g}_{\Theta i} &= \left(\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) + p_{\mathrm{R}} v_i - c_{\partial\Omega} u v_i \right) \boldsymbol{\tau}.
\end{aligned}
$$

The similar regularity for $\boldsymbol{g}_i$ in Theorem 9.8.2 means $\boldsymbol{G}_{\Omega i} \in H^1 \cap L^\infty \left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d \times d} \right)$, $g_{\Omega i} \in H^1 \cap L^\infty \left(\Omega(\boldsymbol{\phi}); \mathbb{R} \right)$ and $\boldsymbol{g}_{\partial\Omega i} \in H^{1/2} \cap L^\infty \left(\partial\Omega(\boldsymbol{\phi}); \mathbb{R}^d \right)$. To obtain the results, from the proof of Theorem 9.8.2, considering that u and v_i are elements of $W^{2,4}(D; \mathbb{R})$, the regularity of known function required in this case is

$$
\begin{aligned}
&c_{\partial\Omega} \in C^1_{\mathrm{S}'} \left(B; C^{1,1}(D; \mathbb{R}) \right), \quad p_{\mathrm{R}} \in C^1_{\mathrm{S}'} \left(B; C^{1,1}(D; \mathbb{R}) \right), \\
&\eta_{\mathrm{R}i} (\boldsymbol{\phi}, u) \in W^{2, q_{\mathrm{R}}}(D; \mathbb{R}), \quad \eta_{\mathrm{R}iu} (\boldsymbol{\phi}, u) [\hat{u}] \in W^{1,4}(D; \mathbb{R})
\end{aligned}
$$

in a neighborhood $B \subset Y$ of $\boldsymbol{\phi} \in \mathcal{D}^\circ$. On the other side, with respect to an opening angle β of a corner point, the condition $\beta < 2\pi/3$ when the corner point is between boundaries of the same type will be applied.

9.3 Let us use Eq. (9.15.3) in order to obtain $\hat{\boldsymbol{g}}_{i\mathrm{C}}$. With respect to the first term in the right-hand integrand of Eq. (9.15.3),

$$
\boldsymbol{\nabla} u = \begin{pmatrix} \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r} \dfrac{\partial}{\partial \theta} \\ \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r} \dfrac{\partial}{\partial \theta} \end{pmatrix} u = \frac{k_j}{2\epsilon^{1/2}} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix},
$$

$$\boldsymbol{\nabla} v_i = \frac{l_{ij}}{2\epsilon^{1/2}} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$$

holds. Here, we obtain

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i = \frac{k_j l_{ij}}{4\epsilon}.$$

Substituting this result into the first term of the right-hand integrand of Eq. (9.15.3) gives

$$\begin{aligned} &-\int_0^{2\pi} (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\, \epsilon \, \mathrm{d}\theta \\ &\quad = \int_0^{2\pi} \frac{k_j l_{ij}}{4} (\varphi_1 \cos\theta + \varphi_2 \sin\theta)\, \mathrm{d}\theta = 0 \end{aligned} \tag{P.9.3}$$

with respect to an arbitrary $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top \in \mathbb{R}^2$. Furthermore, with respect to the second term of the integrand,

$$\begin{aligned} \partial_\nu u = \boldsymbol{\nu} \cdot \boldsymbol{\nabla} u &= \frac{k_j}{2\epsilon^{1/2}} \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \\ &= -\frac{k_j}{2\epsilon^{1/2}} \cos(\theta/2), \\ \partial_\nu u \boldsymbol{\nabla} v_i &= -\frac{k_j l_{ij}}{4\epsilon} \cos(\theta/2) \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \end{aligned}$$

is established. Here the second term of the integrand becomes

$$\begin{aligned} \int_0^{2\pi} \partial_\nu u \boldsymbol{\nabla} v_i \cdot \boldsymbol{\varphi}\, \epsilon \, \mathrm{d}\theta &= \int_0^{2\pi} \partial_\nu v_i \boldsymbol{\nabla} u \cdot \boldsymbol{\varphi}\, \epsilon \, \mathrm{d}\theta \\ &= -\frac{k_j l_{ij}}{4} \begin{pmatrix} \pi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \end{aligned} \tag{P.9.4}$$

with respect to an arbitrary $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top \in \mathbb{R}^2$. The same result holds for the third term of the integrand. Hence, from Eq. (P.9.3) and Eq. (P.9.4),

$$\langle \hat{\boldsymbol{g}}_{i\mathrm{C}}, \boldsymbol{\varphi} \rangle = -\frac{k_j l_{ij}}{2} \begin{pmatrix} \pi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{P.9.5}$$

can be obtained. From Eq. (P.9.5), we see that the shape derivative $\hat{\boldsymbol{g}}_{i\mathrm{C}}$ with respect to a variation of a crack point is in the direction of the crack surface.

$\hat{\boldsymbol{g}}_{i\mathrm{M}}$ becomes as follows. With respect to the first term on the right-hand integrand of Eq. (9.15.3),

$$\boldsymbol{\nabla} u = \begin{pmatrix} \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r} \dfrac{\partial}{\partial \theta} \\ \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r} \dfrac{\partial}{\partial \theta} \end{pmatrix} u = \frac{k_j}{2\epsilon^{1/2}} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix},$$

$$\boldsymbol{\nabla} v_i = \frac{l_{ij}}{2\epsilon^{1/2}} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

holds. Hence,

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i = \frac{k_j l_{ij}}{4\epsilon}$$

is obtained. If this result is substituted into the first term of the right-hand integrand of Eq. (9.15.3), it becomes

$$\begin{aligned} -\int_0^\pi (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\, \epsilon \,\mathrm{d}\theta &= \int_0^\pi \frac{k_j l_{ij}}{4} (\varphi_1 \cos\theta + \varphi_2 \sin\theta)\,\mathrm{d}\theta \\ &= \frac{k_j l_{ij}}{2} \varphi_2 \end{aligned} \tag{P.9.6}$$

with respect to an arbitrary $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top \in \mathbb{R}^2$. Furthermore,

$$\begin{aligned} \partial_\nu u = \boldsymbol{\nu} \cdot \boldsymbol{\nabla} u &= \frac{k_j}{2\epsilon^{1/2}} \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \\ &= -\frac{k_j}{2\epsilon^{1/2}} \sin(\theta/2), \\ \partial_\nu u \boldsymbol{\nabla} v_i &= -\frac{k_j l_{ij}}{4\epsilon} \sin(\theta/2) \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \end{aligned}$$

holds. Here, the second term of the integrand becomes

$$\begin{aligned} \int_0^\pi \partial_\nu u \boldsymbol{\nabla} v_i \cdot \boldsymbol{\varphi}\, \epsilon \,\mathrm{d}\theta &= \int_0^\pi \partial_\nu v_i \boldsymbol{\nabla} u \cdot \boldsymbol{\varphi}\, \epsilon \,\mathrm{d}\theta \\ &= \frac{k_j l_{ij}}{8} \begin{pmatrix} \pi \\ -2 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \end{aligned} \tag{P.9.7}$$

with respect to an arbitrary $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top \in \mathbb{R}^2$. The third term of the integrand gives the same result. Hence, from Eq. (P.9.6) and Eq. (P.9.7),

$$\langle \hat{\boldsymbol{g}}_{i\mathrm{M}}, \boldsymbol{\varphi} \rangle = \frac{k_j l_{ij}}{4} \begin{pmatrix} \pi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{P.9.8}$$

can be obtained. Equation (P.9.8) shows that the shape derivative $\hat{\boldsymbol{g}}_{i\mathrm{M}}$ at a point of a mixed boundary on a smooth boundary is in the direction of the Neumann boundary.