Contents

Chapter 9

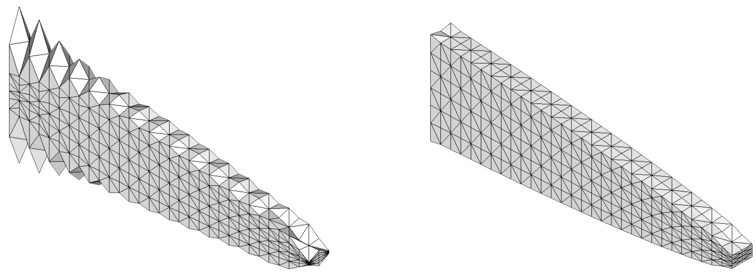
Shape Optimization Problems of Domain Variation Type

In Chap. 8 we looked at problems for obtaining the optimal topologies of continua with the densities of continua set to be the design variable. In this chapter, we shall look at the type of shape optimization problems in which the boundary of a continuum varies. The key theory of numerical solution shown in this chapter is published in the paper [3]. In this book, we shall look at the theory used there by comparing it to the contents shown in Chaps. 1 to 7.

First, let us take an abridged look at the history of research relating to a shape optimization problem of domain variation type. This type of shape optimization problem is also referred to as a domain optimization problem and has been studied since the early 20th century. For example, among the vast works of Hadamard, there is a description relating to a problem seeking the boundary shape of a thin membrane such that the fundamental vibration frequency is maximized. In this description, a notion equivalent to a Fréchet derivative of the fundamental frequency when a boundary is moved in the outward normal direction is presented [28, 87]. Even after that, Fréchet derivatives with respect to shape variations of domain variation type have been referred to as shape derivatives, and many researchers have announced research results relating to it.[1] To add background to this research, there are works relating to optimal control theory assuming a function as a control variable by mathematicians lead by Lions [61].

In this way, theories relating to the calculation methods of shape derivatives have been developed consistently, but research relating to moving the shapes using shape derivatives has not always obtained favorable results. In reality, it is known that if the node coordinates on a boundary of a finite element model

[1] For example, refer to [11–13, 18–23, 25, 26, 29–32, 35, 65, 68, 71–74, 85, 87, 100, 101].



(a) Rippling shape (b) Optimal shape by H^1 gradient method

Fig. 9.1: Numerical examples with respect to the shape optimization problem of a linear elastic body (provided by Quint Corporation).

are chosen to be the design variable, and the Fréchet derivatives with respect to the variation of the design variable are evaluated in order to move the nodes, a numerically unstable phenomenon in which the boundary becomes rippled such as shown in Fig. 9.1 (a) appears [40]. Figure 9.1 (a) shows the result of a numerical analysis with respect to a mean compliance minimization problem (Problem 9.12.2) of a three-dimensional linear elastic body. The boundary condition in the state determination problem constrains the displacement on the back edge, while a uniform downward facing nodal force (external force) on the horizontal central line of the front edge was assumed. The boundary condition in the shape variation problem restrains the variation in the normal direction on the front/back and left/right edges, and the variation on the horizontal central line on the front/back edge. Numerical analysis of a state determination problem uses the first-order finite elements. The calculation method of shape derivatives uses the formula of boundary integration form as shown later.

In order to avoid rippling boundaries such as in this case, there is a method to define the boundary shape as a B-spline curve, Bezier curve, etc. and choose its control variables as the design variables [16, 17]. There is also a method for giving the shape variation as the linear sum of the basic deformation modes and choosing the undetermined multipliers in this case as the design variables (basis vector method) [14, 32, 75, 90, 91]. All these methods have been highlighted and used in actual optimal designs. However, all the methods used derivatives with respect to parametric design variables, such as those explained in the Preface and differ from original shape derivatives.

In this chapter, we will look at a method for evaluating the shape derivatives of cost functions after having constructed a shape optimization problem of a domain variation type in which a function expressing the domain variation defined on an appropriate function space is set to be the design variable based on the framework of the abstract optimal design problem shown in Chap. 7. As a result, the shape gradient does not have enough regularity to create the following domain. This is thought to be one of the factors generating numerical instability. In this situation, even if such a shape derivative is used, if an appropriate gradient method is used, there is the possibility that a shape optimization problem can be solved without facing numerical instability. In this chapter, this method will be the focus of our discussion.

Figure 9.1 (b) shows the results obtained via the algorithm shown in Sect. 9.10. Boundary conditions and calculation method of the shape derivative are the same as in Fig. 9.1 (a). Numerical analysis of the state determination problem used second-order tetrahedral finite elements. Moreover, in numerical analysis of the H^1 gradient method using the Robin condition shown later, the first-order tetrahedral finite elements were used. Moreover, the validity relating to the selection of a finite element such as this is shown in Sect. 9.11.

The fundamental idea relating the gradient method on a function space was presented by Cea [18]. A primitive form of the gradient method can be found in Pironneau's monograph [73, p. 48 (17)]. In addition, a method called asymptotical regularization was proposed by Tautenhahn [89]. In contrast, in the 1990s, the author [2] proposed a gradient method on a function space which was referred to as the traction method, based on an engineering principle. After that, a generalization of the traction method was also introduced [7]. Furthermore, these methods have been applied in various engineering problems.[2] Moreover, the interpretation of the traction method in mathematics was also attempted in an existing report [45]. Here the domain mapping was assumed to be an element of the set of all continuous functions of some class, and the traction method was justified using the Gâteaux derivative of a cost function with respect to the variation of the domain mapping. In this chapter, a gradient method uses the Fréchet derivative of a cost function by defining the variation of the domain mapping in an appropriate Hilbert space. Based on this gradient method, it is apparent that the traction method was indeed a concrete example of that computational procedure.

Furthermore, a different method for constructing a shape optimization problem of domain variation type is proposed. As thought by Hadamard [28], since the next boundary shape can be determined by moving the boundary to the normal direction, one method is choosing the function that represents the amount of movement in the normal direction defined on the boundary as the design variable [64]. This method also uses the gradient method with the functionality of keeping the regularity equivalent to the gradient method shown in this chapter. However, if a finite element method is used for numerical analysis of a state determination problem, after the boundary has been moved by the gradient method, we have to consider a method for moving the finite element mesh within the domain along with the new boundary. In addition, methods using level-set functions for design variables are also being researched [1, 92, 99]. In these methods, a level-set function which is a continuous function with scalar value defined on a fixed domain is used to define the boundary with a set of points in the domain where its value is zero. Using these methods, the topology of the domain can easily be changed through joining the holes together by varying the level-set function. However, since the level-set function is defined using Euler notation (see after Definition 9.1.3), a wider domain is required than

[2] See, for example, references [4, 6, 8–10, 36–39, 41–44, 46–58, 76–84, 93–98].

the actual domain. Moreover, in order to extract a numerical model from the level set of zero, some processes are required. Furthermore, stronger conditions with respect to the regularity of the solution for the state determination problem are required for the aforementioned two methods than for the method shown in this chapter. The reason for this is that when calculating the Fréchet derivatives of cost functions, only the formula of boundary integral type can be applicable.

This chapter is structured as follows. In Sections 9.1 to 9.4, the definitions and formulae relating to functions and functionals defined on a moving domain are summarized. In Sect. 9.1, the definitions of admissible set of a design variable (function representing domain variation) and shape derivatives of functions and functionals are shown. There, attention will be given to the fact that there are two methods of defining the derivatives of functions defined on a moving domain with respect to domain variation. In this book, we shall refer to these notions of derivatives as “shape derivative of a function” and “partial shape derivative of a function”. Using these definitions, the formulae for shape derivatives relating to the Jacobi matrix of the domain mapping will be obtained in Sect. 9.2. Using the formulae, in Sect. 9.3, the propositions relating to shape derivatives of functions and functionals are shown. Here also, we will focus on the fact that the formulae using the shape derivatives of functions and partial shape derivatives of functions can be obtained. Sect. 9.4 defines several rules for variations of functions with respect to domain variation using the shape derivative of a function and the partial shape derivative of a function.

In Sections 9.5 to 9.8, we will consider a shape optimization problem when a Poisson problem is chosen to be the state determination problem and present the process of computing the shape derivatives of cost functions. In Sect. 9.5, a state determination problem will be defined using a Poisson problem using the variation rules for functions shown in Sect. 9.4. The solution to this problem is used in Sect. 9.6 to define a general cost function which is then used to define a shape optimization problem. The existence of a solution to the shape optimization problem of this is shown in Sect. 9.7. In Sect. 9.8, the methods for obtaining the Fréchet derivatives of cost functions shown in Section 7.4 are followed in order to show the methods to obtain shape derivatives and second-order derivatives of cost functions with respect to a domain variation. In this case, we focus on the fact that we can think of two methods: one using formulae based on the shape derivative of a function, and another using formulae based on the partial shape derivative of a function. As a result, it becomes clear that whichever method is used, the shape gradients of the cost functions do not have enough regularity to be able to define the following domain.

Even if the shape gradients have insufficient regularities, by applying the abstract gradient method or the abstract Newton method shown in Section 7.5 to the shape optimization problems, a gradient method and Newton method with the functionality to regularize the shape derivatives of cost functions can be defined. In Sect. 9.9, their abstract definitions and several methods for specifying these are introduced. In Sect. 9.10, algorithms will be considered. However, the basic structures are as per the algorithms shown in Section 3.7. The error evaluation of the numerical solutions obtained using these algorithms

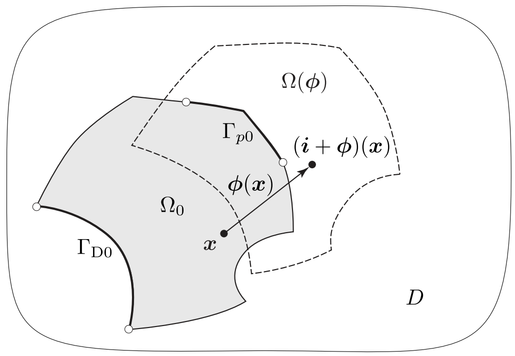


Fig. 9.2: Domain variation (displacement) $\boldsymbol{\phi} : D \to \mathbb{R}^d$.

is shown in Sect. 9.11. Here, the results from the error estimations of numerical analyses shown in Section 6.6 will be used.

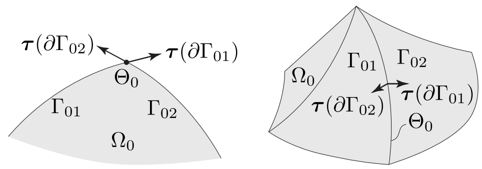
Once we look at the range of solutions with respect to the shape optimization problem of a Poisson problem, the shape derivatives of cost functions with respect to a mean compliance minimization problem of a linear elastic body will be sought in Sect. 9.12. Furthermore, in Sect. 9.13, the mean flow resistance minimization problem of a Stokes flow field will be used as an example to obtain the shape derivatives of the cost functions. The conditions of optimality using these shape derivatives can be seen matching the conditions of optimality with respect to the mean compliance minimization problem for a one-dimensional linear elastic body shown in Section 1.1 and the mean flow resistance minimization problem for a one-dimensional branched Stokes flow field shown in Section 1.3. Moreover, in Sections 9.12.5 and 9.13.5, numerical examples with respect to these simple problems will be shown.

9.1 Set of Domain Variations and Definition of Shape Derivatives

In order to construct a shape optimization problem of domain variation type, let us define the admissible set of design variables. Moreover, the Fréchet derivatives of functions and functionals defined in a moving domain with respect to domain variation will be referred to as shape derivatives. These definitions will be shown in this section.

9.1.1 Initial Domain

Referring to Fig. 9.2, $\Omega_0 \subset \mathbb{R}^d$ is taken to be a $d \in \{2,3\}$-dimensional Lipschitz domain (Section A.5) representing the initial domain. In this chapter, we will assume that this boundary $\partial\Omega_0$ is also $H^2 \cap C^{0,1}$ class. Here, a boundary of $H^{k+2} \cap C^{k,1}$ class ($k \in \{0,1,2,\ldots\}$) is defined as that the function $\boldsymbol{\phi}$



(a) $\Omega_0 \subset \mathbb{R}^2$ (b) $\Omega_0 \subset \mathbb{R}^3$

Fig. 9.3: Set of corner points (when $d = 2$) or edges (when $d = 3$) $\Theta_0 = \partial\Gamma_{01} \cap \partial\Gamma_{02}$ on boundary $\Gamma_0 = \Gamma_{01} \cup \Gamma_{02} \cup (\partial\Gamma_{01} \cap \partial\Gamma_{02}) \subset \partial\Omega_0$ and outward facing tangent $\boldsymbol{\tau}(\partial\Gamma_{01})$ and $\boldsymbol{\tau}(\partial\Gamma_{02})$ on $\partial\Gamma_{01}$ and $\partial\Gamma_{02}$, respectively.

defined in Definition A.5.2 (C^k class domain) belongs to $H^{k+2}\left(B(\boldsymbol{x},\boldsymbol{\alpha});\mathbb{R}^d\right) \cap C^{k,1}\left(B(\boldsymbol{x},\boldsymbol{\alpha});\mathbb{R}^d\right)$ $\left(H^{k+3/2}\left(\partial\Omega_0 \cap B(\boldsymbol{x},\boldsymbol{\alpha});\mathbb{R}^d\right) \cap C^{k,1}\left(\partial\Omega_0 \cap B(\boldsymbol{x},\boldsymbol{\alpha});\mathbb{R}^d\right)\right.$ on boundary). Moreover, hereafter, we denote $H^{k+2}\left(\Omega_0;\mathbb{R}^d\right) \cap C^{k,1}\left(\Omega_0;\mathbb{R}^d\right)$ as $H^{k+2} \cap C^{k,1}\left(\Omega_0;\mathbb{R}^d\right)$.

It is assumed that Ω_0 is given. With respect to the boundary $\partial\Omega_0$ of the initial domain, $\Gamma_{\mathrm{D}0} \subset \partial\Omega_0$ is taken to be a Dirichlet boundary and $\Gamma_{\mathrm{N}0} = \partial\Omega_0 \setminus \bar{\Gamma}_{\mathrm{D}0}$ a Neumann boundary. Moreover, the notation for the set $\bar{(\,\cdot\,)}$ is to represent a closure. Moreover, in this chapter, homogeneous Neumann boundaries and inhomogeneous Neumann boundaries will be distinguished from one another, and the inhomogeneous Neumann boundary of the initial domain will be written as $\Gamma_{p0} \subset \Gamma_{\mathrm{N}0}$. Furthermore, we assume that the integrands used in the boundary integrals in the $m+1$ cost functions f_0 (object cost function) and $f_1, \ldots, f_m$ (constraint cost functions) to be defined by Eq. (9.6.1) later will be denoted as $\eta_{\mathrm{N}i}$ with respect to $i \in \{0, 1, \ldots, m\}$ and these will be non-zero on $\Gamma_{\eta i0} \subset \Gamma_{\mathrm{N}0}$. If Γ_{p0} and $\Gamma_{\eta i0}$ are assumed to vary, these boundaries are piecewise $H^3 \cap C^{1,1}$, and when $d = 3$, boundaries $\partial\Gamma_{p0}$ or $\partial\Gamma_{\eta i0}$ are assumed to be $H^2 \cap C^{0,1}$ class. These hypotheses will be needed to guarantee appropriate regularity of shape derivatives of cost functions obtained on these boundaries. Moreover, when their boundaries are denoted as Γ_0 (Γ_0 is an open set excluding $\partial\Gamma_0$) as shown in Fig. 9.3, the set of corner points (when $d = 2$) or edges (when $d = 3$) on Γ_0 is denoted as Θ_0, and edges included in Θ_0 (when $d = 3$) are assumed to be $H^2 \cap C^{0,1}$ class. For $\Gamma_{(\,\cdot\,)}$, the notation $\Theta_{(\,\cdot\,)}$ is used.

9.1.2 Sets of Domain Variations

Let us define a domain after Ω_0 is perturbed. $\boldsymbol{i}$ will represent the identity mapping. In this case, the domain after Ω_0 is perturbed is assumed to be formed by a continuous bijective mapping $\boldsymbol{i} + \boldsymbol{\phi} : \Omega_0 \to \mathbb{R}^d$ as $(\boldsymbol{i} + \boldsymbol{\phi})(\Omega_0) = \{(\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega_0\}$. In other words, $\boldsymbol{\phi}$ is to represent the displacement in the domain mapping. Since the domain $(\boldsymbol{i} + \boldsymbol{\phi})(\Omega_0)$ is formed by $\boldsymbol{\phi}$, it is denoted by $\Omega(\boldsymbol{\phi})$. Similarly, with respect to an initial domain or boundary $(\,\cdot\,)_0$, $(\,\cdot\,)(\boldsymbol{\phi})$ represents $\{(\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in (\,\cdot\,)_0\}$.

When the design variable $\boldsymbol{\phi}$ is selected as above, even though the domain of $\boldsymbol{\phi}$ is fixed at Ω_0, the domain of the solution to a state determination problem varies with the domain variation. Such a situation is not expected to occur in a general function optimization problem. However, from the Calderón extension theorem (Theorem 4.4.4), if the domain of $\boldsymbol{\phi}$ is expanded to $D \subset \mathbb{R}^d$ large enough via Theorem 4.4.4, the conditions for ordinary function optimization problems are satisfied.

Hence, under conditions satisfying the assumption (with respect to $p > 1$, $\boldsymbol{\phi} \in W^{1,p}\left(\Omega_0; \mathbb{R}^d\right)$) of Theorem 4.4.4, we will expand the domain of $\boldsymbol{\phi}$ from Ω_0 to a bounded domain $D \subset \mathbb{R}^d$ large enough. Furthermore, since we will be considering the gradient method on a function space later, the function space containing the design variable $\boldsymbol{\phi}$ needs to be a Hilbert space. Hence in this chapter, the linear space of design variables is defined as

$$X = \left\{ \boldsymbol{\phi} \in H^1\left(D; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{\phi} = \mathbf{0}_{\mathbb{R}^d} \text{ on } \partial D \cup \bar{\Omega}_{\mathrm{C0}} \right\}, \tag{9.1.1}$$

where $\bar{\Omega}_{\mathrm{C0}} \subset \bar{\Omega}_0$ represents a boundary or closure of the domain in which the domain variation is constrained by a design demand. In continuing discussions in this chapter, it will be viewed as $\bar{\Omega}_{\mathrm{C0}} = \emptyset$ (in other words, $X = H^1\left(D; \mathbb{R}^d\right)$). If it is needed that the measure of $\bar{\Omega}_{\mathrm{C0}}$ is assumed to have a certain positive value, its condition will be clearly presented.

However, when $\boldsymbol{\phi}$ is taken to be an element of X, there is no guarantee that $\Omega(\boldsymbol{\phi})$ is a Lipschitz domain. In order to become a Lipschitz domain, $\boldsymbol{\phi}$ has to be an element of $C^{0,1}\left(D; \mathbb{R}^d\right)$. To repeat domain variations under the same conditions, the condition ($\Gamma_{p0} \cup \Gamma_{\eta 00} \cup \Gamma_{\eta 10} \cup \cdots \cup \Gamma_{\eta m0} \setminus \bar{\Omega}_{\mathrm{C0}}$ belongs to a class of piecewise $H^3 \cap C^{1,1}$) with respect to $\partial\Omega_0$ needs to be satisfied even for the perturbed boundary $\partial\Omega(\boldsymbol{\phi})$. Furthermore, to guarantee the existence of an optimum shape as shown in Sect. 9.7, the admissible set for $\boldsymbol{\phi}$ should be compact in X. Considering those conditions, one needs to take a linear space of $\boldsymbol{\phi}$, in which Fréchet derivatives of functions and functionals with respect to domain variation can be defined, as

$$Y = \begin{cases} X \cap H^2 \cap C^{0,1}\left(D; \mathbb{R}^d\right) & (\tilde{\Gamma}_0 = \emptyset \text{ or } \tilde{\Gamma}_0 \subset \bar{\Omega}_{\mathrm{C0}}) \\ X \cap H^3 \cap C^{1,1}\left(D; \mathbb{R}^d\right) & (\tilde{\Gamma}_0 \not\subset \bar{\Omega}_{\mathrm{C0}}) \end{cases}, \tag{9.1.2}$$

where $\tilde{\Gamma}_0$ denotes $\Gamma_{p0} \cup \Gamma_{\eta 00} \cup \Gamma_{\eta 10} \cup \cdots \cup \Gamma_{\eta m0}$ after Sect. 9.5 and a piecewise $H^3 \cap C^{1,1}$ class boundary before that section, and the admissible set of design variables as

$$\mathcal{D} = \left\{ \boldsymbol{\phi} \in Y \;\middle|\; \begin{cases} |\boldsymbol{\phi}|_{C^{0,1}(D;\mathbb{R}^d)} \leq \sigma, & \\ \|\boldsymbol{\phi}\|_{H^2 \cap C^{0,1}(D;\mathbb{R}^d)} \leq \beta & (\tilde{\Gamma}_0 = \emptyset \text{ or } \tilde{\Gamma}_0 \subset \bar{\Omega}_{\mathrm{C0}}), \\ \|\boldsymbol{\phi}\|_{H^3 \cap C^{1,1}(D;\mathbb{R}^d)} \leq \beta & (\tilde{\Gamma}_0 \not\subset \bar{\Omega}_{\mathrm{C0}}) \end{cases} \right\}. \tag{9.1.3}$$

Here, let $\|\boldsymbol{\phi}\|_{H^2 \cap C^{0,1}(D;\mathbb{R}^d)}$ be defined as $\max\left\{\|\boldsymbol{\phi}\|_{H^2(D;\mathbb{R}^d)}, \|\boldsymbol{\phi}\|_{C^{0,1}(D;\mathbb{R}^d)}\right\}$, and $\sigma \in (0,1)$ and β be positive constants. Norm $|\cdot|_{C^{0,1}(D;\mathbb{R}^d)} = \|\cdot\|_{C^{0,1}(D;\mathbb{R}^d)} - \|\cdot\|_{C(D;\mathbb{R}^d)}$ represents the Lipschitz constant (see Eq. (4.3.2)). The condition

$|\boldsymbol{\phi}|_{C^{0,1}(D;\mathbb{R}^d)} \le \sigma$ represents that $\boldsymbol{i}+\boldsymbol{\phi}$ and its inverse mapping $(\boldsymbol{i}+\boldsymbol{\phi})^{-1}$ become Lipschitz mappings (bi-Lipschitz mappings) on Ω_0 and $\Omega(\boldsymbol{\phi})$ respectively (cf. [59, Proposition 1.41, p. 23], [63]). Indeed, if this condition is satisfied, $\boldsymbol{i}+\boldsymbol{\phi}$ is a surjective Lipschitz mapping on Ω_0. Moreover, using

$$\begin{aligned}\|(\boldsymbol{i}+\boldsymbol{\phi})(\boldsymbol{x}_0)-(\boldsymbol{i}+\boldsymbol{\phi})(\boldsymbol{y}_0)\|_{\mathbb{R}^d} &\ge \|\boldsymbol{x}_0-\boldsymbol{y}_0\|_{\mathbb{R}^d}-\|\boldsymbol{\phi}(\boldsymbol{x}_0)-\boldsymbol{\phi}(\boldsymbol{y}_0)\|_{\mathbb{R}^d}\\ &\ge (1-\sigma)\|\boldsymbol{x}_0-\boldsymbol{y}_0\|_{\mathbb{R}^d}\end{aligned}$$

for arbitrary $\boldsymbol{x}_0, \boldsymbol{y}_0 \in \Omega_0$, and that when $(\boldsymbol{i}+\boldsymbol{\phi})(\boldsymbol{x}_0) = (\boldsymbol{i}+\boldsymbol{\phi})(\boldsymbol{y}_0)$, we obtain that $\boldsymbol{x}_0 = \boldsymbol{y}_0$, $\boldsymbol{i}+\boldsymbol{\phi}$ becomes injective. Then, there exists $(\boldsymbol{i}+\boldsymbol{\phi})^{-1}$ with which

$$\|\boldsymbol{x}_1-\boldsymbol{y}_1\|_{\mathbb{R}^d} \ge (1-\sigma)\left\|(\boldsymbol{i}+\boldsymbol{\phi})^{-1}(\boldsymbol{x}_1)-(\boldsymbol{i}+\boldsymbol{\phi})^{-1}(\boldsymbol{y}_1)\right\|_{\mathbb{R}^d}$$

holds for arbitrary $\boldsymbol{x}_1, \boldsymbol{y}_1 \in \Omega(\boldsymbol{\phi})$. This inequality shows that $(\boldsymbol{i}+\boldsymbol{\phi})^{-1}$ is a Lipschitz mapping on $\Omega(\boldsymbol{\phi})$. On the other hand, that $\mathcal{D}$ is a compact set in X is assured by the Rellich–Kondrachov compact embedding theorem (Theorem 4.4.15).

In future discussions, we assume that $\boldsymbol{\phi}$ is in the interior of $\mathcal{D}$ ($\boldsymbol{\phi} \in \mathcal{D}^\circ$), and a domain perturbed via domain variation $\boldsymbol{\varphi} \in Y$ such as in Fig. 9.4 will be denoted by $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$. In the condition satisfying $\boldsymbol{\varphi} \in \mathcal{D}$, the perturbed domain will be defined as

$$(\Omega(\boldsymbol{\phi}))(\boldsymbol{\varphi}) = ((\boldsymbol{i}+\boldsymbol{\varphi}) \circ (\boldsymbol{i}+\boldsymbol{\phi}))(\Omega_0),$$

where $\circ$ denotes the composite mapping. However, in future discussions, we will define the shape derivatives of function and functional as bounded linear operators of $\boldsymbol{\varphi} \in X$ (Definitions 9.1.1, 9.1.3, 9.1.4). In the Fréchet derivatives of functions and functionals with respect to $\boldsymbol{\varphi} \in X$, $(\Omega(\boldsymbol{\phi}))(\boldsymbol{\varphi})$ is linearized as $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$. Then, in this chapter, we assume that $\boldsymbol{\varphi}$ is originally an element of X, check that $\boldsymbol{\varphi}$ obtained by the proposed methods belongs to Y based on the problem setting and solution used, and confirm that $\boldsymbol{\phi}+\epsilon\boldsymbol{\varphi}$ (ϵ is a positive constant) belongs to $\mathcal{D}$.

9.1.3 Definitions of Shape Derivatives

For problems involving varying domains, the functions and integrals also vary. Here let us define their shape derivatives.

Let $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$ be given. For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $\boldsymbol{\varphi} \in Y$ be an arbitrary variation of $\Omega(\boldsymbol{\phi})$. When the domain varies from $\Omega(\boldsymbol{\phi})$ to $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$, the function defined on it is also assumed to change. In this case, we write the function at $\boldsymbol{\phi}$ as $u(\boldsymbol{\phi})$ and the value at a point $\boldsymbol{x}$ on the expanded domain D of $\Omega(\boldsymbol{\phi})$ as $u(\boldsymbol{\phi})(\boldsymbol{x})$. We use this notation to define the shape derivative of a function in the following way.

Definition 9.1.1 (Shape derivative of a function) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, consider a function $u : B \to L^2(D;\mathbb{R})$.

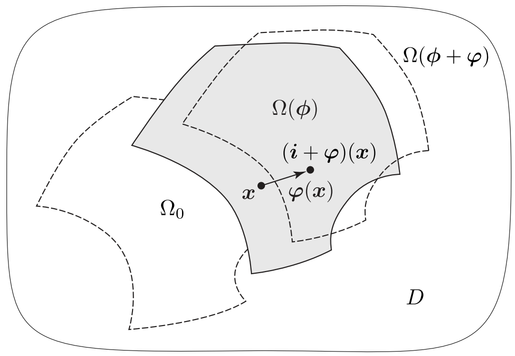


Fig. 9.4: Domain variation $\boldsymbol{\varphi} \in Y$ from $\Omega(\boldsymbol{\phi})$.

The value of $u(\boldsymbol{\phi})$ at $\boldsymbol{x} \in D$ will be written as $u(\boldsymbol{\phi})(\boldsymbol{x})$. If there exists a bounded linear operator $u'(\boldsymbol{\phi})[\,\cdot\,] : Y \to L^2(D;\mathbb{R})$ which satisfies

$$\lim_{\|\boldsymbol{\varphi}\|_Y \to 0} \frac{\|u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})) - u(\boldsymbol{\phi})(\boldsymbol{x}) + u'(\boldsymbol{\phi})[\boldsymbol{\varphi}](\boldsymbol{x})\|_{L^2(D;\mathbb{R})}}{\|\boldsymbol{\varphi}\|_X} = 0$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$, and $u'(\boldsymbol{\phi})[\,\cdot\,] : X \to L^2(D;\mathbb{R})$ is also a bounded linear operator, $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is referred to as the shape derivative at $\boldsymbol{\phi} \in B$ of u. When $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ exists for all $\boldsymbol{\phi} \in B$ and belongs to $C\left(B;\mathcal{L}\left(X;L^2(D;\mathbb{R})\right)\right)$, we write $u \in C^1_{\mathrm{S}'}\left(B;L^2(D;\mathbb{R})\right)$. □

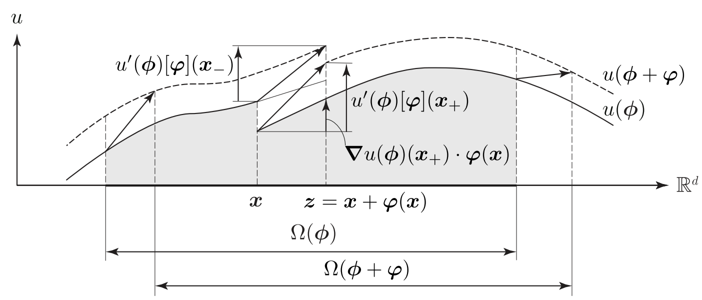
In Definition 9.1.1, we remark the following.

Remark 9.1.2 (Shape derivative) In Definition 9.1.1, at the same time that $u'(\boldsymbol{\phi})[\,\cdot\,]$ is a bounded linear operator on Y, it is assumed that it is also a bounded linear operator in X. When Y is compactly embedded in X, a bounded linear operator in X is automatically a bounded linear operator in Y (Practice **4.4**). The reason to define it in such a way is that, in general, $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$ varied by an arbitrary $\boldsymbol{\varphi} \in X$ is not well-defined. Also, by defining it such a way, stronger regularity is required than when the shape derivatives are defined as bounded linear operators in Y. In some types of definitions regarding shape derivatives shown later, the condition to be a bounded linear operator in X will be needed for the same reason. □

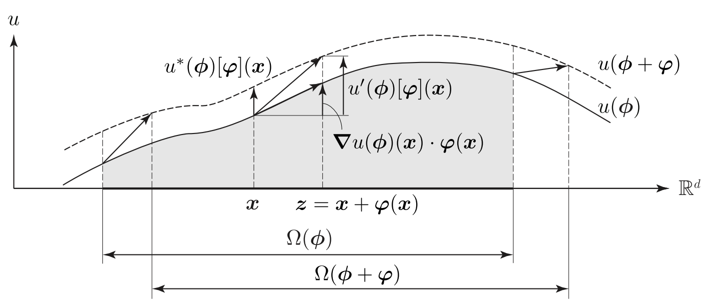
In continuum mechanics, $u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))$ in Definition 9.1.1 is called the Lagrangian description of $u(\boldsymbol{\phi})(\boldsymbol{x})$ and $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is called the material derivative.

Figure 9.5 (a) shows $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$. Here, even if $u \in L^2(D;\mathbb{R})$ is a discontinuous function, if $\boldsymbol{\varphi}$ is a continuous function, it is apparent that $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ can be defined.

Next, let us think about the derivative of $u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x})$ when a point $\boldsymbol{x}$ is fixed on the expanded domain D of $\Omega(\boldsymbol{\phi})$ in the case of the perturbed domain.



(a) When $u(\boldsymbol{\phi})$ is a discontinuous function.



(b) When $u(\boldsymbol{\phi})$ is a continuous function.

Fig. 9.5: The function $u(\boldsymbol{\phi})$ varying with domain.

The Fréchet derivative of u with respect to an arbitrary variation $\boldsymbol{\varphi}$ is called the partial shape derivative of a function and is defined as follows.

Definition 9.1.3 (Partial shape derivative of a function) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, consider a function $u : B \to C^{0,1}(D;\mathbb{R})$. The value of $u(\boldsymbol{\phi})$ at $\boldsymbol{x} \in D$ will be written as $u(\boldsymbol{\phi})(\boldsymbol{x})$. With respect to an arbitrary $\boldsymbol{\varphi} \in Y$, when a bounded linear operator $u^*(\boldsymbol{\phi})[\,\cdot\,] : Y \to C^{0,1}(D;\mathbb{R})$ which satisfies that

$$\lim_{\|\boldsymbol{\varphi}\|_Y \to 0} \frac{\|u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}) - u(\boldsymbol{\phi})(\boldsymbol{x}) + u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}](\boldsymbol{x})\|_{C^{0,1}(D;\mathbb{R})}}{\|\boldsymbol{\varphi}\|_X} = 0$$

for almost every $\boldsymbol{x} \in D$ exists and when $u^*(\boldsymbol{\phi})[\,\cdot\,] : X \to C^{0,1}(D;\mathbb{R})$ is also a bounded linear operator, $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is called the partial shape derivative of u at $\boldsymbol{\phi} \in B$. When $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ exists for all $\boldsymbol{\phi} \in B$ and belongs to $C\left(B;\mathcal{L}\left(X;C^{0,1}(D;\mathbb{R})\right)\right)$, we write $u \in C^1_{\mathrm{S}^*}\left(B;C^{0,1}(D;\mathbb{R})\right)$. □

In Definition 9.1.3, $u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x})$ is called the Euler description of $u(\boldsymbol{\phi})(\boldsymbol{x})$ in continuum mechanics, and $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is called the spatial derivative.

Figure 9.5 (b) shows $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$. Here, it should be noted that since $u \in H^1(D;\mathbb{R})$ is a continuous function, the definition of $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is valid. In reality, looking at Fig. 9.5 (a), if u is a discontinuous function, $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ is not defined for $\boldsymbol{x}$ such that a discontinuity of u crosses in the domain variation due to $\boldsymbol{\varphi}$.

Moreover, when $u \in C^1_{\mathrm{S}^*}\left(B; C^{0,1}(D;\mathbb{R})\right)$,

$$
\begin{aligned}
&u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})) \\
&\quad = u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}) + \boldsymbol{\nabla} u(\boldsymbol{\phi}) \cdot \boldsymbol{\varphi} + o\left(\|\boldsymbol{\varphi}(\boldsymbol{x})\|_X\right) \\
&\quad = u(\boldsymbol{\phi})(\boldsymbol{x}) + u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}](\boldsymbol{x}) + \boldsymbol{\nabla} u(\boldsymbol{\phi}) \cdot \boldsymbol{\varphi} + o\left(\|\boldsymbol{\varphi}(\boldsymbol{x})\|_X\right)
\end{aligned}
$$

holds. Then, we have

$$
u'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}] + \boldsymbol{\nabla} u(\boldsymbol{\phi}) \cdot \boldsymbol{\varphi}. \tag{9.1.4}
$$

with respect to arbitrary $\boldsymbol{\varphi} \in X$. Here $(\partial(\,\cdot\,)/x_1, \ldots, \partial(\,\cdot\,)/x_d)^\top$ is denoted as $\boldsymbol{\nabla}(\,\cdot\,)$ with respect to $\boldsymbol{x} = (x_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d$. The left-hand side of Eq. (9.1.4) becomes $u'(\boldsymbol{\phi})[\,\cdot\,] : X \to L^2(D;\mathbb{R})$. Then, $u \in C^1_{\mathrm{S}'}\left(B; L^2(D;\mathbb{R})\right)$ holds.

Furthermore, the shape derivative of a functional defined on a perturbed domain will be defined as follows. In this chapter, we use the notation $\boldsymbol{\nabla}_z = (\partial(\,\cdot\,)/z_1, \ldots, \partial(\,\cdot\,)/z_d)^\top$ with respect to $\boldsymbol{z} = (\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x})$. Moreover, let $\boldsymbol{\nu}(\boldsymbol{\phi})$ be the outward unit normal defined on the boundary $\partial\Omega(\boldsymbol{\phi})$, $\partial_\nu(\,\cdot\,) = \boldsymbol{\nu}(\boldsymbol{\phi}) \cdot \boldsymbol{\nabla}(\,\cdot\,)$, $\boldsymbol{\mu} = (\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{\nu})$ be the outward unit normal on $\partial\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$, and $\partial_\mu(\,\cdot\,) = \boldsymbol{\mu} \cdot \boldsymbol{\nabla}_z(\,\cdot\,)$. Furthermore, the dual space of X is denoted by X'.

Definition 9.1.4 (Shape derivative of a functional) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u : B \to \mathcal{U} = H^3 \cap C^{1,1}(D;\mathbb{R})$ be given, and $h_0 \in C^1\left(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}\right)$ and $h_1 \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ be defined for $(u, \boldsymbol{\nabla} u, \partial_\nu u) \in \mathcal{U} \times \mathcal{G} \times \mathcal{G}_{\Gamma(\boldsymbol{\phi})}$ ($\mathcal{G} = \{\boldsymbol{\nabla} u \mid u \in \mathcal{U}\}$, $\mathcal{G}_{\Gamma(\boldsymbol{\phi})} = \left\{ \partial_\nu u|_{\Gamma(\boldsymbol{\phi})} \;\middle|\; u \in \mathcal{U}\right\}$) ($\Gamma(\boldsymbol{\phi}) \subseteq \partial\Omega(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$) as

$$
\begin{aligned}
&h_0(u, \boldsymbol{\nabla} u), h_{0u}(u, \boldsymbol{\nabla} u) \in L^2(D;\mathbb{R}), \quad h_{0\boldsymbol{\nabla} u}(u, \boldsymbol{\nabla} u) \in L^2\left(D;\mathbb{R}^d\right), \\
&h_1(u, \partial_\nu u), h_{1u}(u, \partial_\nu u), h_{1\partial_\nu u}(u, \partial_\nu u) \in H^1(D;\mathbb{R}).
\end{aligned}
$$

With respect to an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$
\begin{aligned}
&f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi}), \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi}), \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi})) \\
&\quad = \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} h_0(u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}), \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}))\,\mathrm{d}z \\
&\qquad + \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} h_1(u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}), \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}))\,\mathrm{d}\zeta.
\end{aligned} \tag{9.1.5}
$$

Here, $\Gamma(\boldsymbol{\phi})$ is taken to be the partial set of $\partial\Omega(\boldsymbol{\phi})$ (allowing $\Gamma(\boldsymbol{\phi}) = \partial\Omega(\boldsymbol{\phi})$). Moreover, $\mathrm{d}z$ and $\mathrm{d}\zeta$ represent infinitesimal measures used in domain and boundary integrals over $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$. In this case, if a bounded linear functional $f'(\boldsymbol{\phi}, u(\boldsymbol{\phi}), \boldsymbol{\nabla} u(\boldsymbol{\phi}), \partial_\nu u(\boldsymbol{\phi}))[\,\cdot\,] : Y \to \mathbb{R}$ satisfies

$$
f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi}), \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi}), \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi}))
$$

$$
\begin{aligned}
&= f\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right) \\
&\quad + f'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}\right] + o\left(\|\boldsymbol{\varphi}\|_X\right)
\end{aligned}
$$

and $f'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu u\left(\boldsymbol{\phi}\right)\right)\left[\,\cdot\,\right] : X \to \mathbb{R}$ is also a bounded linear functional, in other words, there exists a $\boldsymbol{g}\left(\boldsymbol{\phi}\right) \in X'$ such that $f'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu u\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}\right] = \left\langle \boldsymbol{g}\left(\boldsymbol{\phi}\right), \boldsymbol{\varphi}\right\rangle$, f is said to be shape differentiable at $\boldsymbol{\phi}$, and $\boldsymbol{g}\left(\boldsymbol{\phi}\right)$ is called the shape gradient of f. Moreover, when there exists $f'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}\right]$ for all $\boldsymbol{\phi} \in B$ and those are in $C\left(B; \mathcal{L}\left(X; \mathbb{R}\right)\right)$, it is expressed as $f \in C^1_{\mathrm{S}}\left(B; \mathbb{R}\right)$.

Furthermore, if with respect to an arbitrary $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2 \in Y$, a bounded bilinear functional $f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] : Y \times Y \to \mathbb{R}$ satisfies

$$
\begin{aligned}
&\left\langle \boldsymbol{g}\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_2\right), \boldsymbol{\varphi}_1 \circ \left(\boldsymbol{i} + \boldsymbol{\varphi}_2\right)^{-1} \right\rangle \\
&\quad = \left\langle \boldsymbol{g}\left(\boldsymbol{\phi}\right), \boldsymbol{\varphi}_1\right\rangle + f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \\
&\qquad + o\left(\|\boldsymbol{\varphi}_1\|_X, \|\boldsymbol{\varphi}_2\|_X\right)
\end{aligned}
$$

and $f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] = h\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] : X \times X \to \mathbb{R}$ is also a bounded bilinear functional, f is said to be second-order shape differentiable, and $h\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right]$ is called the second-order shape derivative or shape Hessian of f. In addition, with respect to all $\boldsymbol{\phi} \in B$, if there exists a second-order shape derivative and $f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \in C\left(B; \mathcal{L}\left(X; \mathcal{L}\left(X; \mathbb{R}\right)\right)\right)$, then we write that $f \in C^2_{\mathrm{S}}\left(B; \mathbb{R}\right)$. □

According to the definition of the second-order shape derivative, $f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right]$ can be divided into two parts as [86]

$$
\begin{aligned}
&f''\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \\
&\quad = \left(f'\right)'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] + \left\langle \boldsymbol{g}\left(\boldsymbol{\phi}\right), \boldsymbol{t}\left(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right)\right\rangle .
\end{aligned}
\tag{9.1.6}
$$

Here, the summands on the right side of the above equation are respectively given as

$$
\begin{aligned}
&\left(f'\right)'\left(\boldsymbol{\phi}, u\left(\boldsymbol{\phi}\right), \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right), \partial_\nu\left(\boldsymbol{\phi}\right)\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \\
&\quad = \lim_{\|\boldsymbol{\varphi}_2\|_X \to 0} \frac{1}{\|\boldsymbol{\varphi}_2\|_X} \left(\left\langle \boldsymbol{g}\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_2\right), \boldsymbol{\varphi}_1\right\rangle - \left\langle \boldsymbol{g}\left(\boldsymbol{\phi}\right), \boldsymbol{\varphi}_1\right\rangle\right) \qquad (9.1.7) \\
&\left\langle \boldsymbol{g}\left(\boldsymbol{\phi}\right), \boldsymbol{t}\left(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right)\right\rangle \\
&\quad = \lim_{\|\boldsymbol{\varphi}_2\|_X \to 0} \frac{1}{\|\boldsymbol{\varphi}_2\|_X} \left\langle \boldsymbol{g}\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_2\right), \boldsymbol{\varphi}_1 \circ \left(\boldsymbol{i} + \boldsymbol{\varphi}_2\right)^{-1} - \boldsymbol{\varphi}_1 \right\rangle . \qquad (9.1.8)
\end{aligned}
$$

Equation (9.1.7) represents the derivative of $\boldsymbol{g}\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_2\right)$ by the variation of $\boldsymbol{\varphi}_2$, and commonly appears in calculations of the second-order derivative of a functional in optimization problems. On the other hand, Eq. (9.1.8) is a specific term in shape optimization problems to correct the variation of $\boldsymbol{\varphi}_1$ by $\boldsymbol{\varphi}_2$. The term $\boldsymbol{\varphi}_1 \circ (\boldsymbol{i} + \boldsymbol{\varphi}_2)^{-1} - \boldsymbol{\varphi}_1$ in Eq. (9.1.8) represents the variation of $\boldsymbol{\varphi}_1$ by the inverse mapping of $\boldsymbol{i} + \boldsymbol{\varphi}_2$. When only this item is calculated, we have

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) &= \lim_{\|\boldsymbol{\varphi}_2\|_X \to 0} \frac{1}{\|\boldsymbol{\varphi}_2\|_X} \left(\boldsymbol{\varphi}_1 \circ (\boldsymbol{i} + \boldsymbol{\varphi}_2)^{-1} - \boldsymbol{\varphi}_1 \right) \\ &= -\left(\boldsymbol{\varphi}_2 \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^\top = -\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^\top \boldsymbol{\varphi}_2. \end{aligned} \tag{9.1.9}$$

The above relation can be obtained in the following way. We notice that the transfer vector of the coordinate system linearizing the inverse mapping of $\boldsymbol{i}+\boldsymbol{\varphi}_2$ becomes $-\boldsymbol{\varphi}_2$, and the varying function is $\boldsymbol{\varphi}_1$. In Eq. (9.1.4), replacing u by $\boldsymbol{\varphi}_1^\top$ and putting $\boldsymbol{\varphi}_1^* (\boldsymbol{\phi}) [-\boldsymbol{\varphi}_2] = \mathbf{0}_{\mathbb{R}^d}$, $\boldsymbol{\varphi}_1' (\boldsymbol{\phi}) (\boldsymbol{\phi}) [-\boldsymbol{\varphi}_2]$ gives the right-hand side of Eq. (9.1.9).

It should be noted that Eq. (9.1.9) holds if $\boldsymbol{\nabla} \boldsymbol{\varphi}^\top$ or $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$ is not used in the shape derivative $\langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{\varphi} \rangle$. In the cases that $\boldsymbol{\nabla} \boldsymbol{\varphi}^\top$ or $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$ is used, those calculations for $\boldsymbol{\nabla} \boldsymbol{\varphi}^\top$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$ will be given in Eq. (9.3.11). In that situation, the inverse mapping of $\boldsymbol{i} + \boldsymbol{\varphi}_2$ is applied to $\boldsymbol{\nabla}$ too.

9.2 Shape Derivatives of Jacobi Determinants

Since the domain variation and shape derivatives of functions and functionals have been defined, let us use them to find the shape derivative of the Jacobi determinant (Jacobian) and the inverse matrix of Jacobi matrix (Jacobi inverse matrix) with respect to domain variation $\boldsymbol{\varphi} \in Y$. These are used when seeking the formulae for the shape derivatives of functionals.

Fix $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$. For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, consider an arbitrary domain variation $\boldsymbol{\varphi} \in Y$ from $\Omega(\boldsymbol{\phi})$. In this case, the Jacobi matrix and Jacobi determinant (Jacobian) with respect to the mapping $\boldsymbol{i} + \boldsymbol{\varphi}$ are expressed as

$$\boldsymbol{F}(\boldsymbol{\varphi}) = \boldsymbol{I} + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right)^\top, \tag{9.2.1}$$

$$\omega(\boldsymbol{\varphi}) = \det \boldsymbol{F}(\boldsymbol{\varphi}), \tag{9.2.2}$$

where $\boldsymbol{I}$ represents the unit matrix.[3] In this case, $\omega(\boldsymbol{\varphi})$ becomes a function which gives $\mathrm{d}z = \omega(\boldsymbol{\varphi})\,\mathrm{d}x$ with respect to an infinitesimal measure $\mathrm{d}z$ on $\Omega(\boldsymbol{\phi} + \boldsymbol{\varphi})$ corresponding to the infinitesimal measure $\mathrm{d}x$ on $\Omega(\boldsymbol{\phi})$. Here, taking up two types of Jacobi determinants defined on the domain and the boundary, let us look at their shape derivatives.

9.2.1 Shape Derivatives of Domain Jacobi Determinant and Domain Jacobi Inverse Matrix

Firstly, the shape derivative of $\omega(\boldsymbol{\varphi})$ defined on Eq. (9.2.2) at $\boldsymbol{\varphi}_0 = \mathbf{0}_{\mathbb{R}^d}$ is given in the following way.

[3] Although it is usually written as $\boldsymbol{F}(\boldsymbol{i} + \boldsymbol{\varphi})$, following the notation for a deformation gradient tensor in elasticity theory, $\boldsymbol{F}(\boldsymbol{\varphi})$ is used.

Proposition 9.2.1 (Derivative of domain Jacobi determinant) For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, we have

$$\omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] = \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Moreover, $\omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]$ also belongs to $C\left(B; \mathcal{L}\left(X; L^2(D;\mathbb{R})\right)\right)$. □

Proof For $\boldsymbol{x} \in D$, we have

$$\begin{aligned}\omega(\boldsymbol{\varphi}) &= \det\left(\boldsymbol{I} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)^\top\right) = \det\begin{pmatrix} 1+\varphi_{1,1} & \cdots & \varphi_{1,d} \\ \vdots & \ddots & \vdots \\ \varphi_{d,1} & \cdots & 1+\varphi_{d,d}\end{pmatrix} \\ &= 1 + \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + \sum_{(i,j)\in\{1,\ldots,d\}^2} o\left(\|\varphi_{i,j}\|_{L^2(D;\mathbb{R})}\right).\end{aligned}$$

□

Moreover, the shape derivative of the Jacobi inverse matrix $\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})$ at $\boldsymbol{\varphi}_0 = \mathbf{0}_{\mathbb{R}^d}$ is as follows.

Proposition 9.2.2 (Derivative of domain Jacobi inverse matrix) For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, we have

$$\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] = -\boldsymbol{\nabla}\boldsymbol{\varphi}^\top$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Moreover, $\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]$ also belongs to $C\left(B; \mathcal{L}\left(X; L^2\left(D;\mathbb{R}^{d\times d}\right)\right)\right)$. □

Proof For $\boldsymbol{x} \in D$, using the differentiability of the inverse map $(\boldsymbol{i}+\boldsymbol{\varphi})^{-1}$, we have

$$\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\left(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right) = \boldsymbol{I}.$$

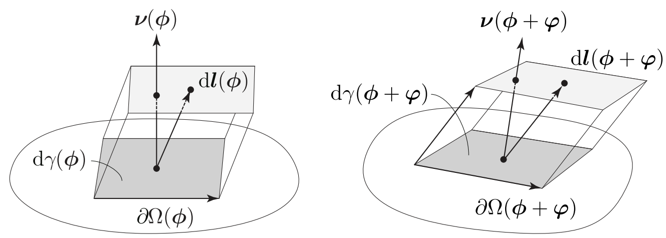
Taking its shape derivative with respect to $\boldsymbol{\varphi}$ at $\boldsymbol{\phi}$, we get

$$\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] + \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0)\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right) = \mathbf{0}_{\mathbb{R}^{d\times d}}.$$

Since $\boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0) = \boldsymbol{I}$, the proposition follows. □

9.2.2 Shape Derivatives of Boundary Jacobi Determinant and the Normal

Next let us obtain the formulae for the shape derivatives relating to the Jacobi determinant on a boundary. In shape optimization problems of domain variation type, boundary integrals appear in the Lagrange functions of state determination problems and cost functions. Hence, when obtaining the shape derivatives of such boundary integrals, the shape derivatives of the boundary Jacobi determinant and the normal are needed.



(a) Before variation (b) After variation

Fig. 9.6: Infinitesimal measures $\mathrm{d}\gamma\left(\boldsymbol{\phi}\right)$ and $\mathrm{d}\gamma\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)$.

Let us represent an infinitesimal measure on $\partial\Omega\left(\boldsymbol{\phi}\right)$ by $\mathrm{d}\gamma\left(\boldsymbol{\phi}\right)$ and an outward unit normal by $\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)$. Furthermore, the normal on a Lipschitz boundary is defined by the normal with respect to the graph defining the boundary as a graph in a local coordinate system around the boundary, and is assumed to be in $L^\infty\left(\partial\Omega\left(\boldsymbol{\phi}\right);\mathbb{R}^d\right)$ [27, 62]. Here, we assume that $\partial\Omega\left(\boldsymbol{\phi}\right)$ is piecewise $H^2\cap C^{0,1}$ and $\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\in H^{1/2}\cap L^\infty\left(\partial\Omega\left(\boldsymbol{\phi}\right);\mathbb{R}^d\right)$.

In this case, with respect to arbitrary $\boldsymbol{\varphi}\in Y$, the relation

$$\varpi\left(\boldsymbol{\varphi}\right)=\frac{\mathrm{d}\gamma\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)}{\mathrm{d}\gamma\left(\boldsymbol{\phi}\right)}=\omega\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)\cdot\Big(\boldsymbol{F}^{-\top}\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\Big) \tag{9.2.3}$$

holds. Here, $\varpi\left(\boldsymbol{\varphi}\right)$ denotes the Jacobi determinant for the boundary. This relationship can be obtained from the following proposition.

Proposition 9.2.3 (Nanson formula) For $\boldsymbol{\phi}$ in a neighborhood $B\subset Y$ of $\boldsymbol{\phi}_0\in\mathcal{D}^\circ$, let $\partial\Omega\left(\boldsymbol{\phi}\right)$ be piecewise $H^2\cap C^{0,1}$. For an arbitrary $\boldsymbol{\varphi}\in Y$, the equation

$$\boldsymbol{\nu}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)\mathrm{d}\gamma\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)=\omega\left(\boldsymbol{\varphi}\right)\boldsymbol{F}^{-\top}\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\,\mathrm{d}\gamma\left(\boldsymbol{\phi}\right) \tag{9.2.4}$$

holds. Moreover, $\omega\left(\boldsymbol{\varphi}\right)\boldsymbol{F}^{-\top}\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)$ belongs to $L^\infty\left(\partial\Omega\left(\boldsymbol{\phi}\right);\mathbb{R}\right)$. □

Proof Let $\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)\in\mathbb{R}^d$ be an arbitrary vector satisfying $\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\cdot\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)>0$ on $\mathrm{d}\gamma\left(\boldsymbol{\phi}\right)$ and $\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)$ a vector obtained through the mapping $\boldsymbol{i}+\boldsymbol{\varphi}$. In this case, the relation

$$\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)\cdot\boldsymbol{\nu}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)\mathrm{d}\gamma\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)=\omega\left(\boldsymbol{\varphi}\right)\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)\cdot\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\mathrm{d}\gamma\left(\boldsymbol{\phi}\right)$$

holds with respect to the volume of a parallelepiped shown in Fig. 9.6. Here, if $\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)=\boldsymbol{F}\left(\boldsymbol{\varphi}\right)\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)$ is substituted into the equation above, one obtains

$$\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)\cdot\Big(\boldsymbol{F}^{\top}\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)\Big)\mathrm{d}\gamma\left(\boldsymbol{\phi}+\boldsymbol{\varphi}\right)=\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)\cdot\left(\omega\left(\boldsymbol{\varphi}\right)\boldsymbol{\nu}\left(\boldsymbol{\phi}\right)\right)\mathrm{d}\gamma\left(\boldsymbol{\phi}\right).$$

Since $\mathrm{d}\boldsymbol{l}\left(\boldsymbol{\phi}\right)$ is arbitrary, Eq. (9.2.4) follows. □

Equation (9.2.4) can be obtained by using the Piola transformation giving the correspondence between second-order tensor functions defined over the deformed and initial domains [24, Theorem 1.7-1]. The Piola transformation $\boldsymbol{A}(\boldsymbol{\varphi})$ of an arbitrary second-order tensor-valued function $\boldsymbol{A} \in C^1\left(D;\mathbb{R}^{d\times d}\right)$ with respect to an arbitrary $\boldsymbol{\varphi} \in Y$ is defined as

$$\boldsymbol{A} = \omega(\boldsymbol{\varphi})\, \boldsymbol{A}(\boldsymbol{\varphi})\, \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}),$$

where $\omega(\boldsymbol{\varphi})\, \boldsymbol{F}^{-\top}(\boldsymbol{\varphi})$ is the cofactor matrix of the Jacobi matrix $\boldsymbol{F}(\boldsymbol{\varphi})$. Letting $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{\varphi}(\boldsymbol{x}) = (\boldsymbol{i} + \boldsymbol{\varphi})(\boldsymbol{x})$ on D be an admissible perturbation $\boldsymbol{\varphi}$ of a point $\boldsymbol{x}$ from $\Omega(\boldsymbol{\phi})$ after deformation, and $\boldsymbol{\nabla}_z$ represents $\partial(\,\cdot\,)/\partial \boldsymbol{z}$, we have

$$\begin{aligned}\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}\cdot \boldsymbol{A}\,\mathrm{d}x &= \int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{A}\boldsymbol{\nu}(\boldsymbol{\phi})\,\mathrm{d}\gamma(\boldsymbol{\phi})\\ &= \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}_z\cdot \boldsymbol{A}(\boldsymbol{\varphi})\,\omega(\boldsymbol{\varphi})\,\mathrm{d}x\\ &= \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} \boldsymbol{\nabla}_z\cdot \boldsymbol{A}(\boldsymbol{\varphi})\,\mathrm{d}z\\ &= \int_{\partial\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} \boldsymbol{A}(\boldsymbol{\varphi})\,\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})\,\mathrm{d}\gamma(\boldsymbol{\phi}+\boldsymbol{\varphi}).\end{aligned}$$

Applying the Piola transformation to the above equality with respect to the boundary integral equation and putting $\boldsymbol{A}(\boldsymbol{\varphi}) = \boldsymbol{I}$, we obtain Eq. (9.2.4).

Taking the inner product of both sides of Eq. (9.2.4) and $\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})$ leads to Eq. (9.2.3). Moreover, from the fact that $\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})$ is a unit vector in the direction of $\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\,\boldsymbol{\nu}(\boldsymbol{\phi})$, then by Eq. (9.2.4), the following holds:

$$\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi}) = \frac{\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\,\boldsymbol{\nu}(\boldsymbol{\phi})}{\left\|\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\,\boldsymbol{\nu}(\boldsymbol{\phi})\right\|_{\mathbb{R}^d}}. \tag{9.2.5}$$

Based on these relationships, the shape derivative of $\varpi(\boldsymbol{\varphi})$ of Eq. (9.2.3) can be obtained in Proposition 9.2.4. In the sequel, the tangent (Definition A.5.3) on $\partial\Omega(\boldsymbol{\phi})$ will be written as $\boldsymbol{\tau}_1(\boldsymbol{\phi}), \ldots, \boldsymbol{\tau}_{d-1}(\boldsymbol{\phi})$. On the Lipschitz boundary, the tangent is defined as a tangent on the graph defining the boundary as a graph of the local coordinate system near the boundary, in a similar way to the normal, and is assumed to be included in $L^\infty\left(\partial\Omega(\boldsymbol{\phi});\mathbb{R}^d\right)$. Moreover, $d-1$ times of the mean curvature (Definition A.5.5) (sum of principle curvatures) is given by $\kappa(\boldsymbol{\phi}) = \boldsymbol{\nabla}\cdot\boldsymbol{\nu}(\boldsymbol{\phi})$ on a piecewise $C^{1,1}$ class boundary in a similar way to the derivative of the normal, and is assumed to be included in $L^\infty(\partial\Omega(\boldsymbol{\phi});\mathbb{R})$. Here, we assume that $\partial\Omega(\boldsymbol{\phi})$ is piecewise $H^3\cap C^{1,1}$ and $\kappa(\boldsymbol{\phi}) \in H^{1/2}\cap L^\infty(\partial\Omega(\boldsymbol{\phi});\mathbb{R})$. Moreover, $\boldsymbol{\nabla}_\tau(\,\cdot\,) = (\boldsymbol{\tau}_j(\boldsymbol{\phi})\cdot\boldsymbol{\nabla})_{j\in\{1,\ldots,d-1\}}(\,\cdot\,) \in \mathbb{R}^{d-1}$ and $\boldsymbol{\varphi}_\tau = (\boldsymbol{\tau}_j(\boldsymbol{\phi})\cdot\boldsymbol{\varphi})_{j\in\{1,\ldots,d-1\}} \in \mathbb{R}^{d-1}$. From now on, $\boldsymbol{\nu}(\boldsymbol{\phi})$, $\boldsymbol{\tau}_1(\boldsymbol{\phi})$, ... , $\boldsymbol{\tau}_{d-1}(\boldsymbol{\phi})$ and $\kappa(\boldsymbol{\phi})$ are to be written simply as $\boldsymbol{\nu}$, $\boldsymbol{\tau}_1$, ... , $\boldsymbol{\tau}_{d-1}$ and κ.

Proposition 9.2.4 (Derivative of boundary Jacobi determinant) For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $\partial\Omega(\boldsymbol{\phi})$ be piecewise $H^2\cap C^{0,1}$. In this case, we have the identity

$$\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] = (\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau = \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} - \boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nu}\right) \tag{9.2.6}$$

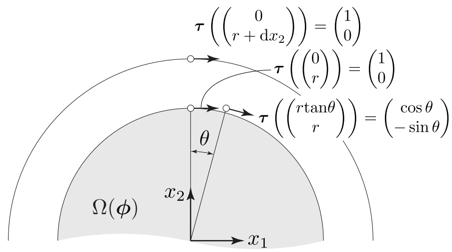


Fig. 9.7: Distribution of a tangent near a circle.

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$, where $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_{\tau}$ is defined by the right-hand side of Eq. (9.2.6). Furthermore, if $\partial\Omega(\boldsymbol{\phi})$ is a piecewise $H^3 \cap C^{1,1}$ boundary, one has

$$\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] = \kappa\boldsymbol{\nu}\cdot\boldsymbol{\varphi} + \boldsymbol{\nabla}_{\tau}\cdot\boldsymbol{\varphi}_{\tau}. \tag{9.2.7}$$

Moreover, $\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]$ belongs to $C(B; \mathcal{L}(Y; L^{\infty}(\partial\Omega(\boldsymbol{\phi}); \mathbb{R})))$. □

Proof From Eq. (9.2.3) and Eq. (9.2.5), we have

$$\varpi(\boldsymbol{\varphi}) = \omega(\boldsymbol{\varphi}) \left\| \boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\boldsymbol{\nu} \right\|_{\mathbb{R}^d}.$$

Eq. (9.2.6) can be obtained from Propositions 9.2.1 and 9.2.2 as

$$\begin{aligned}\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] &= \omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] \left\| \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0)\boldsymbol{\nu} \right\|_{\mathbb{R}^d} \\ &\quad + \omega(\boldsymbol{\varphi}_0)\left(\boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0)\boldsymbol{\nu}\right)\cdot\left(\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]\boldsymbol{\nu}\right) \Big/ \left\| \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0)\boldsymbol{\nu} \right\|_{\mathbb{R}^d} \\ &= \boldsymbol{\nabla}\cdot\boldsymbol{\varphi} - \boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nu}\right).\end{aligned}$$

Furthermore, if its boundary is piecewise $H^3 \cap C^{1,1}$, we can define $\kappa = \boldsymbol{\nabla}\cdot\boldsymbol{\nu}$ almost everywhere and write

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} &= \boldsymbol{\nabla}\cdot\left\{(\boldsymbol{\nu}\cdot\boldsymbol{\varphi})\boldsymbol{\nu} + \sum_{j\in\{1,\ldots,d-1\}} (\boldsymbol{\tau}_j\cdot\boldsymbol{\varphi})\boldsymbol{\tau}_j\right\} \\ &= \partial_{\nu}(\boldsymbol{\nu}\cdot\boldsymbol{\varphi}) + \kappa(\boldsymbol{\nu}\cdot\boldsymbol{\varphi}) + \boldsymbol{\nabla}_{\tau}\cdot\boldsymbol{\varphi}_{\tau}. \end{aligned} \tag{9.2.8}$$

Here, the notation $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_1 = 0, \ldots, \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{d-1} = 0$ is used. This is because when $\Omega(\boldsymbol{\phi})$ is a circle (a two-dimensional domain) with radius r, such as in Fig. 9.7, one has

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_1 = \boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \frac{\partial\tau_1}{\partial x_1} + \frac{\partial\tau_2}{\partial x_2} = \lim_{\theta\to 0}\frac{\cos\theta - 1}{r\tan\theta} = 0$$

at $\boldsymbol{x} = (0, r)^{\top}$. A similar relationship holds even when $\Omega(\boldsymbol{\phi})$ is a three-dimensional domain.

Moreover,

$$\boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nu}\right) = \boldsymbol{\nu}\cdot\left[\boldsymbol{\nabla}\left\{(\boldsymbol{\nu}\cdot\boldsymbol{\varphi})\boldsymbol{\nu} + \sum_{j\in\{1,\ldots,d-1\}}(\boldsymbol{\tau}_j\cdot\boldsymbol{\varphi})\boldsymbol{\tau}_j\right\}^{\top}\boldsymbol{\nu}\right]$$

$$= \partial_\nu (\boldsymbol{\nu} \cdot \boldsymbol{\varphi}) \tag{9.2.9}$$

holds. Here, the following equalities are used:

$$\boldsymbol{\nabla} (\boldsymbol{\nu} \cdot \boldsymbol{\varphi}) \boldsymbol{\nu}^\top \boldsymbol{\nu} = \boldsymbol{\nabla} (\boldsymbol{\nu} \cdot \boldsymbol{\varphi}), \quad \boldsymbol{\nabla} \boldsymbol{\nu}^\top \boldsymbol{\nu} = \mathbf{0}_{\mathbb{R}^d},$$

$$\boldsymbol{\nabla} (\boldsymbol{\tau}_j \cdot \boldsymbol{\varphi}) \boldsymbol{\tau}_j^\top \boldsymbol{\nu} = \mathbf{0}_{\mathbb{R}^d}, \quad \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\tau}_j^\top \boldsymbol{\nu} \right) = 0.$$

The fact that $\boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\tau}_1^\top \boldsymbol{\nu} \right) = 0, \ldots, \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\tau}_{d-1}^\top \boldsymbol{\nu} \right) = 0$ holds follows from the fact that when $\Omega (\boldsymbol{\phi})$ is a circle with radius r, such as that in Fig. 9.7, at $\boldsymbol{x} = (0, r)^\top$, we have

$$\begin{aligned} \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\tau}_j^\top \boldsymbol{\nu} \right) &= \begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix} \begin{pmatrix} \partial \tau_1 / \partial x_1 & \partial \tau_2 / \partial x_1 \\ \partial \tau_1 / \partial x_2 & \partial \tau_2 / \partial x_2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \end{aligned}$$

Even in a case when $\Omega (\boldsymbol{\phi})$ is a three-dimensional domain, a similar relationship holds.

Here if Eq. (9.2.8) and Eq. (9.2.9) are substituted into Eq. (9.2.6), then one obtains Eq. (9.2.7). □

Moreover, the following formula can be obtained with respect to the shape derivative of a normal.

Proposition 9.2.5 (Derivative of the normal) For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $\partial \Omega (\boldsymbol{\phi})$ be piecewise $H^2 \cap C^{0,1}$. In this case, we have the identity

$$\boldsymbol{\nu}' (\boldsymbol{\phi}) [\boldsymbol{\varphi}] = - \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} + \left\{ \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \boldsymbol{\nu} \right) \right\} \boldsymbol{\nu}$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Moreover, $\boldsymbol{\nu}' (\boldsymbol{\phi}) [\boldsymbol{\varphi}]$ belongs to $C \left(B; \mathcal{L} \left(Y; L^\infty \left(\partial \Omega (\boldsymbol{\phi}); \mathbb{R}^d \right) \right) \right)$. □

Proof The outward unit normal on $\Omega (\boldsymbol{\phi} + \boldsymbol{\varphi})$ can be expressed as Eq. (9.2.5); that is

$$\boldsymbol{\nu} (\boldsymbol{\phi} + \boldsymbol{\varphi}) = \frac{\boldsymbol{F}^{-\top} (\boldsymbol{\varphi}) \boldsymbol{\nu}}{\left\| \boldsymbol{F}^{-\top} (\boldsymbol{\varphi}) \boldsymbol{\nu} \right\|_{\mathbb{R}^d}} = \frac{\boldsymbol{h} (\boldsymbol{\varphi})}{\left\| \boldsymbol{h} (\boldsymbol{\varphi}) \right\|_{\mathbb{R}^d}}.$$

In this case, we have

$$\begin{aligned} \boldsymbol{\nu}' (\boldsymbol{\varphi}_0) [\boldsymbol{\varphi}] &= \frac{1}{\left\| \boldsymbol{h} (\boldsymbol{\varphi}_0) \right\|_{\mathbb{R}^d}^2} \left\{ \boldsymbol{h}' (\boldsymbol{\varphi}_0) [\boldsymbol{\varphi}] \left\| \boldsymbol{h} (\boldsymbol{\varphi}_0) \right\|_{\mathbb{R}^d} - \frac{\boldsymbol{h} (\boldsymbol{\varphi}_0)^\top (\boldsymbol{h}' (\boldsymbol{\varphi}_0) [\boldsymbol{\varphi}]) \boldsymbol{h} (\boldsymbol{\varphi}_0)}{\left\| \boldsymbol{h} (\boldsymbol{\varphi}_0) \right\|_{\mathbb{R}^d}} \right\} \\ &= - \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} + \left[\boldsymbol{\nu} \cdot \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} \right\} \right] \boldsymbol{\nu}. \end{aligned}$$

□

9.3 Shape Derivatives of Functionals

Let us use the results in Sect. 9.2 to obtain the formulae of the shape derivatives of domain and boundary integrals over a moving domain. In this case, one has to be cautious with the two types of formulae of the shape derivatives of domain and boundary integrals: the first one using the shape derivative of a function and the second one using the partial shape derivative of a function.

9.3.1 Formulae Using Shape Derivative of a Function

Firstly, let us consider finding the formulae using the shape derivative u' of a function u. From Definition 9.1.1, the following proposition holds. Here, we assume that $u(\boldsymbol{\phi})$ denotes a function u when $\boldsymbol{\phi}$, and $f(\boldsymbol{\phi}, u(\boldsymbol{\phi}))$ denotes a functional f when $\boldsymbol{\phi}$ and $u(\boldsymbol{\phi})$. Furthermore, write $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ based on Definition 9.1.1 as u'.

Proposition 9.3.1 (Derivative of domain integral of u using u') For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, suppose $u \in C^1_{\mathrm{S}'}\left(B; L^2(D;\mathbb{R})\right)$. For an arbitrary $\boldsymbol{\varphi} \in Y$, we set

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} u(\boldsymbol{\phi}+\boldsymbol{\varphi})\,\mathrm{d}z.$$

Then,

$$f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} \left(u' + u\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\right)\mathrm{d}x \tag{9.3.1}$$

holds. Moreover, $f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$ also belongs to $C(B; \mathcal{L}(X;\mathbb{R}))$. □

Proof If the domain $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$ of f is pulled back to $\Omega(\boldsymbol{\phi})$, we get

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Omega(\boldsymbol{\phi})} u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))\,\omega(\boldsymbol{\varphi})(\boldsymbol{x})\,\mathrm{d}x.$$

If Definition 9.1.1 is used, one obtains

$$f'(\boldsymbol{\phi}, u(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} \left(u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]\,\omega(\boldsymbol{\varphi}_0) + u(\boldsymbol{\phi})\,\omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]\right)\mathrm{d}x.$$

Using Proposition 9.2.1, the desired result follows. □

Next, let us think about the domain integral when a derivative of a function is the integrand. Firstly, let us focus on the following result. Below we write that the point $\boldsymbol{x}$ on domain D to which $\Omega(\boldsymbol{\phi})$ is extended moves to $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{\varphi}(\boldsymbol{x}) = (\boldsymbol{i}+\boldsymbol{\varphi})(\boldsymbol{x})$ with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Moreover, $\boldsymbol{\nabla}_z$ represents $\partial(\,\cdot\,)/\partial\boldsymbol{z}$.

Proposition 9.3.2 (Pullback of derivative) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C\left(B; H^1(D;\mathbb{R})\right)$. Suppose

$$u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}) = u(\boldsymbol{\phi})\left((\boldsymbol{i}+\boldsymbol{\varphi})^{-1}(\boldsymbol{z})\right) = u(\boldsymbol{\phi})(\boldsymbol{x}) \tag{9.3.2}$$

holds with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. In this case, we have

$$\boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}) = \boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\,\boldsymbol{\nabla} u(\boldsymbol{\phi})(\boldsymbol{x}) \in L^1\left(D;\mathbb{R}^d\right).$$

Moreover, $\boldsymbol{F}^{-\top}(\boldsymbol{\varphi})\,\boldsymbol{\nabla} u(\boldsymbol{\phi})$ belongs to $C\left(B; \mathcal{L}\left(X; L^1\left(D;\mathbb{R}^d\right)\right)\right)$. □

Proof The chain rule of derivatives gives

$$\frac{\partial u(\boldsymbol{\phi}+\boldsymbol{\varphi})}{\partial \boldsymbol{z}}(\boldsymbol{z}) = \frac{\partial \boldsymbol{x}^\top}{\partial \boldsymbol{z}} \frac{\partial u(\boldsymbol{\phi})}{\partial \boldsymbol{x}}(\boldsymbol{x}) = \left(\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}^\top}\right)^{-\top} \frac{\partial u(\boldsymbol{\phi})}{\partial \boldsymbol{x}}(\boldsymbol{x}).$$

□

Here, if the derivative of a function is included in the integrand of a domain integral, the following formula is obtained [59, 60, 69].

Proposition 9.3.3 (Derivative of domain integral of $\boldsymbol{\nabla} u$ using u') In a neighborhood $B \subset Y$ of $\boldsymbol{\phi} \in \mathcal{D}^\circ$, suppose $u \in C^1_{S'}\left(B; H^1(D;\mathbb{R})\right)$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})\, \mathrm{d}z.$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, \boldsymbol{\nabla} u)[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{\nabla} u' - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right) \boldsymbol{\nabla} u + (\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}) \boldsymbol{\nabla} u \right\} \mathrm{d}x. \tag{9.3.3}$$

Moreover, $f'(\boldsymbol{\phi}, \boldsymbol{\nabla} u)[\boldsymbol{\varphi}]$ also belongs to $C(B; \mathcal{L}(X;\mathbb{R}))$. □

Proof In Proposition 9.3.2, if we assume Eq. (9.3.2), then we obtain

$$\begin{aligned}
&f(\boldsymbol{\phi}+\boldsymbol{\varphi}, \boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})) \\
&\quad = \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} \big[\boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z})|_* \\
&\qquad + \boldsymbol{\nabla}_z \left\{ u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}) - u(\boldsymbol{\phi})\left((\boldsymbol{i}+\boldsymbol{\varphi})^{-1}(\boldsymbol{z})\right)\right\}\big]\, \mathrm{d}z \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \big\{ \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}) \boldsymbol{\nabla} u(\boldsymbol{\phi})(\boldsymbol{x}) + u_{\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})}(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})) \\
&\qquad - u_{\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})}(\boldsymbol{\phi})(\boldsymbol{x}) \big\} \omega(\boldsymbol{\varphi})\, \mathrm{d}x.
\end{aligned}$$

Here $\boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z})|_*$ is taken to be $\boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z})$ in view of Eq. (9.3.2). From the definition of the shape derivative of f (Definition 9.1.4) and the definition of $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ (Definition 9.1.1), we obtain

$$\begin{aligned}
f'(\boldsymbol{\phi}, \boldsymbol{\nabla} u(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} \Big\{ &\Big(\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] \boldsymbol{\nabla} u(\boldsymbol{\phi}) + \boldsymbol{\nabla} u'(\boldsymbol{\phi})[\boldsymbol{\varphi}] \Big) \omega(\boldsymbol{\varphi}_0) \\
&+ \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0) \boldsymbol{\nabla} u(\boldsymbol{\phi}) \omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] \Big\} \mathrm{d}x.
\end{aligned}$$

If Propositions 9.2.1 and 9.2.2 are used in this result, then the conclusion follows. □

A comparison of Proposition 9.3.1 and Proposition 9.3.3 suggests the following. With respect to the terms relating to the shape derivative of the domain measure (term containing $\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}$), since the domain measures are only multiplied by $\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}$, both are treated in the same way. On the other hand, with respect to the terms relating to the integrands, the treatments of the two are different. If the integrand does not contain any differential term, u simply changes to u', but if there is a derivative, ∇u changes to $\nabla u' - \left(\nabla \varphi^{\top}\right) \nabla u$. If attention is given to this point, the following can be obtained if the integrand is given by a function of u and ∇u.

Proposition 9.3.4 (Derivative of domain integral using u') For all ϕ in a neighborhood $B \subset Y$ of $\phi_0 \in \mathcal{D}^{\circ}$, let $u \in C^1_{\mathrm{S}'}(B;\mathcal{U})$ ($\mathcal{U} = H^2(D;\mathbb{R})$) and $h \in C^1\left(\mathbb{R} \times \mathbb{R}^d;\mathbb{R}\right)$ be defined as

$$h(u, \nabla u), h_u(u, \nabla u) \in L^2(D;\mathbb{R}), \quad h_{\nabla u}(u, \nabla u) \in L^2\left(D;\mathbb{R}^d\right)$$

with respect to $(u, \nabla u) \in \mathcal{U} \times \mathcal{G}$ ($\mathcal{G} = \{\nabla u \mid u \in \mathcal{U}\}$). Let

$$\begin{aligned} & f(\phi + \varphi, u(\phi + \varphi), \nabla_z u(\phi + \varphi)) \\ & \quad = \int_{\Omega(\phi+\varphi)} h(u(\phi + \varphi), \nabla_z u(\phi + \varphi))\, \mathrm{d}z. \end{aligned}$$

In this case, the shape derivative of f becomes

$$\begin{aligned} & f'(\phi, u, \nabla u)[\varphi] \\ & \quad = \int_{\Omega(\phi)} \left\{ h_u(u, \nabla u)[u'] + h_{\nabla u}(u, \nabla u)\left[\nabla u' - \left(\nabla \varphi^{\top}\right) \nabla u\right] \right. \\ & \qquad \left. + h(u, \nabla u) \nabla \cdot \varphi \right\} \mathrm{d}x. \end{aligned} \tag{9.3.4}$$

Furthermore, $f'(\phi, u, \nabla u)[\varphi]$ also belongs to $C(B;\mathcal{L}(X;\mathbb{R}))$. □

The formula obtained in Proposition 9.3.4 becomes a key identity used in seeking the shape derivatives of cost functions in Sect. 9.8.1. From the next section onward, $f(\phi, u, \nabla u)$ will be written as $f(\phi, u)$ and Eq. (9.3.4) will be expressed as

$$f'(\phi, u, \nabla u)[\varphi] = f'(\phi, u)[\varphi, u'] = f_{\phi'}(\phi, u)[\varphi] + f_u(\phi, u)[u']. \tag{9.3.5}$$

Here,

$$\begin{aligned} & f_{\phi'}(\phi, u)[\varphi] \\ & \quad = \int_{\Omega(\phi)} \left\{ h_{\nabla u}(u, \nabla u)\left[-\left(\nabla \varphi^{\top}\right) \nabla u\right] + h(u, \nabla u) \nabla \cdot \varphi \right\} \mathrm{d}x, \end{aligned} \tag{9.3.6}$$

$$f_u(\phi, u)[u'] = \int_{\Omega(\phi)} \left\{ h_u(u, \nabla u)[u'] + h_{\nabla u}(u, \nabla u)[\nabla u'] \right\} \mathrm{d}x. \tag{9.3.7}$$

In the expression of Eq. (9.3.5), all the terms are divided into the linear forms of φ and u'. This formulation will be used when we calculate the shape derivative of the Lagrange function with respect to each cost function. In this situation, we will obtain the shape derivative of each cost function from the linear form of φ and the weak form of adjoint problem from linear form of u'. In Eq. (9.3.5), the subscript $(\,\cdot\,)_{\boldsymbol{\phi}'}$ is used to distinguish the similar partial shape derivative shown in Sect. 9.3.2, where $(\,\cdot\,)_{\boldsymbol{\phi}^*}$ will be used.

Regarding the second-order shape derivative of the domain integral, we will only check the formulation for the shape derivative of the function to use it later. Here, we focus only on $f_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$, and will show the formulation of $f_{\boldsymbol{\phi}'\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]$. According to Definition 9.1.4, it could be expressed in two parts and is given by

$$f_{\boldsymbol{\phi}'\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = \left(f_{\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle, \tag{9.3.8}$$

where

$$\left(f_{\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = \lim_{\|\boldsymbol{\varphi}_2\|_X \to 0} \frac{1}{\|\boldsymbol{\varphi}_2\|_X} \left(\langle \boldsymbol{g}(\boldsymbol{\phi} + \boldsymbol{\varphi}_2, u), \boldsymbol{\varphi}_1\rangle - \langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{\varphi}_1\rangle\right), \tag{9.3.9}$$

$$\langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle = \lim_{\|\boldsymbol{\varphi}_2\|_X \to 0} \frac{1}{\|\boldsymbol{\varphi}_2\|_X} \left\langle \boldsymbol{g}(\boldsymbol{\phi} + \boldsymbol{\varphi}_2, u), \boldsymbol{\varphi}_1 \circ (\boldsymbol{i} + \boldsymbol{\varphi}_2)^{-1} - \boldsymbol{\varphi}_1 \right\rangle. \tag{9.3.10}$$

Equation (9.3.9) represents the derivative of $\langle \boldsymbol{g}(\boldsymbol{\phi} + \boldsymbol{\varphi}_2, u), \boldsymbol{\varphi}_1\rangle$ with respect to a variation of $\boldsymbol{\varphi}_2$ fixing $\boldsymbol{\varphi}_1$. On the other hand, Eq. (9.3.10) is the element to correct the variation of $\boldsymbol{\varphi}_1$ by $\boldsymbol{\varphi}_2$ using the inverse mapping of $\boldsymbol{i} + \boldsymbol{\varphi}_2$. The calculation of only the term of $\boldsymbol{\varphi}_1 \circ (\boldsymbol{i} + \boldsymbol{\varphi}_2)^{-1} - \boldsymbol{\varphi}_1$ yields Eq. (9.1.9). However, $f_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$ in Eq. (9.3.5) uses $\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$. Then, we need another formulation shown in the following.

According to the explanation given after Eq. (9.1.9), with respect to $-\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}$ in Eq. (9.3.6), we replace $\boldsymbol{\varphi}$ by $\boldsymbol{\varphi}_1$ and add the variation $-\boldsymbol{\varphi}_2$ which is a linearization of the inverse mapping of $\boldsymbol{i}+\boldsymbol{\varphi}_2$. By this variation, $-\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}$ becomes $\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2\right)\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}$ using Proposition 9.3.3 in which $\boldsymbol{\varphi}$ is changed by $-\boldsymbol{\varphi}_2$, and u is replaced by $\boldsymbol{\varphi}_1^{\top}$. Moreover, we apply the same variation to $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$ in Eq. (9.3.6). Using Proposition 9.3.3 in which $\boldsymbol{\varphi}$ is changed by $-\boldsymbol{\varphi}_2$, and u is replaced by $\cdot\boldsymbol{\varphi}_1$, $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}$ becomes $\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1$.

Let us confirm this thing to the functional

$$f(\boldsymbol{\phi}, u) = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} u \,\mathrm{d}x.$$

We put the shape derivative of $f(\boldsymbol{\phi}, u)$ as

$$\begin{aligned}\langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{\varphi}_1\rangle &= \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\boldsymbol{\nabla} u \,\mathrm{d}x - \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \boldsymbol{\nabla} u \,\mathrm{d}x \\ &= f_1(\boldsymbol{\phi}, \boldsymbol{\varphi}_1) + f_2(\boldsymbol{\phi}, \boldsymbol{\varphi}_1).\end{aligned}$$

For the shape derivative, Eq. (9.3.10) is written as

$$\langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle = \langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{\varphi}_1\rangle_{\boldsymbol{\varphi}_1}[-\boldsymbol{\varphi}_2]$$

$$= f_{1\boldsymbol{\varphi}_1}(\boldsymbol{\phi}, \boldsymbol{\varphi}_1)[-\boldsymbol{\varphi}_2] + f_{2\boldsymbol{\varphi}_1}(\boldsymbol{\phi}, \boldsymbol{\varphi}_1)[-\boldsymbol{\varphi}_2].$$

Each term on the right-hand side is obtained using Proposition 9.3.3 as follows.

$$f_{1\boldsymbol{\varphi}_1}(\boldsymbol{\phi}, \boldsymbol{\varphi}_1)[-\boldsymbol{\varphi}_2] = \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right) \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} u \, \mathrm{d}x,$$

$$f_{2\boldsymbol{\varphi}_1}(\boldsymbol{\phi}, \boldsymbol{\varphi}_1)[-\boldsymbol{\varphi}_2] = \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right) \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1 \boldsymbol{\nabla} u \, \mathrm{d}x$$
$$= \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2 \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \boldsymbol{\nabla} u \, \mathrm{d}x.$$

Using these relations, we have

$$\begin{aligned}
&\langle \boldsymbol{g}(\boldsymbol{\phi}, u), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big\{ h_{\boldsymbol{\nabla} u}(u, \boldsymbol{\nabla} u)\left[\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right)\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} u\right] \\
&\qquad + h(u, \boldsymbol{\nabla} u)\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2 \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)\Big\} \mathrm{d}x. \qquad (9.3.11)
\end{aligned}$$

In calculating the second-order derivatives of cost functions, one has to pay attention to the term in Eq. (9.3.11) added to the expression given in equation Eq. (9.3.9).

Next, let us think about the case when the functional is given by a boundary integral. Suppose $\Gamma(\boldsymbol{\phi})$ is a partial set of $\partial\Omega(\boldsymbol{\phi})$ (allowing $\Gamma(\boldsymbol{\phi}) = \partial\Omega(\boldsymbol{\phi})$). Moreover, let $\Theta(\boldsymbol{\phi})$ be corner points (when $d = 2$) or edges (when $d = 3$) on $\partial\Omega(\boldsymbol{\phi})$ (Fig. 9.3). Also, let $\boldsymbol{\tau}$ be a tangent of $\Gamma(\boldsymbol{\phi})$ (when $d = 2$) or tangent of $\Gamma(\boldsymbol{\phi})$ and outward normal of $\partial\Gamma(\boldsymbol{\phi})$ (when $d = 3$). Note that $\boldsymbol{\tau}$ at $\Theta(\boldsymbol{\phi})$ exists on both sides of $\Theta(\boldsymbol{\phi})$ as shown in Fig. 9.3. Lastly, let $\mathrm{d}\varsigma$ express the measure of $\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})$.

Proposition 9.3.5 (Derivative of boundary integral of u using u') For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}\left(B; H^1(D; \mathbb{R})\right)$ and $\Gamma(\boldsymbol{\phi})$ be piecewise $H^2 \cap C^{0,1}$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi} + \boldsymbol{\varphi}, u(\boldsymbol{\phi} + \boldsymbol{\varphi})) = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} u(\boldsymbol{\phi} + \boldsymbol{\varphi}) \, \mathrm{d}\zeta.$$

Then, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left\{u' + u(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau\right\} \mathrm{d}\gamma,$$

where $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau$ follows Eq. (9.2.6). Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$,

$$f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left(u' + \kappa u \boldsymbol{\nu}\cdot\boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau u \cdot \boldsymbol{\varphi}_\tau\right) \mathrm{d}\gamma + \int_{\partial\Gamma(\boldsymbol{\phi})\cup\Theta(\boldsymbol{\phi})} u \boldsymbol{\tau}\cdot\boldsymbol{\varphi} \, \mathrm{d}\varsigma$$

holds, where $\boldsymbol{\nabla}_\tau(\,\cdot\,) = \left(\boldsymbol{\tau}_j(\boldsymbol{\phi})\cdot\boldsymbol{\nabla}\right)_{j\in\{1,\ldots,d-1\}}(\,\cdot\,) \in \mathbb{R}^{d-1}$ and $\boldsymbol{\varphi}_\tau = \left(\boldsymbol{\tau}_j(\boldsymbol{\phi})\cdot\boldsymbol{\varphi}\right)_{j\in\{1,\ldots,d-1\}} \in \mathbb{R}^{d-1}$. Moreover, $f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$ also belongs to $C(B; \mathcal{L}(X; \mathbb{R}))$. □

Proof If the integral domain $\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})$ of f is pulled back to $\Gamma(\boldsymbol{\phi})$, we get

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Gamma(\boldsymbol{\phi})} u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))\, \varpi(\boldsymbol{\varphi})\,\mathrm{d}\gamma.$$

From the definition of the shape derivative of f (Definition 9.1.4) and the definition of $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ (Definition 9.1.1), we obtain

$$f'(\boldsymbol{\phi}, u(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left\{u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]\, \varpi(\boldsymbol{\varphi}_0) + u(\boldsymbol{\phi})\, \varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]\right\} \mathrm{d}\gamma.$$

If Proposition 9.2.4 is applied, the first part of the proposition can be obtained. Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, and if the Gauss–Green theorem (Theorem A.8.2) is applied to $\int_{\Gamma(\boldsymbol{\phi})} u(\boldsymbol{\phi})\, \boldsymbol{\nabla}_\tau \cdot \boldsymbol{\varphi}_\tau\, \mathrm{d}\gamma$, the remaining part is established. □

Furthermore, if the integrand of the boundary integral is a derivative in the direction of the normal, we get the following.

Proposition 9.3.6 (Derivative of boundary integral of $\partial_\nu u$ using u') For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}\left(B; H^2(D;\mathbb{R})\right)$ and $\Gamma(\boldsymbol{\phi})$ be piecewise $H^2 \cap C^{0,1}$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi})\,\mathrm{d}\zeta.$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left\{\partial_\nu u' + w(\boldsymbol{\varphi}, u) + \partial_\nu u\, (\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau\right\} \mathrm{d}\gamma,$$

where

$$w(\boldsymbol{\varphi}, u) = \left[\left\{\boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nu}\right)\right\}\boldsymbol{\nu} - \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)^\top\right\}\boldsymbol{\nu}\right]\cdot\boldsymbol{\nabla}u, \tag{9.3.12}$$

and $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau$ follows Eq. (9.2.6). Moreover, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, the identity

$$\begin{aligned} f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] &= \int_{\Gamma(\boldsymbol{\phi})} \left\{\partial_\nu u' + w(\boldsymbol{\varphi}, u) + \kappa\partial_\nu u \boldsymbol{\nu}\cdot\boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau(\partial_\nu u)\cdot\boldsymbol{\varphi}_\tau\right\}\mathrm{d}\gamma \\ &\quad + \int_{\partial\Gamma(\boldsymbol{\phi})\cup\Theta(\boldsymbol{\phi})} \partial_\nu u \boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\mathrm{d}\varsigma \end{aligned} \tag{9.3.13}$$

holds. Moreover, $f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}]$ belongs to $C(B; \mathcal{L}(Y;\mathbb{R}))$. □

Proof If we assume Eq. (9.3.2) to hold in Proposition 9.3.2, then we get

$$\begin{aligned} &f(\boldsymbol{\phi}+\boldsymbol{\varphi}, \partial_\mu u(\boldsymbol{\phi}+\boldsymbol{\varphi})) \\ &\quad = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} \left[\left.\boldsymbol{\nabla}_z u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z})\right|_* \right. \\ &\qquad \left. + \boldsymbol{\nabla}_z\left\{u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z}) - u(\boldsymbol{\phi})\left((\boldsymbol{i}+\boldsymbol{\varphi})^{-1}(\boldsymbol{z})\right)\right\}\right]\cdot\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{z})\,\mathrm{d}\zeta \end{aligned}$$

$$= \int_{\Gamma(\phi)} \left\{ \left(\boldsymbol{F}^{-\top}(\boldsymbol{\varphi}) \boldsymbol{\nabla} u(\phi) \right) \cdot \left(\boldsymbol{\nu} + \boldsymbol{\nu}'(\phi)[\boldsymbol{\varphi}] + o\left(\|\boldsymbol{\varphi}\|_X\right) \right) \right.$$
$$\left. + \partial_\mu u(\phi + \boldsymbol{\varphi})(\boldsymbol{x} + \boldsymbol{\varphi}(\boldsymbol{x})) - \partial_\mu u(\boldsymbol{x}) \right\} \varpi(\boldsymbol{\varphi}) \, \mathrm{d}\gamma,$$

where $\boldsymbol{\nabla}_z u(\phi + \boldsymbol{\varphi})(\boldsymbol{z})|_*$ equates to $\boldsymbol{\nabla}_z u(\phi + \boldsymbol{\varphi})(\boldsymbol{z})$ under the assumption of Eq. (9.3.2). From the definition of the shape derivative of f (Definition 9.1.4) and the definition of $u'(\phi)[\boldsymbol{\varphi}]$ (Definition 9.1.1), we get

$$f'(\phi, \partial_\nu u(\phi))[\boldsymbol{\varphi}] = \int_{\Gamma(\phi)} \left[\left\{ \left(\boldsymbol{F}^{-\top\prime}(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] \boldsymbol{\nabla} u \right) \cdot \boldsymbol{\nu} + \partial_\nu u'(\phi)[\boldsymbol{\varphi}] \right. \right.$$
$$\left. + \left(\boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0) \boldsymbol{\nabla} u(\phi) \right) \cdot \boldsymbol{\nu}'(\phi)[\boldsymbol{\varphi}] \right\} \varpi(\boldsymbol{\varphi}_0)$$
$$\left. + \boldsymbol{F}^{-\top}(\boldsymbol{\varphi}_0) \partial_\nu u(\phi) \varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}] \right] \mathrm{d}\gamma.$$

Using Propositions 9.2.2, 9.2.4 and 9.2.5, we have

$$f'(\phi, \partial_\nu u(\phi))[\boldsymbol{\varphi}] = \int_{\Gamma(\phi)} \left[-\left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nabla} u(\phi) \right\} \cdot \boldsymbol{\nu} + \partial_\nu u'(\phi)[\boldsymbol{\varphi}] \right.$$
$$+ \left[-\left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} + \left\{ \boldsymbol{\nu} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} \right\} \boldsymbol{\nu} \right] \cdot \boldsymbol{\nabla} u(\phi)$$
$$\left. + \partial_\nu u(\phi) \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} - \boldsymbol{\nu} \cdot \left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \boldsymbol{\nu} \right) \right\} \right] \mathrm{d}\gamma.$$

From this, the first part of the proposition can be obtained. The remaining part can be obtained in a similar way to the proof of Proposition 9.3.5. □

When the integrand of a boundary integral is given by the function of u and $\partial_\nu u$, if the chain rule for derivatives is used in the proof of Propositions 9.3.5 and 9.3.6, the following results can be obtained.

Proposition 9.3.7 (Derivative of boundary integral using u') For all ϕ in a neighborhood $B \subset Y$ of $\phi_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}(B; \mathcal{U})$ ($\mathcal{U} = H^2(D; \mathbb{R})$), and $h \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ be defined as

$$h(u, \partial_\nu u) \in H^2(D; \mathbb{R}), \quad h_u(u, u), h_{\partial_\nu u}(u, \boldsymbol{\nabla} u) \in H^1\left(D; \mathbb{R}^d\right)$$

with respect to $(u, \partial_\nu u) \in \mathcal{U} \times \mathcal{G}_{\Gamma(\phi)}$ ($\mathcal{G}_{\Gamma(\phi)} = \left\{ \partial_\nu u|_{\Gamma(\phi)} \;\middle|\; u \in \mathcal{U} \right\}$) and $\Gamma(\phi)$ be piecewise $H^2 \cap C^{0,1}$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\phi + \boldsymbol{\varphi}, u(\phi + \boldsymbol{\varphi}), \partial_\mu u(\phi + \boldsymbol{\varphi}))$$
$$= \int_{\Gamma(\phi + \boldsymbol{\varphi})} h(u(\phi + \boldsymbol{\varphi}), \partial_\mu u(\phi + \boldsymbol{\varphi})) \, \mathrm{d}\zeta.$$

In this case, the shape derivative of f becomes

$$f'(\phi, u, \partial_\nu u)[\boldsymbol{\varphi}]$$
$$= \int_{\Gamma(\phi)} \left\{ h_u(u, \partial_\nu u)[u'] + h_{\partial_\nu u}(u, \partial_\nu u)[\partial_\nu u' + w(\boldsymbol{\varphi}, u)] \right.$$

$$+ h(u, \partial_\nu u) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau \big\} \mathrm{d}\gamma.$$

Here, $w(\boldsymbol{\varphi}, u)$ and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ are given by Eq. (9.3.12) and Eq. (9.2.6), respectively. Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, we have

$$\begin{aligned} & f'(\boldsymbol{\phi}, u, \partial_\nu u)[\boldsymbol{\varphi}] \\ & \quad = \int_{\Gamma(\boldsymbol{\phi})} \Big\{ h_u(u, \partial_\nu u)[u'] + h_{\partial_\nu u}(u, \partial_\nu u) \left[\partial_\nu u' + w(\boldsymbol{\varphi}, u) \right] \\ & \qquad + \kappa h(u, \partial_\nu u) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau h(u, \partial_\nu u) \cdot \boldsymbol{\varphi}_\tau \Big\} \mathrm{d}\gamma \\ & \quad + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} h(u, \partial_\nu u) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma. \end{aligned} \tag{9.3.14}$$

Moreover, $f'(\boldsymbol{\phi}, u, \partial_\nu u)[\boldsymbol{\varphi}]$ belongs to $C(B; \mathcal{L}(Y; \mathbb{R}))$. □

In Propositions 9.3.6 and 9.3.7, we remark the following.

Remark 9.3.8 (Derivative of boundary integral of $\partial_\nu u$ using u') For a boundary integral that included the derivative of a function, $w(\boldsymbol{\varphi}, u)$ of Eq. (9.3.12) was contained in the shape derivatives of the boundary integral (Eq. (9.3.13) and Eq. (9.3.14)). For that reason, we had $f'(\boldsymbol{\phi}, u, \partial_\nu u)[\,\cdot\,] \in \mathcal{L}(Y; \mathbb{R})$ ($\notin \mathcal{L}(X; \mathbb{R})$). As shown in Sect. 9.1.3, the shape derivatives were defined as bounded linear operators with respect to an arbitrary $\boldsymbol{\varphi} \in X$. Hence in future discussions, when defining the cost functions, the shape derivatives of cost functions must be constructed so that $w(\boldsymbol{\varphi}, u)$ is not left in there. In actual fact, if the cost function is defined as Eq. (9.6.1), the desired results can be obtained. □

The formula obtained in Proposition 9.3.7 is the key identity for obtaining the shape derivative of the cost function in Sect. 9.8.1. From the next section onward, we will write $f(\boldsymbol{\phi}, u, \partial_\nu u)$ as $f(\boldsymbol{\phi}, u)$, and Eq. (9.3.14) as

$$f'(\boldsymbol{\phi}, u, \partial_\nu u)[\boldsymbol{\varphi}] = f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}, u'] \qquad = f_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] + f_u(\boldsymbol{\phi}, u)[u']. \tag{9.3.15}$$

Here stands

$$f_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \{ h_{\partial_\nu u}(u, \partial_\nu u)[w(\boldsymbol{\varphi}, u)] + h(u, \partial_\nu u)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau \} \, \mathrm{d}\gamma,$$

$$f_u(\boldsymbol{\phi}, u)[u'] = \int_{\Gamma(\boldsymbol{\phi})} (h_u(u, \partial_\nu u)[u'] + h_{\partial_\nu u}(u, \partial_\nu u)[\partial_\nu u']) \, \mathrm{d}\gamma.$$

Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, we have

$$\begin{aligned} f_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = & \int_{\Gamma(\boldsymbol{\phi})} \Big(h_{\partial_\nu u}(u, \partial_\nu u)[w(\boldsymbol{\varphi}, u)] \\ & \qquad + \kappa h(u, \partial_\nu u) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau h(u, \partial_\nu u) \cdot \boldsymbol{\varphi}_\tau \Big) \mathrm{d}\gamma \\ & + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} h(u, \partial_\nu u) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma. \end{aligned}$$

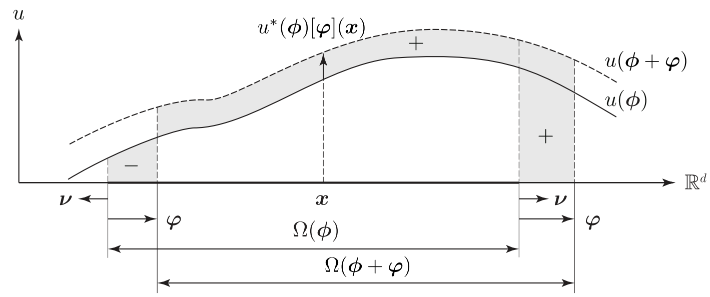


Fig. 9.8: Shape derivative of a domain integral when the partial shape derivative u^* of a function is used.

9.3.2 Formulae Using Partial Shape Derivative of a Function

Next, let us use the partial shape derivative u^* of the function u (Definition 9.1.3) to obtain the formulae for seeking the shape derivatives of domain and boundary integrals. Again, we express a function and a functional as $u(\boldsymbol{\phi}+\boldsymbol{\varphi})$ and $f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi}))$, respectively, when $\boldsymbol{\phi}+\boldsymbol{\varphi}$, and simply by u and $f(\boldsymbol{\phi}, u)$ when $\boldsymbol{\phi}$. Furthermore, we write $u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ in Definition 9.1.3 as u^*.

Firstly, in view of Proposition 9.3.1, the following result holds.

Proposition 9.3.9 (Derivative of domain integral of u using u^*) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}^*}\left(B; H^1(D;\mathbb{R})\right)$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, u(\boldsymbol{\phi}+\boldsymbol{\varphi})) = \int_{\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})} u(\boldsymbol{\phi}+\boldsymbol{\varphi})\, \mathrm{d}z.$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} u^* \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} u\boldsymbol{\nu}\cdot\boldsymbol{\varphi}\, \mathrm{d}\gamma. \tag{9.3.16}$$

Moreover, $f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$ also belongs to $C(B; \mathcal{L}(X;\mathbb{R}))$. □

Proof The proposition is easily proved by substituting $u'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$ in Eq. (9.1.4) of Proposition 9.3.1 together with the Gauss–Green theorem (Theorem A.8.2). □

Figure 9.8 shows the areas corresponding to each integral on the right-hand side of Eq. (9.3.16). The first term on the right-hand side corresponds to the shaded area in $\Omega(\boldsymbol{\phi}) \cap \Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$, while the second term corresponds to the shaded areas on the left and right sides of the figure. Here, it should be noted that since the area on the right side has the outward unit normal $\boldsymbol{\nu}$ pointing to the right, $\boldsymbol{\nu}\cdot\boldsymbol{\varphi} > 0$, and since the area on the left side has $\boldsymbol{\nu}$ pointing to the left, then $\boldsymbol{\nu}\cdot\boldsymbol{\varphi} < 0$.

Moreover, the formula corresponding to Proposition 9.3.3 which represents the shape derivative of the domain integral with a differential term as integrand can be obtained by viewing $\boldsymbol{\nabla} u \in C^1_{\mathrm{S}^*}\left(B; H^1\left(D; \mathbb{R}^d\right)\right)$ as u in Proposition 9.3.9. In this case, from the fact that $\left(\boldsymbol{\nabla} u\right)^* \left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] = \boldsymbol{\nabla} u^* \left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}]$ can be established based on Definition 9.1.3, if $\boldsymbol{\nabla} u^* \left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}]$ is written as $\boldsymbol{\nabla} u^*$, then we have

$$f'\left(\boldsymbol{\phi}, \boldsymbol{\nabla} u\right)[\boldsymbol{\varphi}] = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} u^* \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\nu} \cdot \boldsymbol{\varphi}\right) \boldsymbol{\nabla} u \, \mathrm{d}\gamma. \tag{9.3.17}$$

Furthermore, Eq. (9.3.17) can be written as

$$\begin{aligned} f'\left(\boldsymbol{\phi}, \boldsymbol{\nabla} u\right)[\boldsymbol{\varphi}] &= \int_{\Omega(\boldsymbol{\phi})} \left[\boldsymbol{\nabla} u^* + \left\{\boldsymbol{\nabla}^\top \left(\boldsymbol{\nabla} u \, \boldsymbol{\varphi}^\top\right)^\top\right\}^\top\right] \mathrm{d}x \\ &= \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\nabla} u^* + \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \boldsymbol{\nabla} u + \Delta u \boldsymbol{\varphi}\right) \mathrm{d}x \end{aligned} \tag{9.3.18}$$

using the Gauss–Green theorem. Hence, if it is compared with the results of Proposition 9.3.3, we get the identity

$$\boldsymbol{\nabla} u'\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] = \boldsymbol{\nabla} u^*\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top\right) \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right) + \Delta u\left(\boldsymbol{\phi}\right) \boldsymbol{\varphi}. \tag{9.3.19}$$

Equation (9.3.19) can be obtained also from

$$\boldsymbol{\nabla} u'\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] = \boldsymbol{\nabla} u^*\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right) \cdot \boldsymbol{\varphi}\right)$$

by using Eq. (9.1.4).

If the integrand is given by a function of u and $\boldsymbol{\nabla} u$, then by using the chain rule for derivatives on Proposition 9.3.9, the following result can be obtained.

Proposition 9.3.10 (Derivative of domain integral using u^*) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}^*}\left(B; \mathcal{U}\right)$ ($\mathcal{U} = H^2\left(D; \mathbb{R}\right)$) and $h \in C^1\left(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}\right)$ be defined as

$$h\left(u, \boldsymbol{\nabla} u\right) \in H^1\left(D; \mathbb{R}\right), \quad h_u\left(u, \boldsymbol{\nabla} u\right), h_{\boldsymbol{\nabla} u}\left(u, \boldsymbol{\nabla} u\right) \in L^2\left(D; \mathbb{R}^d\right)$$

with respect to $\left(u, \boldsymbol{\nabla} u\right) \in \mathcal{U} \times \mathcal{G}$ ($\mathcal{G} = \{\boldsymbol{\nabla} u \mid u \in \mathcal{U}\}$). For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$\begin{aligned} &f\left(\boldsymbol{\phi} + \boldsymbol{\varphi}, u\left(\boldsymbol{\phi} + \boldsymbol{\varphi}\right), \boldsymbol{\nabla}_z u\left(\boldsymbol{\phi} + \boldsymbol{\varphi}\right)\right) \\ &\quad = \int_{\Omega(\boldsymbol{\phi} + \boldsymbol{\varphi})} h\left(u\left(\boldsymbol{\phi} + \boldsymbol{\varphi}\right), \boldsymbol{\nabla}_z u\left(\boldsymbol{\phi} + \boldsymbol{\varphi}\right)\right) \mathrm{d}z. \end{aligned}$$

Then, the shape derivative of f in this case becomes

$$\begin{aligned} f'\left(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u\right)[\boldsymbol{\varphi}] &= \int_{\Omega(\boldsymbol{\phi})} \left\{h_u\left(u, \boldsymbol{\nabla} u\right)[u^*] + h_{\boldsymbol{\nabla} u}\left(u, \boldsymbol{\nabla} u\right)[\boldsymbol{\nabla} u^*]\right\} \mathrm{d}x \\ &\quad + \int_{\partial\Omega(\boldsymbol{\phi})} h\left(u, \boldsymbol{\nabla} u\right) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma. \end{aligned} \tag{9.3.20}$$

Moreover, $f'\left(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u\right)[\boldsymbol{\varphi}]$ also belongs to $C\left(B; \mathcal{L}\left(X; \mathbb{R}\right)\right)$. □

The formula obtained in Proposition 9.3.10 is the key identity for obtaining the shape derivative of the cost function in Sect. 9.8.4. From now on, we will write $f(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)$ as $f(\boldsymbol{\phi}, u)$ and Eq. (9.3.20) as

$$f'(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\boldsymbol{\varphi}] = f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}, u^*] = f_{\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] + f_u(\boldsymbol{\phi}, u)[u^*]. \tag{9.3.21}$$

Here, stands

$$f_{\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\partial\Omega(\boldsymbol{\phi})} h(u, \boldsymbol{\nabla} u)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\, \mathrm{d}\gamma,$$
$$f_u(\boldsymbol{\phi}, u)[u^*] = \int_{\Omega(\boldsymbol{\phi})} \{h_u(u, \boldsymbol{\nabla} u)[u^*] + h_{\boldsymbol{\nabla} u}(u, \boldsymbol{\nabla} u)[\boldsymbol{\nabla} u^*]\}\, \mathrm{d}x.$$

If a functional is given by a boundary integral, the following formula is obtained by substituting Eq. (9.1.4) into Proposition 9.3.5.

Proposition 9.3.11 (Derivative of boundary integral of u using u^*) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}^*}\left(B; H^2(D; \mathbb{R})\right)$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi} + \boldsymbol{\varphi}, u(\boldsymbol{\phi} + \boldsymbol{\varphi})) = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} u(\boldsymbol{\phi} + \boldsymbol{\varphi})\, \mathrm{d}\zeta.$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left(u^* + \boldsymbol{\nabla} u \cdot \boldsymbol{\varphi} + u (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau\right) \mathrm{d}\gamma, \tag{9.3.22}$$

where $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ obeys Eq. (9.2.6). Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, we have

$$\begin{aligned} f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] = {} & \int_{\Gamma(\boldsymbol{\phi})} \{u^* + (\partial_\nu + \kappa)\, u \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\}\, \mathrm{d}\gamma \\ & + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} u \boldsymbol{\tau} \cdot \boldsymbol{\varphi}\, \mathrm{d}\varsigma. \end{aligned} \tag{9.3.23}$$

Moreover, $f'(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}]$ also belongs to $C(B; \mathcal{L}(X; \mathbb{R}))$. □

Moreover, if the integrand of the boundary integral is $\partial_\nu u$, the following result is obtained.

Proposition 9.3.12 (Derivative of boundary integral of $\partial_\nu u$ using u^*) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}^*}\left(B; H^3(D; \mathbb{R})\right)$. For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$f(\boldsymbol{\phi} + \boldsymbol{\varphi}, \partial_\mu u(\boldsymbol{\phi} + \boldsymbol{\varphi})) = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} \partial_\mu u(\boldsymbol{\phi} + \boldsymbol{\varphi})\, \mathrm{d}\zeta.$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left(\partial_\nu u^* + \bar{w}(\boldsymbol{\varphi}, u) + \partial_\nu u \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}\right)_\tau\right) \mathrm{d}\gamma,$$

where

$$\bar{w}(\boldsymbol{\varphi}, u) = -\left[\sum_{i\in\{1,\dots,d-1\}} \left\{\boldsymbol{\tau}_i \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top \boldsymbol{\nu}\right)\right\} \boldsymbol{\tau}_i\right] \cdot \boldsymbol{\nabla} u + (\boldsymbol{\nu} \cdot \boldsymbol{\varphi}) \Delta u, \tag{9.3.24}$$

and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ obeys Eq. (9.2.6). Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, we have

$$\begin{aligned} f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} &\left\{\partial_\nu u^* + \bar{w}(\boldsymbol{\varphi}, u) + \kappa \partial_\nu u \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\right. \\ &\left. - \boldsymbol{\nabla}_\tau (\partial_\nu u) \cdot \boldsymbol{\varphi}_\tau\right\} \mathrm{d}\gamma + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} \partial_\nu u \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma. \end{aligned}$$

Moreover, $f'(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}]$ belongs to $C(B; \mathcal{L}(Y; \mathbb{R}))$. □

Proof From Eq. (9.3.19), we have the equation

$$\partial_\nu u'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \partial_\nu u^*(\boldsymbol{\phi})[\boldsymbol{\varphi}] + \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)^\top \boldsymbol{\nu}\right\} \cdot \boldsymbol{\nabla} u(\boldsymbol{\phi}) + \Delta u(\boldsymbol{\phi}) \boldsymbol{\nu} \cdot \boldsymbol{\varphi}. \tag{9.3.25}$$

Substituting the above equation into the result of Proposition 9.3.6, we arrive at the desired result. □

Here, if the integrand of a boundary integral is given by a function of u and $\partial_\nu u$, the following result can be obtained by using the chain rule for derivatives on Propositions 9.3.11 and 9.3.12.

Proposition 9.3.13 (Derivative of boundary integral using u^*) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{S^*}(B; \mathcal{U})$ ($\mathcal{U} = H^3(D; \mathbb{R})$), and $h \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ be defined as

$$h(u, \partial_\nu u) \in H^2(D; \mathbb{R}), \quad h_u(u, u), h_{\partial_\nu u}(u, \boldsymbol{\nabla} u) \in H^1\left(D; \mathbb{R}^d\right)$$

with respect to $(u, \partial_\nu u) \in \mathcal{U} \times \mathcal{G}_{\Gamma(\boldsymbol{\phi})}$ ($\mathcal{G}_{\Gamma(\boldsymbol{\phi})} = \left\{ \partial_\nu u|_{\Gamma(\boldsymbol{\phi})} \;\middle|\; u \in \mathcal{U}\right\}$). For an arbitrary $\boldsymbol{\varphi} \in Y$, let

$$\begin{aligned} &f(\boldsymbol{\phi} + \boldsymbol{\varphi}, u(\boldsymbol{\phi} + \boldsymbol{\varphi}), \partial_\mu u(\boldsymbol{\phi} + \boldsymbol{\varphi})) \\ &\quad = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} h(u(\boldsymbol{\phi} + \boldsymbol{\varphi}), \partial_\mu u(\boldsymbol{\phi} + \boldsymbol{\varphi})) \, \mathrm{d}\zeta. \end{aligned}$$

In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, u, \partial_\nu u)[\boldsymbol{\varphi}]$$

$$
\begin{aligned}
&= \int_{\Gamma(\boldsymbol{\phi})} \bigl\{ h_u \left(u, \partial_\nu u\right) [u^*] + \boldsymbol{\nabla} h \left(u, \partial_\nu u\right) \cdot \boldsymbol{\varphi} \\
&\qquad + h_{\partial_\nu u} \left(u, \partial_\nu u\right) [\partial_\nu u^* + \bar{w} \left(\boldsymbol{\varphi}, u\right)] + h \left(u, \partial_\nu u\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}\right)_\tau \bigr\} \mathrm{d}\gamma,
\end{aligned}
$$

where $\bar{w}(\boldsymbol{\varphi}, u)$ and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ obey Eq. (9.3.24) and Eq. (9.2.6), respectively. Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$, we have

$$
\begin{aligned}
&f' \left(\boldsymbol{\phi}, u, \partial_\nu u\right) [\boldsymbol{\varphi}] \\
&\quad = \int_{\Gamma(\boldsymbol{\phi})} \bigl\{ h_u \left(u, \partial_\nu u\right) [u^*] + h_{\partial_\nu u} \left(u, \partial_\nu u\right) [\partial_\nu u^* + \bar{w} \left(\boldsymbol{\varphi}, u\right)] \\
&\qquad\quad + \left(\partial_\nu + \kappa\right) h \left(u, \partial_\nu u\right) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \bigr\} \mathrm{d}\gamma \\
&\qquad + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} h \left(u, \partial_\nu u\right) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma.
\end{aligned} \tag{9.3.26}
$$

Moreover, $f' (\boldsymbol{\phi}, u, \partial_\nu u) [\boldsymbol{\varphi}]$ belongs to $C (B; \mathcal{L} (Y; \mathbb{R}))$. □

In Propositions 9.3.12 and 9.3.13, let us recall the similar situation in Remark 9.3.8.

The formula given in Proposition 9.3.13 is the key identity for obtaining the shape derivative of the cost function in Sect. 9.8.4. From the next section onward, by writing $f (\boldsymbol{\phi}, u, \partial_\nu u)$ as $f (\boldsymbol{\phi}, u)$, Eq. (9.3.26) is expressed as

$$
f' \left(\boldsymbol{\phi}, u, \partial_\nu u\right) [\boldsymbol{\varphi}] = f' \left(\boldsymbol{\phi}, u\right) [\boldsymbol{\varphi}, u^*] = f_{\boldsymbol{\phi}^*} \left(\boldsymbol{\phi}, u\right) [\boldsymbol{\varphi}] + f_u \left(\boldsymbol{\phi}, u\right) [u^*], \tag{9.3.27}
$$

where

$$
\begin{aligned}
f_{\boldsymbol{\phi}^*} \left(\boldsymbol{\phi}, u\right) [\boldsymbol{\varphi}] &= \int_{\Gamma(\boldsymbol{\phi})} \bigl\{ h_u \left(u, \partial_\nu u\right) [\boldsymbol{\nabla} u \left(\boldsymbol{\phi}\right) \cdot \boldsymbol{\varphi}] + h_{\partial_\nu u} \left(u, \partial_\nu u\right) [\bar{w} \left(\boldsymbol{\varphi}, u\right)] \\
&\qquad + h \left(u \left(\boldsymbol{\phi}\right), \partial_\nu u \left(\boldsymbol{\phi}\right)\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}\right)_\tau \bigr\} \mathrm{d}\gamma, \\
f_u \left(\boldsymbol{\phi}, u\right) [u^*] &= \int_{\Gamma(\boldsymbol{\phi})} \left(h_u \left(u, \partial_\nu u\right) [u^*] + h_{\partial_\nu u} \left(u, \partial_\nu u\right) [\partial_\nu u^*] \right) \mathrm{d}\gamma.
\end{aligned}
$$

Furthermore, if $\Gamma(\boldsymbol{\phi})$ is piecewise $H^3 \cap C^{1,1}$ class,

$$
\begin{aligned}
f_{\boldsymbol{\phi}^*} \left(\boldsymbol{\phi}, u\right) [\boldsymbol{\varphi}] &= \int_{\Gamma(\boldsymbol{\phi})} \bigl\{ h_{\partial_\nu u} \left(u, \partial_\nu u\right) [\bar{w} \left(\boldsymbol{\varphi}, u\right)] \\
&\qquad + \left(\partial_\nu + \kappa\right) h \left(u \left(\boldsymbol{\phi}\right), \partial_\nu u \left(\boldsymbol{\phi}\right)\right) \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \bigr\} \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} h \left(u \left(\boldsymbol{\phi}\right), \partial_\nu u \left(\boldsymbol{\phi}\right)\right) \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma.
\end{aligned}
$$

9.4 Variation Rules of Functions

In Sect. 9.5, a state determination problem (boundary value problem of partial differential equation) will be defined. In this case, one has to be aware of how

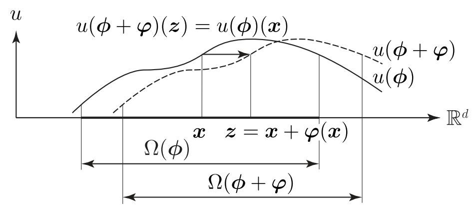


Fig. 9.9: The function $u : D \to \mathbb{R}$ fixed with material.

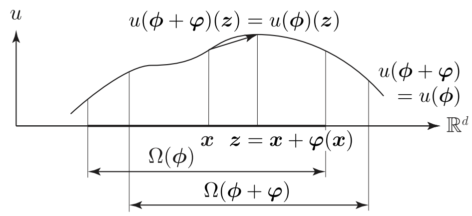


Fig. 9.10: The function $u : D \to \mathbb{R}$ fixed in space.

the known function behaves with respect to the moving domain. Here, let us define typical variation rules using the results obtained up to the end of Sect. 9.3. Also, in this section, we will fix $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$ and consider an arbitrary domain variation $\boldsymbol{\varphi} \in Y$ for $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$.

Firstly, we think about the case when the function value moves along with the movement of a point on the domain, as shown in Fig. 9.9. The variation rule for the function in this case is defined as follows.

Definition 9.4.1 (Function fixed with material) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}\left(B; L^2\left(D; \mathbb{R}\right)\right)$, and suppose

$$u'\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}\right] = 0$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Then, u is referred to as a function fixed with material. □

Moreover, the variation rule for a function not depending on the domain variation such as that in Fig. 9.10 is defined as follows.

Definition 9.4.2 (Function fixed in space) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}^*}\left(B; H^1\left(D; \mathbb{R}\right)\right)$, and suppose

$$u'\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}\right] - \boldsymbol{\nabla} u\left(\boldsymbol{\phi}\right) \cdot \boldsymbol{\varphi} = u^*\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}\right] = 0$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Then, u is referred to as a function fixed in space. □

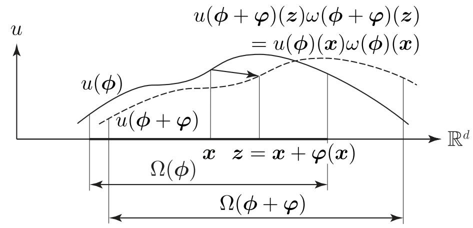


Fig. 9.11: The function $u : D \to \mathbb{R}$ varying with domain measure.

Furthermore, consider the case when along with the movement of a point on the domain, its function value changes inversely proportionate to the Jacobian $\omega(\boldsymbol{\varphi})$ of the domain. Here, the equation

$$\begin{aligned} & u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})) \\ & \quad = \frac{u(\boldsymbol{\phi})(\boldsymbol{x})}{\omega(\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))} \\ & \quad = u(\boldsymbol{\phi})(\boldsymbol{x})(1-\omega'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}](\boldsymbol{x})+o(\|\boldsymbol{\varphi}(\boldsymbol{x})\|_{\mathbb{R}^d})) \end{aligned} \tag{9.4.1}$$

holds at almost everywhere $\boldsymbol{x} \in D$, see Figure 9.11 for an illustration. Hence, using Proposition 9.2.1, the variation rule in this case is defined as follows.

Definition 9.4.3 (Function varying with domain measure) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}\left(B; L^2(D;\mathbb{R})\right)$, and suppose

$$u'(\boldsymbol{\phi})[\boldsymbol{\varphi}] + u(\boldsymbol{\phi})\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} = 0 \tag{9.4.2}$$

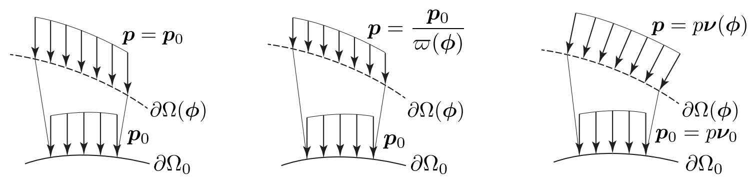
with respect to an arbitrary $\boldsymbol{\varphi} \in Y$. Then, u is called a function varying with domain measure. □

If Eq. (9.4.2) is substituted into Proposition 9.3.1, then we obtain $f'(\boldsymbol{\phi}, u(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = 0$. Hence, a function varying with a domain measure indicates that the domain integral of the function would be fixed even when the domain varies.

Moreover, if along with the movement of a point on the boundary, its function takes a value inversely proportional to the Jacobian $\varpi(\boldsymbol{\varphi})$ on the boundary, the equation

$$\begin{aligned} & u(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x})) \\ & \quad = \frac{u(\boldsymbol{\phi})(\boldsymbol{x})}{\varpi(\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))} \\ & \quad = u(\boldsymbol{\phi})(\boldsymbol{x})(1-\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}](\boldsymbol{x})+o(\|\boldsymbol{\varphi}(\boldsymbol{x})\|_{\mathbb{R}^d})) \end{aligned} \tag{9.4.3}$$

holds. Here, Proposition 9.2.4 is used in order to define the variation rule given in the following definition.



(a) $\boldsymbol{p}_0$ is constant fixed with material or fixed in space

(b) $\boldsymbol{p}_0$ is constant varying with boundary measure

(c) p is fixed with space, $\boldsymbol{\nu}$ is fixed with material (hydrostatic pressure)

Fig. 9.12: Typical variation patterns of traction $\boldsymbol{p}$ in linear elastic problem.

Definition 9.4.4 (Function varying with boundary measure) For all $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let $u \in C^1_{\mathrm{S}'}\left(B; H^1(D;\mathbb{R})\right)$ and $\partial\Omega(\boldsymbol{\phi})$ be piecewise $H^2 \cap C^{0,1}$, and suppose

$$u'(\boldsymbol{\phi})[\boldsymbol{\varphi}] + u(\boldsymbol{\phi})(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau = 0 \tag{9.4.4}$$

with respect to an arbitrary $\boldsymbol{\varphi} \in Y$ at almost every $\boldsymbol{x} \in \partial\Omega(\boldsymbol{\phi})$. Then, u is referred to as a function varying with boundary measure. Here, $\boldsymbol{\nabla}_\tau \cdot \boldsymbol{\varphi}$ follows Eq. (9.2.6). □

If Eq. (9.4.4) is substituted into Proposition 9.3.5, then we obtain $f'(\boldsymbol{\phi}, u(\boldsymbol{\phi}))[\boldsymbol{\varphi}] = 0$. In this case, it indicates the fact that the boundary integral of u remains unchanged.

Let us think about a specific problem using the definition above. Figure 9.12 shows the representative variation patterns when the traction $\boldsymbol{p}_0$ in a linear elastic problem moves to $\boldsymbol{p}$ along with the movement of the boundary. Figure 9.12 (c) represents the change in traction on a boundary when the hydrostatic pressure is acting on it. In this book, although the assumption of hydrostatic pressure will not be used directly, in order to use it in future discussion, let us obtain the shape derivative with respect to the boundary integral of hydrostatic pressure.

Proposition 9.4.5 (Derivative of integral using hydrostatic pressure) Let $p \in H^2(D;\mathbb{R})$ be a function fixed in space. For $\boldsymbol{\phi}$ in a neighborhood $B \subset Y$ of $\boldsymbol{\phi}_0 \in \mathcal{D}^\circ$, let

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi}, p) = \int_{\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})} p\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})\,\mathrm{d}\zeta$$

where $\boldsymbol{\varphi} \in Y$ is arbitrary. In this case, the shape derivative of f becomes

$$f'(\boldsymbol{\phi}, p)[\boldsymbol{\varphi}] = \int_{\Gamma(\boldsymbol{\phi})} \left\{(\boldsymbol{\nabla}p\cdot\boldsymbol{\varphi})\boldsymbol{\nu} - p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)\boldsymbol{\nu} + p(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})\boldsymbol{\nu}\right\}\mathrm{d}\gamma.$$

Moreover, $f'(\boldsymbol{\phi}, p)[\boldsymbol{\varphi}]$ belongs to $C(B;\mathcal{L}(Y;\mathbb{R}))$. □

Proof If $\Gamma(\boldsymbol{\phi}+\boldsymbol{\varphi})$ is pulled back to $\Gamma(\boldsymbol{\phi})$, then we have

$$f(\boldsymbol{\phi}+\boldsymbol{\varphi},p)=\int_{\Gamma(\boldsymbol{\phi})} p(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))\,\boldsymbol{\nu}(\boldsymbol{\phi}+\boldsymbol{\varphi})(\boldsymbol{x}+\boldsymbol{\varphi}(\boldsymbol{x}))\,\varpi(\boldsymbol{\varphi})(\boldsymbol{x})\,\mathrm{d}\gamma.$$

From the definition of the shape derivative of f,

$$f'(\boldsymbol{\phi},p)[\boldsymbol{\varphi}]=\int_{\Gamma(\boldsymbol{\phi})}\left\{\left(p'(\boldsymbol{\phi})[\boldsymbol{\varphi}]\,\boldsymbol{\nu}+p\boldsymbol{\nu}'(\boldsymbol{\phi})[\boldsymbol{\varphi}]\right)\varpi(\boldsymbol{\varphi}_0)+p\boldsymbol{\nu}\varpi'(\boldsymbol{\varphi}_0)[\boldsymbol{\varphi}]\right\}\mathrm{d}\gamma$$

can be obtained. Here, since p is fixed in space (Definition 9.4.2), then the equation $p'(\boldsymbol{\phi})[\boldsymbol{\varphi}]=\boldsymbol{\nabla}p\cdot\boldsymbol{\varphi}$ holds and if Propositions 9.2.4 and 9.2.5 are used, the desired result then follows. □

9.5 State Determination Problem

Since the definitions and formulas of shape derivatives of functions and functionals have been obtained, let us use them to define a boundary value problem of a partial differential equation which would be a state determination problem. In this chapter, a Poisson problem will be considered first for ease.

In a shape optimization problem of domain variation type, the domains of known functions and the solution function vary along with each other. Let $b_0 : D \to \mathbb{R}$, $p_{\mathrm{N}0} : D \to \mathbb{R}$, $u_{\mathrm{D}0} : D \to \mathbb{R}$ be known functions over the reference domain Ω_0, which can then be recovered through a specified variation rule with the functions $b(\boldsymbol{\phi}) : D \to \mathbb{R}$, $p_{\mathrm{N}}(\boldsymbol{\phi}) : D \to \mathbb{R}$, $u_{\mathrm{D}}(\boldsymbol{\phi}) : D \to \mathbb{R}$ defined over the perturbed domain $\Omega(\boldsymbol{\phi})$. We shall use their respective variation rules when we eventually deal with computing the shape derivative of an associated cost function.

With respect to the solution function, since it is a function of H^1 class, the Calderón extension theorem (Theorem 4.4.4) can be used to view it as a function defined on D. Hence, we define the real Hilbert space (linear space of state variables in optimal design problem) containing the homogeneous solution (given by $\tilde{u}=u-u_{\mathrm{D}}$ with a known function u_{D} providing the Dirichlet condition) for the solution of a state determination problem by

$$U(\boldsymbol{\phi})=\left\{u\in H^1(D;\mathbb{R})\;\middle|\;u=0\text{ on }\Gamma_{\mathrm{D}}(\boldsymbol{\phi})\right\} \tag{9.5.1}$$

with respect to $\boldsymbol{\phi}\in\mathcal{D}$. Furthermore, in order for the domain variation obtained from the gradient method shown later to be in $\mathcal{D}$ of Eq. (9.1.3), the admissible set of state variables for the homogeneous solution $\tilde{u}$ with respect to a state determination problem is taken to be

$$\mathcal{S}(\boldsymbol{\phi})=U(\boldsymbol{\phi})\cap W^{2,4}(D;\mathbb{R}). \tag{9.5.2}$$

The regularity which is needed in addition to the condition of $\mathcal{S}(\boldsymbol{\phi})$ will be specified when required.

The following two types of hypotheses are set with respect to regularity of known functions. When the shape derivatives are sought using formulae based on the shape derivative of a function, the following hypothesis is used later.

Hypothesis 9.5.1 (Known functions (shape derivative)) With respect to the given known functions, in a neighborhood $B \subset Y$ of $\boldsymbol{\phi} \in \mathcal{D}^{\circ}$, we assume

$$b \in C^1_{\mathrm{S}'}\left(B; C^{0,1}(D;\mathbb{R})\right), \quad p_{\mathrm{N}} \in C^1_{\mathrm{S}'}\left(B; C^{1,1}(D;\mathbb{R})\right),$$
$$u_{\mathrm{D}} \in C^1_{\mathrm{S}'}\left(B; W^{2,4}(D;\mathbb{R})\right)$$

and denote their shape derivatives as $(\,\cdot\,)'(\boldsymbol{\phi})[\boldsymbol{\varphi}]$. □

On the other hand, the following hypothesis is used when seeking the shape derivatives using the formulae based on the partial shape derivative of a function.

Hypothesis 9.5.2 (Known functions (partial shape derivative)) With respect to the given known functions, in a neighborhood $B \subset Y$ of $\boldsymbol{\phi} \in \mathcal{D}^{\circ}$, we assume

$$b \in C^1_{\mathrm{S}^*}\left(B; C^{0,1}(D;\mathbb{R})\right), \quad p_{\mathrm{N}} \in C^1_{\mathrm{S}^*}\left(B; C^{1,1}(D;\mathbb{R})\right),$$
$$u_{\mathrm{D}} \in C^1_{\mathrm{S}'}\left(B; W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})\right)$$

where $q_{\mathrm{R}} > d$, and denote their partial shape derivatives as $(\,\cdot\,)^*(\boldsymbol{\phi})[\boldsymbol{\varphi}]$. □

The following hypothesis is established with respect to regularity of the boundary.

Hypothesis 9.5.3 (Opening angle of corner point) Let $\Omega(\boldsymbol{\phi})$ be a two-dimensional domain and consider a corner point on the boundary. When $\Omega(\boldsymbol{\phi})$ is a three-dimensional domain, we consider a plane which is perpendicular to the corner line on the boundary and the corner point on the boundary in the plane. Let β be the opening angle of the corner point between two boundaries that are a Dirichlet boundary or Neumann boundary,

(1) if the boundaries are same of the type, assume $\beta < 2\pi/3$,

(2) if the boundaries are of mixed type, assume $\beta < \pi/3$.

□

If Hypotheses 9.5.1 and 9.5.3 hold, the fact that u is in $\mathcal{S}$ is shown by Proposition 5.3.1.

Using the hypotheses above, a Poisson problem of domain variation type will be defined as follows. Here, we write $\partial_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\nabla}$.

Problem 9.5.4 (Poisson problem of domain variation type) Let $\boldsymbol{\phi} \in \mathcal{D}$ and $b(\boldsymbol{\phi})$, $p_{\mathrm{N}}(\boldsymbol{\phi})$, $u_{\mathrm{D}}(\boldsymbol{\phi})$ be given. Find $u : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$\begin{aligned}
-\Delta u &= b(\boldsymbol{\phi}) \quad \text{in } \Omega(\boldsymbol{\phi}), \\
\partial_\nu u &= p_{\mathrm{N}}(\boldsymbol{\phi}) \quad \text{on } \Gamma_p(\boldsymbol{\phi}), \\
\partial_\nu u &= 0 \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}) \setminus \bar{\Gamma}_p(\boldsymbol{\phi}), \\
u &= u_{\mathrm{D}}(\boldsymbol{\phi}) \quad \text{on } \Gamma_{\mathrm{D}}(\boldsymbol{\phi}).
\end{aligned}$$

□

Here and in what follows, $b(\boldsymbol{\phi})$ or $u_{\mathrm{D}}(\boldsymbol{\phi})$ and $U(\boldsymbol{\phi})$ or $\mathcal{S}(\boldsymbol{\phi})$, etc. will be written respectively as b or u_{D} and U or $\mathcal{S}$, etc.

Problem 9.5.4 will be used as an equality constraint in the shape optimization problem (Problem 9.6.3) of domain variation type shown later. In a later argument, an equality constraint will be replaced with stationary conditions for a Lagrange function. Here, as a preparation for this, we define the Lagrange function of Problem 9.5.4 as

$$\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v) = \int_{\Omega(\boldsymbol{\phi})} (-\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + bv)\,\mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} p_{\mathrm{N}} v\,\mathrm{d}\gamma + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \{(u - u_{\mathrm{D}})\,\partial_\nu v + v\partial_\nu u\}\,\mathrm{d}\gamma, \tag{9.5.3}$$

where u is not necessarily the solution of Problem 9.5.4 and v is an element of $\mathcal{S}$ introduced as a Lagrange multiplier. In Eq. (9.5.3), the third term on the right-hand side was added in order to make the later discussions easier in a similar way to Eq. (8.2.4) in Chap. 8 defining the Lagrange function with respect to a θ-type Poisson problem. Moreover, in a similar manner to Eq. (7.2.3) defining the Lagrange function with respect to the abstract variational problem in Chap. 7, using $\tilde{u} = u - u_{\mathrm{D}}$, we write

$$\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v) = -a(\boldsymbol{\phi})(u, v) + l(\boldsymbol{\phi})(v) = -a(\boldsymbol{\phi})(\tilde{u}, v) + \hat{l}(\boldsymbol{\phi})(v), \tag{9.5.4}$$

where

$$a(\boldsymbol{\phi})(u, v) = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v\ \mathrm{d}x, \tag{9.5.5}$$

$$l(\boldsymbol{\phi})(v) = \int_{\Omega(\boldsymbol{\phi})} bv\ \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} p_{\mathrm{N}} v\ \mathrm{d}\gamma, \tag{9.5.6}$$

$$\hat{l}(\boldsymbol{\phi})(v) = l(\boldsymbol{\phi})(v) + a(\boldsymbol{\phi})(u_{\mathrm{D}}, v). \tag{9.5.7}$$

When u is the solution to Problem 9.5.4,

$$\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v) = 0$$

holds for all $v \in U$. This equation is equivalent to the weak form of Problem 9.5.4.

Following the notation in Sect. 9.3, $\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v)$ should be written as $\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u, \partial_\nu u, v, \boldsymbol{\nabla} v, \partial_\nu v)$. However, from now on, it will be written as $\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v)$.

9.6 Shape Optimization Problem of Domain Variation Type

In Sect. 9.5, we saw how the state variable $\tilde{u} = u - u_{\mathrm{D}} \in \mathcal{S}$ is determined as the solution of a state determination problem when a design variable $\boldsymbol{\phi} \in \mathcal{D}$ is given. These variables are used to define a shape optimization problem.

Here, the cost functions are set to

$$\begin{aligned} f_i(\boldsymbol{\phi}, u) = &\int_{\Omega(\boldsymbol{\phi})} \zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)\,\mathrm{d}x + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i}(\boldsymbol{\phi}, u)\,\mathrm{d}\gamma \\ &- \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u)\,\mathrm{d}\gamma - c_i, \end{aligned} \tag{9.6.1}$$

for every $i \in \{0, 1, \dots, m\}$, respectively. Here $c_1, \dots, c_m$ are constants and have to be determined such that there exists some $(\boldsymbol{\phi}, \tilde{u}) \in \mathcal{D} \times \mathcal{S}$ which satisfies $f_i \le 0$ for all $i \in \{1, \dots, m\}$. Moreover, ζ_i, $\eta_{\mathrm{N}i}$ and $\eta_{\mathrm{D}i}$ are assumed to be given and satisfy two types of hypotheses as follows. Those hypotheses will be needed to obtain an appropriate regularity in the solution of a adjoint problem (Problem 9.8.1) shown later. To calculate the second-order shape derivatives of cost functions, additional hypotheses are required. However, details of these conditions will be omitted and we shall only tacitly assume that they were already satisfied to carry out a second-order differentiation of the costs.

The following assumption is used when employing the formulae based on the shape derivative of a function.

Hypothesis 9.6.1 (Cost functions (shape derivative)) With respect to cost function f_i $(i \in \{0, 1, \dots, m\})$ of Eq. (9.6.1), let $\zeta_i \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$, $\eta_{\mathrm{N}i} \in C^1(\mathbb{R}; \mathbb{R})$, $\eta_{\mathrm{D}i} \in C^1(\mathbb{R}; \mathbb{R})$ be functions fixed with material satisfying

$$\begin{aligned} &\zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u), \zeta_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\boldsymbol{\varphi}] \in H^1 \cap L^\infty(D; \mathbb{R}), \\ &\zeta_{iu}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\hat{u}] \in L^4(D; \mathbb{R}), \quad \zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\boldsymbol{\nabla} \hat{u}] \in W^{1,4}(D; \mathbb{R}^d), \\ &\eta_{\mathrm{N}i}(\boldsymbol{\phi}, u), \eta_{\mathrm{N}i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] \in W^{2,q_{\mathrm{R}}}(D; \mathbb{R}), \quad \eta_{\mathrm{N}iu}(\boldsymbol{\phi}, u)[\hat{u}] \in W^{1,4}(D; \mathbb{R}), \\ &\eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u), \eta_{\mathrm{D}i\boldsymbol{\phi}'}(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] \in W^{1,q_{\mathrm{R}}}(D; \mathbb{R}), \\ &\eta_{\mathrm{D}i\partial_\nu u}(\boldsymbol{\phi}, \partial_\nu u)[\partial_\nu \hat{u}] \in W^{2,4}(D; \mathbb{R}) \end{aligned}$$

with respect to $(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u, \partial_\nu u) \in \mathcal{D} \times \mathcal{S} \times \mathcal{G} \times \mathcal{G}_{\Gamma_{\mathrm{D}}}$ $(\mathcal{G} = \{\boldsymbol{\nabla} u \mid u \in \mathcal{D}\}$, $\mathcal{G}_{\Gamma_{\mathrm{D}}} = \{\partial_\nu u|_{\Gamma_{\mathrm{D}}} \mid u \in \mathcal{D}\})$ and arbitrary $(\boldsymbol{\varphi}, \hat{u}) \in Y \times U$. Let $\eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u)$ be a linear function of $\partial_\nu u$. When $\eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u)$ is a nonlinear function of $\partial_\nu u$, we assume $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau = 0$ on $\Gamma_{\mathrm{D}}(\boldsymbol{\phi})$. Moreover, $(\,\cdot\,)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \,\cdot\,)[\boldsymbol{\varphi}]$ represents the shape derivatives of functions (Definition 9.1.1). □

Moreover, if the formulae based on the partial shape derivative of a function are used, the following hypothesis will be used.

Hypothesis 9.6.2 (Cost functions (partial shape derivative)) With respect to cost function f_i $(i \in \{0, 1, \dots, m\})$ of Eq. (9.6.1), let $\zeta_i \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$, $\eta_{\mathrm{N}i} \in C^1(\mathbb{R}; \mathbb{R})$, $\eta_{\mathrm{D}i} \in C^1(\mathbb{R}; \mathbb{R})$ be functions fixed in space satisfying

$$\begin{aligned} &\zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u), \zeta_{i\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\boldsymbol{\varphi}] \in W^{1,q_{\mathrm{R}}(D;\mathbb{R})}, \\ &\zeta_{iu}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\hat{u}] \in L^{2q_{\mathrm{R}}}(D; \mathbb{R}), \end{aligned}$$

$$\zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)[\boldsymbol{\nabla}\hat{u}] \in W^{1,2q_{\mathrm{R}}}\left(D;\mathbb{R}^d\right),$$
$$\eta_{\mathrm{N}i}(\boldsymbol{\phi}, u), \eta_{\mathrm{N}i\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u)[\boldsymbol{\varphi}] \in W^{2,q_{\mathrm{R}}}(D;\mathbb{R}), \quad \eta_{\mathrm{N}iu}(\boldsymbol{\phi}, u)[\hat{u}] \in W^{1,2q_{\mathrm{R}}}(D;\mathbb{R}),$$
$$\eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u), \eta_{\mathrm{D}i\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \partial_\nu u)[\boldsymbol{\varphi}] \in W^{1,q_{\mathrm{R}}}(D;\mathbb{R}),$$
$$\eta_{\mathrm{D}i\partial_\nu u}(\boldsymbol{\phi}, \partial_\nu u)[\partial_\nu\hat{u}] \in W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})$$

with respect to $(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u, \partial_\nu u) \in \mathcal{D} \times \mathcal{S} \times \mathcal{G} \times \mathcal{G}_{\Gamma_{\mathrm{D}}}$ ($\mathcal{G} = \{\boldsymbol{\nabla} u \mid u \in \mathcal{D}\}$, $\mathcal{G}_{\Gamma_{\mathrm{D}}} = \left\{ \partial_\nu u|_{\Gamma_{\mathrm{D}}} \mid u \in \mathcal{D}\right\}$) and arbitrary $(\boldsymbol{\varphi}, \hat{u}) \in Y \times U$. Let $\eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u)$ be a linear function of $\partial_\nu u$ and be written as $\eta_{\mathrm{D}i\partial_\nu u}(\boldsymbol{\phi}, \partial_\nu u) = v_{\mathrm{D}i}$. Moreover, $(\,\cdot\,)_{\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \,\cdot\,)[\boldsymbol{\varphi}]$ represents the partial shape derivatives of functions (Definition 9.1.3). □

These cost functions are used to define a shape optimization problem of domain variation type as follows.

Problem 9.6.3 (Shape optimization of domain variation type) Let $\mathcal{D}$ and $\mathcal{S}$ be defined as Eq. (9.1.3) and Eq. (9.5.2), respectively. Also, let f_0, ..., f_m is defined by Eq. (9.6.1). Find $\Omega(\boldsymbol{\phi})$ which satisfies

$$\min_{(\boldsymbol{\phi}, u-u_{\mathrm{D}}) \in \mathcal{D}\times\mathcal{S}} \Big\{ f_0(\boldsymbol{\phi}, u) \mid f_1(\boldsymbol{\phi}, u) \le 0, \ldots, f_m(\boldsymbol{\phi}, u) \le 0, \text{ Problem 9.5.4} \Big\}.$$

□

In what follows, we will look at the Fréchet derivatives of cost functions and the KKT conditions with respect to a shape optimization problem (Problem 9.6.3) of domain variation type. In this respect, Lagrange functions based on several definitions will be used. Here, their relationships are summarized in order to avoid confusion. Let the Lagrange function with respect to the shape optimization problem (Problem 9.6.3) of domain variation type be

$$\begin{aligned}&\mathscr{L}(\boldsymbol{\phi}, u, v_0, v_1, \ldots, v_m, \lambda_1, \ldots, \lambda_m) \\ &\quad = \mathscr{L}_0(\boldsymbol{\phi}, u, v_0) + \sum_{i\in\{1,\ldots,m\}} \lambda_i \mathscr{L}_i(\boldsymbol{\phi}, u, v_i),\end{aligned} \tag{9.6.2}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_m\}^\top \in \mathbb{R}^m$ is a Lagrange multiplier with respect to $f_1(\boldsymbol{\phi}, u) \le 0, \ldots, f_m(\boldsymbol{\phi}, u) \le 0$. Furthermore, if f_i is a functional of u for all $i \in \{0, 1, \ldots, m\}$, and in view of the fact that the state determination problem (Problem 9.5.4) is an equality constraint, the functional

$$\begin{aligned}&\mathscr{L}_i(\boldsymbol{\phi}, u, v_i) \\ &\quad = f_i(\boldsymbol{\phi}, u) + \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v_i) \\ &\quad = \int_{\Omega(\boldsymbol{\phi})} \left(\zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u) - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i + b v_i\right) \mathrm{d}x \\ &\qquad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i}(\boldsymbol{\phi}, u)\,\mathrm{d}\gamma + \int_{\Gamma_p(\boldsymbol{\phi})} p_{\mathrm{N}} v_i\,\mathrm{d}\gamma\end{aligned}$$

$$+\int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})}\left\{\left(u-u_{\mathrm{D}}\right)\partial_{\nu}v_i+v_i\partial_{\nu}u-\eta_{\mathrm{D}i}\left(\boldsymbol{\phi},\partial_{\nu}u\right)\right\}\mathrm{d}\gamma-c_i \tag{9.6.3}$$

is called the Lagrange function of $f_i(\boldsymbol{\phi},u)$. Here, $\mathscr{L}_{\mathrm{S}}$ is the Lagrange function of the state determination problem defined by Eq. (9.5.3). Moreover, v_i is introduced as a Lagrange multiplier with respect to the state determination problem corresponding to f_i and $\tilde{v}_i=v_i-\eta_{\mathrm{D}i\partial_{\nu}u}$ is assumed to be an element of $\mathcal{S}$. Similarly to u, if a variation $\hat{v}_i$ of $\tilde{v}_i$ is to be considered, $\hat{v}_i$ is contained in U.

9.7 Existence of an Optimum Solution

The existence of an optimum solution of Problem 9.6.3 can be confirmed in the same fashion as in Chap. 8. To use Theorem 7.4.4 in Chap. 7, we will show the compactness of

$$\mathcal{F}=\left\{\left(\boldsymbol{\phi},u\left(\boldsymbol{\phi}\right)\right)\in\mathcal{D}\times\mathcal{S}\mid\text{Problem 9.5.4}\right\} \tag{9.7.1}$$

and the continuity of f_0. Hereinafter, we let $\tilde{u}=u-u_{\mathrm{D}}\in U$.

The compactness of $\mathcal{F}$ is presented in the following lemma [29, Lemma 2.5, p. 27, Lemma 2.15, p. 55, Lemma 2.20, p. 63].

Lemma 9.7.1 (Compactness of $\mathcal{F}$) Suppose that Hypothesis 9.5.1 and Hypothesis 9.5.3 are satisfied. Moreover, $\bar{\Gamma}_0=\Gamma_{p0}\cup\Gamma_{\eta 00}\cup\Gamma_{\eta 10}\cup\cdots\cup\Gamma_{\eta m0}$ is (not piecewise) $H^3\cap C^{1,1}$ class. With respect to an arbitrary Cauchy sequence $\boldsymbol{\phi}_n\to\boldsymbol{\phi}$ which is uniformly convergent in $\mathcal{D}$ and their solutions $\tilde{u}_n=\tilde{u}\left(\boldsymbol{\phi}_n\right)\in U$ $(n\to\infty)$ of Problem 9.5.4, the convergence

$$\tilde{u}_n\to\tilde{u}\quad\text{strongly in }U$$

holds, and $\tilde{u}=\tilde{u}\left(\boldsymbol{\phi}\right)\in U$ solves Problem 9.5.4. □

Proof Concerning the solution $\tilde{u}_n$ of Problem 9.5.4 for $\boldsymbol{\phi}_n$,

$$\alpha_n\left\|\tilde{u}_n\right\|_U^2\leq a\left(\boldsymbol{\phi}_n\right)\left(\tilde{u}_n,\tilde{u}_n\right)=\hat{l}\left(\boldsymbol{\phi}_n\right)\left(\tilde{u}_n\right)\leq\left\|\hat{l}\left(\boldsymbol{\phi}_n\right)\right\|_{U'}\left\|\tilde{u}_n\right\|_U$$

holds, where $a\left(\boldsymbol{\phi}_n\right)$ and $\hat{l}\left(\boldsymbol{\phi}_n\right)$ are defined in Eq. (9.5.4), and α_n is a positive constant used in the definition of coerciveness for $a\left(\boldsymbol{\phi}_n\right)$ (see (1) in the answer to Exercise 5.2.5). When $\phi_n\to\phi$ is uniformly convergent in $\mathcal{D}$, α_n can be replaced by a positive constant α not depending on n. The norm $\left\|\hat{l}\left(\boldsymbol{\phi}_n\right)\right\|_{U'}=\left\|l\left(\boldsymbol{\phi}_n\right)+a\left(\boldsymbol{\phi}_n\right)\left(u_{\mathrm{D}},\,\cdot\right)\right\|_{U'}$ ($l\left(\boldsymbol{\phi}_n\right)$ defined in Eq. (9.5.4)) being bounded can be shown using (3) in the answer to Exercise 5.2.5 by replacing $\hat{l}(v)$ and Ω in Exercise 5.2.5 by $\hat{l}\left(\boldsymbol{\phi}_n\right)(v)$ and $\Omega\left(\boldsymbol{\phi}_n\right)$, respectively. Hence, there exists a subsequence such that $\tilde{u}_n\to\tilde{u}$ weakly in U.

Next, we will show that $\tilde{u}$ solves Problem 9.5.4 for $\boldsymbol{\phi}$. From the definition of Problem 9.5.4,

$$\lim_{n\to\infty}a\left(\boldsymbol{\phi}_n\right)\left(\tilde{u}_n,v\right)=\lim_{n\to\infty}\hat{l}\left(\boldsymbol{\phi}_n\right)(v) \tag{9.7.2}$$

holds with respect to an arbitrary $v \in U$. From Hypothesis 9.5.2, the right-hand side of Eq. (9.7.2) becomes

$$\lim_{n \to \infty} \hat{l}(\phi_n)(v) = \hat{l}(\phi)(v). \tag{9.7.3}$$

Indeed,

$$\begin{aligned}
&\left| \hat{l}(\phi_n)(v) - \hat{l}(\phi)(v) \right| \\
&\quad \le \left| \int_{\Omega(\phi_n)} b(\phi_n) v \, \mathrm{d}x - \int_{\Omega(\phi)} b(\phi) v \, \mathrm{d}x \right| \\
&\qquad + \left| \int_{\Gamma_p(\phi_n)} p_{\mathrm{N}}(\phi_n) v \, \mathrm{d}\gamma - \int_{\Gamma_p(\phi)} p_{\mathrm{N}}(\phi) v \, \mathrm{d}\gamma \right| \\
&\qquad + \left| \int_{\Omega(\phi_n)} \nabla u_{\mathrm{D}}(\phi_n) \cdot \nabla v \, \mathrm{d}x - \int_{\Omega(\phi)} \nabla u_{\mathrm{D}}(\phi) \cdot \nabla v \, \mathrm{d}x \right|
\end{aligned} \tag{9.7.4}$$

holds. The first term in the right-hand side of Eq. (9.7.4) becomes

$$\begin{aligned}
&\left| \int_D \left(\chi_{\Omega(\phi_n)} b(\phi_n) - \chi_{\Omega(\phi)} b(\phi) \right) v \, \mathrm{d}x \right| \\
&\quad \le \left| \int_D \chi_{\Omega(\phi)} \left(b(\phi_n) - b(\phi) \right) v \, \mathrm{d}x \right| + \left| \int_D \left(\chi_{\Omega(\phi_n)} - \chi_{\Omega(\phi)} \right) b(\phi_n) v \, \mathrm{d}x \right|,
\end{aligned}$$

where χ_Ω denotes the characteristic function such that $\chi_\Omega : D \to \mathbb{R}$ ($\chi_\Omega(\Omega) = 1$, $\chi_\Omega(D \setminus \bar{\Omega}) = 0$). Using $b \in C^1_{S'}(B; C^{0,1}(D; \mathbb{R}))$ in Hypothesis 9.5.1 and the property [34, Proposition 2.2.28, p. 45]

$$\chi_{\Omega(\phi_n)} \to \chi_{\Omega(\phi)} \quad \text{in } L^\infty(D; \mathbb{R})\text{-weak}^*, \tag{9.7.5}$$

the first term in the right-hand side of Eq. (9.7.4) converges to zero. It can also be shown that the third term in the right-hand side of Eq. (9.7.4) converges to zero using $u_{\mathrm{D}} \in C^1_{S'}(B; W^{2,4}(D; \mathbb{R}))$ and Eq. (9.7.5).

The convergence to zero of the second term in the right-hand side of Eq. (9.7.4) can be confirmed in the following way. Here, we modify the condition for $\Gamma_p(\phi_n)$ in $\mathcal{D}$ defined in Eq. (9.1.3) (a class of $H^3 \cap C^{1,1}$) as follows. $\Gamma_p(\phi_n)$ can be defined using a function $\sigma(\phi_n)(\xi) = \sigma_n(\xi)$ of a parameter $\xi \in \Xi = (0,1)^{d-1}$ as

$$\begin{aligned}
\Gamma_p(\phi_n) &= \tilde{\Gamma}_p(\sigma_n) \\
&= \Bigg\{ \sigma_n \in H^3 \cap C^{1,1}\left(\Xi; \mathbb{R}^d\right) \;\Bigg|\; \|\sigma_n\|_{\mathbb{R}^d} \le c_0, \\
&\qquad c_1 \le \left\| \nabla_\xi \sigma_n^\top \right\|_{\mathbb{R}^{(d-1) \times d}} \le c_2, \\
&\qquad \left\| \nabla_\xi^{|\beta|} \sigma_n^\top \right\|_{\mathbb{R}^{(d-1)^2 \times d}} \le c_3 \ (|\beta| = 2) \text{ a.e. in } \Xi \Bigg\},
\end{aligned} \tag{9.7.6}$$

where $\nabla_\xi = (\partial / \partial \xi_i)_i$ and $c_0, \ldots, c_3$ are positive constants. Hereafter, $\omega_{\Xi n}$ and ω_Ξ denote $\|\nabla_\xi \sigma_n^\top\|_{\mathbb{R}^{(d-1) \times d}}$ and $\|\nabla_\xi \sigma^\top\|_{\mathbb{R}^{(d-1) \times d}}$, respectively. Moreover, let $\tilde{p}_{\mathrm{N}}(t) = p_{\mathrm{N}}(t\phi_n + (1-t)\phi)$ ($t \in [0,1]$). Here, using $\phi_n \to \phi$ (uniformly convergent in $\mathcal{D}$), boundedness of the trace operator $\|\gamma_{\Gamma_p(\phi)}\|$ (Eq. (5.2.4)), the result in [15, Corollary

1] and $p_{\mathrm{N}} \in C^1_{\mathrm{S}'}\left(B; C^{1,1}\left(D;\mathbb{R}\right)\right)$ in Hypothesis 9.5.1, the second term of the right-hand side of Eq. (9.7.4) becomes

$$
\begin{aligned}
&\left| \int_{\tilde{\Gamma}_p(\boldsymbol{\sigma}_n)} p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) v \,\mathrm{d}\gamma - \int_{\tilde{\Gamma}_p(\boldsymbol{\sigma})} p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) v \,\mathrm{d}\gamma \right| \\
&\quad \leq \left| \int_{\Xi} \left\{ \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}_n\right)\left(v \circ \boldsymbol{\sigma}_n\right) \omega_{\Xi n} - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right)\left(v \circ \boldsymbol{\sigma}\right) \omega_{\Xi} \right\} \mathrm{d}\sigma \right| \\
&\quad \leq \left| \int_{\Xi} \left\{ \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}_n\right) - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}\right) \right\} \left(v \circ \boldsymbol{\sigma}_n\right) \omega_{\Xi n} \,\mathrm{d}\sigma \right| \\
&\qquad + \left| \int_{\Xi} \left\{ \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}\right) - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right) \right\} \left(v \circ \boldsymbol{\sigma}_n\right) \omega_{\Xi n} \,\mathrm{d}\sigma \right| \\
&\qquad + \left| \int_{\Xi} \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right)\left(v \circ \boldsymbol{\sigma}_n - v \circ \boldsymbol{\sigma}\right) \omega_{\Xi n} \,\mathrm{d}\sigma \right| \\
&\qquad + \left| \int_{\Xi} \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right)\left(v \circ \boldsymbol{\sigma}\right)\left(\omega_{\Xi n} - \omega_{\Xi}\right) \mathrm{d}\sigma \right| \\
&\quad \leq \sqrt{c_2} \left\| v \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi}_n);\mathbb{R}\right)} \left\| \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}_n\right) - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}\right) \right\|_{L^2(\Xi;\mathbb{R})} \\
&\qquad + \sqrt{\frac{c_2}{c_1}} \left\| v \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi}_n);\mathbb{R}\right)} \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) - p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi});\mathbb{R}\right)} \\
&\qquad + \sqrt{c_2} \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi});\mathbb{R}\right)} \left\| v \circ \boldsymbol{\sigma}_n - v \circ \boldsymbol{\sigma} \right\|_{L^2(\Xi;\mathbb{R})} \\
&\qquad + \frac{1}{c_1} \left\| \omega_{\Xi n} - \omega_{\Xi} \right\|_{H^2 \cap C^{0,1}\left(\Xi;\mathbb{R}^d\right)} \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi});\mathbb{R}\right)} \left\| v \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi});\mathbb{R}\right)} \\
&\quad \leq \sqrt{c_2} \left\| \gamma_{\Gamma_p(\boldsymbol{\phi})} \right\|^2 \left\| v \right\|_U \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \right\|_{C^{1,1}(D;\mathbb{R})} \left\| \boldsymbol{\sigma}_n - \boldsymbol{\sigma} \right\|_{H^3 \cap C^{1,1}\left(\Xi;\mathbb{R}^d\right)} \\
&\qquad + \sqrt{\frac{c_2}{c_1}} \left\| \gamma_{\Gamma_p(\boldsymbol{\phi})} \right\|^2 \left\| v \right\|_U \sup_{t \in [0,1]} \left\| \tilde{p}_{\mathrm{N}}'\left(t\right) \right\|_{C^{1,1}(D;\mathbb{R})} \left\| \boldsymbol{\phi}_n - \boldsymbol{\phi} \right\|_X \\
&\qquad + \sqrt{c_2} \left\| \gamma_{\Gamma_p(\boldsymbol{\phi})} \right\|^2 \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{C^{1,1}(D;\mathbb{R})} \left\| v \right\|_U \left\| \boldsymbol{\sigma}_n - \boldsymbol{\sigma} \right\|_{H^3 \cap C^{1,1}\left(\Xi;\mathbb{R}^d\right)} \\
&\qquad + \frac{1}{c_1} \left\| \gamma_{\Gamma_p(\boldsymbol{\phi})} \right\|^2 \left\| \omega_{\Xi n} - \omega_{\Xi} \right\|_{H^2 \cap C^{0,1}\left(\Xi;\mathbb{R}^d\right)} \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{C^{1,1}(D;\mathbb{R})} \left\| v \right\|_U \\
&\quad \to 0 \quad (n \to \infty).
\end{aligned} \tag{9.7.7}
$$

In Eq. (9.7.7), we used the relations

$$
\begin{aligned}
&\left| \int_{\Xi} \left(v \circ \boldsymbol{\sigma}\right) \omega_{\Xi} \,\mathrm{d}\sigma \right| \leq \sqrt{c_2} \left(\int_{\Xi} \left(v \circ \boldsymbol{\sigma}\right)^2 \omega_{\Xi} \,\mathrm{d}\sigma \right)^{1/2} = \sqrt{c_2} \left\| v \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi}_n);\mathbb{R}\right)}, \\
&\left| \int_{\Xi} \left\{ \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}\right) - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right) \right\} \mathrm{d}\sigma \right| \\
&\quad \leq \frac{1}{\sqrt{c_1}} \left(\int_{\Xi} \left\{ \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) \circ \boldsymbol{\sigma}\right) - \left(p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \circ \boldsymbol{\sigma}\right) \right\}^2 \omega_{\Xi} \,\mathrm{d}\sigma \right)^{1/2} \\
&\quad = \frac{1}{\sqrt{c_1}} \left\| p_{\mathrm{N}}\left(\boldsymbol{\phi}_n\right) - p_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \right\|_{L^2\left(\Gamma_p(\boldsymbol{\phi});\mathbb{R}\right)}.
\end{aligned}
$$

Using the results above, Eq. (9.7.3) is proved.

The left-hand side of Eq. (9.7.2) becomes

$$
\lim_{n \to \infty} a\left(\boldsymbol{\phi}_n\right)\left(u_n, v\right) = a\left(\boldsymbol{\phi}\right)\left(u, v\right). \tag{9.7.8}
$$

It can be confirmed by

$$\begin{aligned}
&\left|a\left(\boldsymbol{\phi}_n\right)\left(\tilde{u}_n, v\right)-a\left(\boldsymbol{\phi}\right)\left(\tilde{u}, v\right)\right| \\
&\quad=\left|\int_{\Omega\left(\boldsymbol{\phi}_n\right)} \boldsymbol{\nabla} \tilde{u}_n \cdot \boldsymbol{\nabla} v \, \mathrm{d} x-\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} \tilde{u} \cdot \boldsymbol{\nabla} v \, \mathrm{d} x\right| \\
&\quad=\left|\int_D\left(\chi_{\Omega\left(\boldsymbol{\phi}_n\right)} \boldsymbol{\nabla} \tilde{u}_n-\chi_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla} \tilde{u}\right) \cdot \boldsymbol{\nabla} v \, \mathrm{d} x\right| \\
&\quad\leq\left|\int_D \chi_{\Omega(\boldsymbol{\phi})}\left(\boldsymbol{\nabla} \tilde{u}_n-\boldsymbol{\nabla} \tilde{u}\right) \cdot \boldsymbol{\nabla} v \, \mathrm{d} x\right| \\
&\qquad+\left|\int_D\left(\chi_{\Omega\left(\boldsymbol{\phi}_n\right)}-\chi_{\Omega(\boldsymbol{\phi})}\right) \boldsymbol{\nabla} \tilde{u}_n \cdot \boldsymbol{\nabla} v \, \mathrm{d} x\right| .
\end{aligned} \tag{9.7.9}$$

To the right-hand side of Eq. (9.7.9), we adopt $\tilde{u}_n \rightarrow \tilde{u}$ weakly in U and the property Eq. (9.7.5) for the characteristic function, and obtain Eq. (9.7.8). Substituting Eq. (9.7.3) and Eq. (9.7.8) into Eq. (9.7.2), the weak form of Problem 9.5.4 can be obtained. Namely, $\tilde{u}=\tilde{u}(\boldsymbol{\phi}) \in U$ is the solution of Problem 9.5.4.

Since the weak convergence was shown, then to prove the strong convergence of $\{u_n\}_{n \in \mathbb{N}}$ to u, it is sufficient to show that

$$\|u_n\|_U \rightarrow\|u\|_U \quad(n \rightarrow \infty). \tag{9.7.10}$$

Indeed, when using $a(\boldsymbol{\phi})$ in Eq. (9.5.5) and taking

$$|||v|||=a(\boldsymbol{\phi})(v, v)$$

as a norm on U, we have

$$\begin{aligned}
|||u_n||| &=a(\boldsymbol{\phi})\left(u_n, u_n\right)=\int_D\left(\chi_{\Omega(\boldsymbol{\phi})}-\chi_{\Omega\left(\boldsymbol{\phi}_n\right)}\right) \boldsymbol{\nabla} u_n \cdot \boldsymbol{\nabla} u_n \, \mathrm{d} x+a\left(\boldsymbol{\phi}_n\right)\left(u_n, u_n\right) \\
&=\int_D\left(\chi_{\Omega(\boldsymbol{\phi})}-\chi_{\Omega\left(\boldsymbol{\phi}_n\right)}\right) \boldsymbol{\nabla} u_n \cdot \boldsymbol{\nabla} u_n \, \mathrm{d} x+l\left(\boldsymbol{\phi}_n\right)\left(u_n\right) \\
&\rightarrow l(\boldsymbol{\phi})(u)=|||u||| \quad(n \rightarrow \infty).
\end{aligned} \tag{9.7.11}$$

Then, $u_n \rightarrow u$ strongly in U is proved. □

We consider that the condition of $\tilde{u}(\boldsymbol{\phi})$ included in $\mathcal{S}$ is guaranteed in the setting of Problem 9.5.4 satisfying Hypotheses 9.5.1 and 9.5.3.

The latter assumption in Theorem 7.4.4 (continuity of f_0) means that f_0 is continuous on

$$S=\left\{(\boldsymbol{\phi}, \tilde{u}(\boldsymbol{\phi})) \in \mathcal{F} \mid f_1(\boldsymbol{\phi}, u(\boldsymbol{\phi})) \leq 0, \cdots, f_m(\boldsymbol{\phi}, u(\boldsymbol{\phi})) \leq 0\right\}. \tag{9.7.12}$$

S depends on the problem setting. Then, we will confirm the continuity of f_0 by showing the continuity of f_i $(i \in \{0,1, \ldots, m\})$ by the following lemma and assuming that S is not empty.

Lemma 9.7.2 (Continuity of f_i) Let f_i be defined as in Eq. (9.6.1) under Hypothesis 9.6.1. Let $u_n \rightarrow u$ strongly in U be determined by Lemma 9.7.1 with respect to an arbitrary Cauchy sequence $\boldsymbol{\phi}_n \rightarrow \boldsymbol{\phi}$ in X which is uniformly convergent in $\mathcal{D}$, and satisfy $\|\partial_\nu u_n-\partial_\nu u\|_{L^2(\Gamma_{\mathrm{D}} ; \mathbb{R})} \rightarrow 0$ $(n \rightarrow \infty)$ on Γ_{D}. Then, f_i is continuous with respect to $\boldsymbol{\phi} \in \mathcal{D}$. □

Proof The proof will be completed when

$$
\begin{aligned}
&|f_i(\boldsymbol{\phi}_n, u_n) - f_i(\boldsymbol{\phi}, u)| \\
&\quad \le \left| \int_{\Omega(\boldsymbol{\phi}_n)} \zeta_i(\boldsymbol{\phi}_n, u_n, \boldsymbol{\nabla} u_n)\,\mathrm{d}x - \int_{\Omega(\boldsymbol{\phi})} \zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)\,\mathrm{d}x \right| \\
&\qquad + \left| \int_{\Gamma_{\eta i}(\boldsymbol{\phi}_n)} \eta_{\mathrm{N}i}(\boldsymbol{\phi}_n, u_n)\,\mathrm{d}\gamma - \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i}(\boldsymbol{\phi}, u)\,\mathrm{d}\gamma \right| \\
&\qquad + \left| \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi}_n)} \eta_{\mathrm{D}i}(\boldsymbol{\phi}_n, \partial_\nu u_n)\,\mathrm{d}\gamma - \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \eta_{\mathrm{D}i}(\boldsymbol{\phi}, \partial_\nu u)\,\mathrm{d}\gamma \right| \\
&\quad = e_\Omega + e_{\Gamma_\eta} + e_{\Gamma_{\mathrm{D}}} \to 0 \quad (n \to \infty)
\end{aligned} \tag{9.7.13}
$$

is shown with respect to $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ which is uniformly convergent in $\mathcal{D}$. For e_Ω,

$$
\begin{aligned}
e_\Omega &\le \left| \int_D \left(\chi_{\Omega(\boldsymbol{\phi}_n)} - \chi_{\Omega(\boldsymbol{\phi})}\right) \zeta_i(\boldsymbol{\phi}_n, u_n, \boldsymbol{\nabla} u_n)\,\mathrm{d}x \right| \\
&\quad + \left| \int_D \chi_{\Omega(\boldsymbol{\phi})} \left(\zeta_i(\boldsymbol{\phi}_n, u_n, \boldsymbol{\nabla} u_n) - \zeta_i(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)\right)\mathrm{d}x \right| \\
&= e_{\Omega 1} + e_{\Omega 2}
\end{aligned}
$$

holds. $e_{\Omega 1}$ converges to zero by Eq. (9.7.5). For $e_{\Omega 2}$, using $\tilde{u}_n \to \tilde{u}$ weakly in U and ζ_i, notation $\tilde{\zeta}_i(t) = \zeta_i(t\boldsymbol{\phi}_n + (1-t)\boldsymbol{\phi}, tu_n + (1-t)n, t\boldsymbol{\nabla} u_n + (1-t)\boldsymbol{\nabla} u)$ $(t \in [0,1])$ and $\zeta_i \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ in Hypothesis 9.6.1,

$$
\begin{aligned}
e_{\Omega 2} &\le \sup_{t\in[0,1]} \left| \int_{\Omega(\boldsymbol{\phi})} \tilde{\zeta}_{i\boldsymbol{\phi}'}(t)\,[\boldsymbol{\phi}_n - \boldsymbol{\phi}]\,\mathrm{d}x \right| + \sup_{t\in[0,1]} \left| \int_{\Omega(\boldsymbol{\phi})} \tilde{\zeta}_{iu}(t)\,[u_n - u]\,\mathrm{d}x \right| \\
&\quad + \sup_{t\in[0,1]} \left| \int_{\Omega(\boldsymbol{\phi})} \tilde{\zeta}_{i\boldsymbol{\nabla} u}(t)\,[\boldsymbol{\nabla} u_n - \boldsymbol{\nabla} u]\,\mathrm{d}x \right| \\
&\le \sup_{t\in[0,1]} \left\| \tilde{\zeta}_{i\boldsymbol{\phi}'}(t) \right\|_{H^1 \cap L^\infty(D;\mathbb{R})} \|\boldsymbol{\phi}_n - \boldsymbol{\phi}\|_X \\
&\quad + \sup_{t\in[0,1]} \left\| \tilde{\zeta}_{iu}(t) \right\|_{L^4(D;\mathbb{R})} \|u_n - u\|_U \\
&\quad + \sup_{t\in[0,1]} \left\| \tilde{\zeta}_{i\boldsymbol{\nabla} u}(t) \right\|_{W^{1,4}(D;\mathbb{R})} \|\boldsymbol{\nabla} u_n - \boldsymbol{\nabla} u\|_{L^2(D;\mathbb{R})} \\
&\to 0 \quad (n \to \infty)
\end{aligned}
$$

holds. The convergence of e_{Γ_η} to zero can be shown as follows. Assuming a similar condition to Eq. (9.7.6) for $\Gamma_{\eta i}(\boldsymbol{\phi}_n)$, $\Gamma_{\eta i}(\boldsymbol{\phi}_n)$ can be represented with the parameter $\boldsymbol{\sigma}_n(\boldsymbol{\xi})$ $(\boldsymbol{\xi} \in \Xi = (0,1)^{d-1})$ as $\tilde{\Gamma}_{\eta i}(\boldsymbol{\sigma}_n)$. Using notation $\tilde{\eta}_{\mathrm{N}i}(t) = \eta_{\mathrm{N}i}(t\boldsymbol{\phi}_n + (1-t)\boldsymbol{\phi}, tu_n + (1-t)u)$ $(t \in [0,1])$, $\tilde{u}_n \to \tilde{u}$ weakly in U, boundedness of the trace operator, the result in [15, Corollary 1] and $\eta_{\mathrm{N}i} \in W^{2,q_{\mathrm{R}}}(D;\mathbb{R})$ in Hypothesis 9.6.1, we have

$$
\begin{aligned}
e_{\Gamma_\eta} &= \left| \int_{\tilde{\Gamma}_{\eta i}(\boldsymbol{\sigma}_n)} \eta_{\mathrm{N}i}(\boldsymbol{\phi}_n, u_n)\,\mathrm{d}\gamma - \int_{\tilde{\Gamma}_{\eta i}(\boldsymbol{\sigma})} \eta_{\mathrm{N}i}(\boldsymbol{\phi}, u)\,\mathrm{d}\gamma \right| \\
&\le \left| \int_\Xi \left\{ (\eta_{\mathrm{N}i}(\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}_n)\,\omega_{\Xi n} - (\eta_{\mathrm{N}i}(\boldsymbol{\phi}, u) \circ \boldsymbol{\sigma})\,\omega_\Xi \right\} \mathrm{d}\sigma \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left| \int_{\Xi} \{ (\eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}_n) - (\eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}) \} \, \omega_{\Xi n} \, \mathrm{d}\sigma \right| \\
&\quad + \left| \int_{\Xi} \{ (\eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}) - (\eta_{\mathrm{N}i} (\boldsymbol{\phi}, u_n) \circ \boldsymbol{\sigma}) \} \, \omega_{\Xi n} \, \mathrm{d}\sigma \right| \\
&\quad + \left| \int_{\Xi} \{ (\eta_{\mathrm{N}i} (\boldsymbol{\phi}, u_n) \circ \boldsymbol{\sigma}) - (\eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \circ \boldsymbol{\sigma}) \} \, \omega_{\Xi n} \, \mathrm{d}\sigma \right| \\
&\quad + \left| \int_{\Xi} (\eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \circ \boldsymbol{\sigma}) \left(\omega_{\Xi n} - \omega_{\Xi} \right) \mathrm{d}\sigma \right| \\
&\leq \sqrt{c_2} \left\| (\eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}_n) - (\eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \circ \boldsymbol{\sigma}) \right\|_{L^2(\Xi;\mathbb{R})} \\
&\quad + \sqrt{\frac{c_2}{c_1}} \left\| \eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) - \eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \right\|_{L^2\left(\Gamma_{\eta i}(\boldsymbol{\phi});\mathbb{R}\right)} \\
&\quad + \sqrt{\frac{c_2}{c_1}} \left\| \eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) - \eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \right\|_{L^2\left(\Gamma_{\eta i}(\boldsymbol{\phi});\mathbb{R}\right)} \\
&\quad + \frac{1}{c_1} \left\| \omega_{\Xi n} - \omega_{\Xi} \right\|_{H^2 \cap C^{0,1}\left(\Xi;\mathbb{R}^d\right)} \left\| \eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \right\|_{L^2\left(\Gamma_{\eta i}(\boldsymbol{\phi});\mathbb{R}\right)} \\
&\leq \sqrt{c_2} \left\| \gamma_{\Gamma_{\eta i}(\boldsymbol{\phi})} \right\| \left\| \eta_{\mathrm{N}i} (\boldsymbol{\phi}_n, u_n) \right\|_{W^{2,q_{\mathrm{R}}}(D;\mathbb{R})} \left\| \boldsymbol{\sigma}_n - \boldsymbol{\sigma} \right\|_{C^{1,1}\left(\Xi;\mathbb{R}^d\right)} \\
&\quad + \sqrt{\frac{c_2}{c_1}} \left\| \gamma_{\Gamma_{\eta i}(\boldsymbol{\phi})} \right\| \sup_{t \in [0,1]} \left\| \tilde{\eta}_{\mathrm{N}i\boldsymbol{\phi}} (t) \right\|_{W^{2,q_{\mathrm{R}}}(D;\mathbb{R})} \left\| \boldsymbol{\phi}_n - \boldsymbol{\phi} \right\|_X \\
&\quad + \sqrt{\frac{c_2}{c_1}} \left\| \gamma_{\Gamma_{\eta i}(\boldsymbol{\phi})} \right\| \sup_{t \in [0,1]} \left\| \tilde{\eta}_{\mathrm{N}iu} (t) \right\|_{W^{2,q_{\mathrm{R}}}(D;\mathbb{R})} \left\| u_n - u \right\|_U \\
&\quad + \frac{1}{c_1} \left\| \gamma_{\Gamma_{\eta i}(\boldsymbol{\phi})} \right\| \left\| \omega_{\Xi n} - \omega_{\Xi} \right\|_{H^2 \cap C^{0,1}\left(\Xi;\mathbb{R}^d\right)} \left\| \eta_{\mathrm{N}i} (\boldsymbol{\phi}, u) \right\|_{W^{2,q_{\mathrm{R}}}(D;\mathbb{R})} \\
&\to 0 \quad (n \to \infty),
\end{aligned}
\tag{9.7.14}
$$

where c_1 and c_2 are positive constants when Eq. (9.7.6) is rewritten for $\Gamma_{\eta i}(\boldsymbol{\phi}_n)$. $e_{\Gamma_{\mathrm{D}}} \to 0$ $(n \to \infty)$ can be shown using $\|\partial_\nu u_n - \partial_\nu u\|_{L^2(\Gamma_{\mathrm{D}};\mathbb{R})} \to 0$ $(n \to \infty)$ in a similar way. Based on the results above, Eq. (9.7.13) is shown. □

In Theorem 7.4.4 showing the existence of a solution in the abstract optimum design problem, the first assumption (compactness of $\mathcal{F}$) was confirmed by Lemma 9.7.1. The second assumption (continuity of f_0) can be satisfied with the conditions for Lemma 9.7.2 and the assumption that S is not empty. Then, under the conditions, it can be assured that there exists an optimum solution of Problem 9.6.3.

Regarding the solution of Problem 9.6.3, let us recall the similar situation of Remark 8.4.3 in Chap. 8. In the definition of $\mathcal{D}$ shown in Eq. (9.1.3), a side constraint $\|\boldsymbol{\phi}\|_{H^2 \cap C^{0,1}(D;\mathbb{R}^d)} \leq \beta$ is added. When this condition becomes active, we have to deal this condition as an inequality condition. Depending on the setting of the problem, we may meet a situation such that a boundary converges to a shape with sharp corners which is not a Lipschitz boundary. In this case, a converged shape can be obtained by activating the side constraint. Moreover, regarding the selection of X and $\mathcal{D}$, the same situation as Remark 8.4.4 holds.

9.8 Derivatives of Cost Functions

In this chapter, we consider the solution of the shape optimization problem (Problem 9.6.3) of domain variation type using a gradient method and a Newton method. In order to use the gradient method, the first-order shape derivatives of cost functions are necessary. Moreover, if the Newton method is to be used, the second-order shape derivatives (Hessians) of the cost functions are required. Here, let us obtain the first and second-order shape derivatives of the cost functions f_i using the Lagrange multiplier method shown in Section 7.5.2 and the method shown in 7.5.3, respectively. In this case, let us look at the methods using the formulae based on the shape derivative of a function separately from the method using the formulae based on the partial shape derivative of a function shown in Sect. 9.3. However, with respect to the second-order shape derivatives, only the results using the method with the formulae based on the shape derivative of a function will be shown.

9.8.1 Shape Derivative of f_i Using Formulae Based on Shape Derivative of a Function

Firstly, let us use the formulae based on the shape derivative of a function (Sect. 9.3.1) to obtain the Fréchet derivative of $\mathscr{L}_i$ and use its stationary conditions to seek the shape derivative of f_i.

The Fréchet derivative of $\mathscr{L}_i\left(\boldsymbol{\phi}, u, v_i\right)$ is written as

$$\begin{aligned}&\mathscr{L}_i'\left(\boldsymbol{\phi}, u, v_i\right)\left[\boldsymbol{\varphi}, \hat{u}, \hat{v}_i\right]\\&\quad=\mathscr{L}_{i\boldsymbol{\phi}'}\left(\boldsymbol{\phi}, u, v_i\right)\left[\boldsymbol{\varphi}\right]+\mathscr{L}_{iu}\left(\boldsymbol{\phi}, u, v_i\right)\left[\hat{u}\right]+\mathscr{L}_{iv_i}\left(\boldsymbol{\phi}, u, v_i\right)\left[\hat{v}_i\right]\end{aligned} \tag{9.8.1}$$

with respect to an arbitrary $(\boldsymbol{\varphi}, \hat{u}, \hat{v}_i) \in X \times U \times U$. Here, the notations in Eq. (9.3.5) and Eq. (9.3.15) are used. In this case, the shape derivative u' used in Eq. (9.3.5) and Eq. (9.3.15) following Definition 9.1.1 was replaced with an arbitrary $\hat{u} \in X$, because it was assumed that u is not necessarily the solution of Problem 9.5.4 in the definition of the Lagrange function. Let us look at each term in detail below.

The third term on the right-hand side of Eq. (9.8.1) becomes

$$\mathscr{L}_{iv_i}\left(\boldsymbol{\phi}, u, v_i\right)\left[\hat{v}_i\right]=\mathscr{L}_{\mathrm{S}v_i}\left(\boldsymbol{\phi}, u, v_i\right)\left[\hat{v}_i\right]=\mathscr{L}_{\mathrm{S}}\left(\boldsymbol{\phi}, u, \hat{v}_i\right). \tag{9.8.2}$$

Equation (9.8.2) is the Lagrange function of the state determination problem (Problem 9.5.4). Hence, if u is a weak solution of the state determination problem, its term is zero.

Moreover, the second term on the right-hand side of Eq. (9.8.1) becomes

$$\begin{aligned}&\mathscr{L}_{iu}\left(\boldsymbol{\phi}, u, v_i\right)\left[\hat{u}\right]\\&\quad=\int_{\Omega(\boldsymbol{\phi})}\left(-\boldsymbol{\nabla}\hat{u}\cdot\boldsymbol{\nabla}v_i+\zeta_{iu}\left(\boldsymbol{\phi}, u, \boldsymbol{\nabla}u\right)\hat{u}+\zeta_{i(\boldsymbol{\nabla}u)^{\top}}\left(\boldsymbol{\phi}, u, \boldsymbol{\nabla}u\right)\boldsymbol{\nabla}\hat{u}\right)\mathrm{d}x\\&\qquad+\int_{\Gamma_{\eta i}(\boldsymbol{\phi})}\eta_{\mathrm{N}iu}\left(\boldsymbol{\phi}, u\right)\hat{u}\,\mathrm{d}\gamma\end{aligned}$$

$$+ \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \{\hat{u}\partial_\nu v_i + (v_i - \eta_{\mathrm{D}i\partial_\nu u}(\boldsymbol{\phi}, \partial_\nu u))\partial_\nu \hat{u}\} \,\mathrm{d}\gamma. \tag{9.8.3}$$

Here, if v_i can be determined so that Eq. (9.8.3) equates to zero with respect to an arbitrary $\hat{u} \in U$, the second term on the right-hand side of Eq. (9.8.1) also vanishes. From the fact that

$$\begin{aligned}
&\int_{\Omega(\boldsymbol{\phi})} \left(\zeta_{i(\boldsymbol{\nabla} u)^\top}(u, \boldsymbol{\nabla} u)\,\boldsymbol{\nabla}\hat{u} - \boldsymbol{\nabla}\hat{u}\cdot\boldsymbol{\nabla} v_i\right) \mathrm{d}x \\
&\quad = \int_{\partial\Omega(\boldsymbol{\phi})} \hat{u}\left(\zeta_{i(\boldsymbol{\nabla} u)^\top} - \boldsymbol{\nabla} v_i\right)\cdot\boldsymbol{\nu}\,\mathrm{d}\gamma - \int_{\Omega(\boldsymbol{\phi})} \hat{u}\boldsymbol{\nabla}\cdot\left(\zeta_{i(\boldsymbol{\nabla} u)^\top} - \boldsymbol{\nabla} v_i\right)\mathrm{d}x
\end{aligned}$$

holds if $v_i \in W^{2,4}(D;\mathbb{R})$ is assumed, its strong form can be written as follows.

Problem 9.8.1 (Adjoint problem with respect to f_i) When the solution u to Problem 9.5.4 with respect to $\boldsymbol{\phi} \in \mathcal{D}$ is obtained, find $v_i : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$\begin{aligned}
-\Delta v_i &= \zeta_{iu}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u) - \boldsymbol{\nabla}\cdot\zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u) \quad \text{in } \Omega(\boldsymbol{\phi}), \\
\partial_\nu v_i &= \eta_{\mathrm{N}iu}(\boldsymbol{\phi}, u) + \zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)\cdot\boldsymbol{\nu} \quad \text{on } \Gamma_{\eta i}(\boldsymbol{\phi}), \\
\partial_\nu v_i &= \zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\phi}, u, \boldsymbol{\nabla} u)\cdot\boldsymbol{\nu} \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}) \setminus \bar{\Gamma}_{\eta i}(\boldsymbol{\phi}), \\
v_i &= \eta_{\mathrm{D}i\partial_\nu u}(\boldsymbol{\phi}, \partial_\nu u) \quad \text{on } \Gamma_{\mathrm{D}}(\boldsymbol{\phi}).
\end{aligned}$$

□

Here, the admissible set of adjoint variables for $v_i - \eta_{\mathrm{D}i\partial_\nu u}$ is taken to be $\mathcal{S}$ in order to obtain a regular solution of the shape optimization problem of domain variation type. In Hypothesis 9.6.1, the regularities for ζ_{iu}, $\zeta_{i(\boldsymbol{\nabla} u)^\top}$, $\eta_{\mathrm{N}iu}$ and $\eta_{\mathrm{D}i\partial_\nu u}$ were given to obtain this result.

Furthermore, the first term on the right-hand side of Eq. (9.8.1) becomes

$$\begin{aligned}
&\mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\boldsymbol{\nabla} u\cdot\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)\boldsymbol{\nabla} v_i\right\} + \boldsymbol{\nabla} v_i\cdot\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)\boldsymbol{\nabla} u\right\} \\
&\qquad - \zeta_{i(\boldsymbol{\nabla} u)^\top}\cdot\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)\boldsymbol{\nabla} u\right\} + (\zeta_i - \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v_i + b v_i)\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} \\
&\qquad + \left(\zeta_{i\boldsymbol{\phi}'} + u b'\right)\cdot\boldsymbol{\varphi}\Big]\,\mathrm{d}x \\
&\quad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \left(\kappa\eta_{\mathrm{N}i}\boldsymbol{\nu}\cdot\boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau\eta_{\mathrm{N}i}\cdot\boldsymbol{\varphi}_\tau + \eta_{\mathrm{N}i\boldsymbol{\phi}'}\cdot\boldsymbol{\varphi}\right)\mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_{\eta i}(\boldsymbol{\phi})\cup\Theta_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\mathrm{d}\varsigma \\
&\quad + \int_{\Gamma_p(\boldsymbol{\phi})} \left\{\kappa p_{\mathrm{N}} v_i\boldsymbol{\nu}\cdot\boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau(p_{\mathrm{N}} v_i)\cdot\boldsymbol{\varphi}_\tau + v_i p_{\mathrm{N}}'\cdot\boldsymbol{\varphi}\right\}\mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta_p(\boldsymbol{\phi})} p_{\mathrm{N}} v_i\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\mathrm{d}\varsigma
\end{aligned}$$

$$+ \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \left[\{ (u - u_{\mathrm{D}}) \, w(\boldsymbol{\varphi}, v_i) + (v_i - \eta_{\mathrm{D}i\partial_\nu u}) \, w(\boldsymbol{\varphi}, u) \} \right.$$
$$\left. + \{ (u - u_{\mathrm{D}}) \, \partial_\nu v_i + v_i \partial_\nu u - \eta_{\mathrm{D}i} (\boldsymbol{\phi}, \partial_\nu u) \} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau + \eta_{\mathrm{D}i\boldsymbol{\phi}'} \cdot \boldsymbol{\varphi} \right] \mathrm{d}\gamma \tag{9.8.4}$$

using the formulae of Eq. (9.3.5), representing the result of Proposition 9.3.4, and Eq. (9.3.15), representing the result of Proposition 9.3.7. Here, $w(\boldsymbol{\varphi}, u)$ and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ follow Eq. (9.3.12) and Eq. (9.2.6), respectively. Moreover, the fact that $\Gamma_p(\boldsymbol{\phi})$ and $\Gamma_{\eta i}(\boldsymbol{\phi})$ are piecewise $H^3 \cap C^{1,1}$ (assumed in the definition of $\mathcal{D}$) was used to obtain the integral on $\Gamma_p(\boldsymbol{\phi})$ and $\Gamma_{\eta i}(\boldsymbol{\phi})$.

Bearing the above results in mind, when u and v_i are the weak solutions of Problem 9.5.4 and Problem 9.8.1, respectively, and the Dirichlet conditions corresponding to these problems, as well as the condition for $\eta_{\mathrm{D}i}$ in Hypothesis 9.6.1 hold, the integral on $\Gamma_{\mathrm{D}}(\boldsymbol{\phi})$ on Eq. (9.8.4) will be zero except the term of $\eta_{\mathrm{D}i\boldsymbol{\phi}'} \cdot \boldsymbol{\varphi}$. Hence, using the notation of Eq. (7.5.15) for $\tilde{f}_i$, we obtain

$$\begin{aligned} \tilde{f}_i'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_i, \boldsymbol{\varphi} \rangle \\ &= \int_{\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{G}_{\Omega i} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) + g_{\Omega i} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} + \boldsymbol{g}_{\zeta b i} \cdot \boldsymbol{\varphi} \right\} \mathrm{d}x \\ &\quad + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{g}_{pi} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta_p(\boldsymbol{\phi})} \boldsymbol{g}_{\partial pi} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma \\ &\quad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \boldsymbol{g}_{\eta i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\partial\Gamma_{\eta i}(\boldsymbol{\phi}) \cup \Theta_{\eta i}(\boldsymbol{\phi})} \boldsymbol{g}_{\partial \eta i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma \\ &\quad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \boldsymbol{g}_{\mathrm{D}i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma, \end{aligned} \tag{9.8.5}$$

where

$$\boldsymbol{G}_{\Omega i} = \boldsymbol{\nabla} u (\boldsymbol{\nabla} v_i)^\top + \boldsymbol{\nabla} v_i (\boldsymbol{\nabla} u)^\top - \zeta_{i(\boldsymbol{\nabla} u)^\top} (\boldsymbol{\nabla} u)^\top, \tag{9.8.6}$$
$$g_{\Omega i} = \zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i + b v_i, \tag{9.8.7}$$
$$\boldsymbol{g}_{\zeta b i} = \zeta_{i\boldsymbol{\phi}'} + u b', \tag{9.8.8}$$
$$\boldsymbol{g}_{pi} = \kappa p_{\mathrm{N}} v_i \boldsymbol{\nu} - \sum_{j \in \{1,\dots,d-1\}} \{ \boldsymbol{\tau}_j \cdot \boldsymbol{\nabla} (p_{\mathrm{N}} v_i) \} \boldsymbol{\tau}_j + v_i p_{\mathrm{N}}', \tag{9.8.9}$$
$$\boldsymbol{g}_{\partial pi} = p_{\mathrm{N}} v_i \boldsymbol{\tau}, \tag{9.8.10}$$
$$\boldsymbol{g}_{\eta i} = \kappa \eta_{\mathrm{N}i} \boldsymbol{\nu} - \sum_{j \in \{1,\dots,d-1\}} (\boldsymbol{\tau}_j \cdot \boldsymbol{\nabla} \eta_{\mathrm{N}i}) \boldsymbol{\tau}_j + \eta_{\mathrm{N}i\boldsymbol{\phi}'}, \tag{9.8.11}$$
$$\boldsymbol{g}_{\partial \eta i} = \eta_{\mathrm{N}i} \boldsymbol{\tau}, \tag{9.8.12}$$
$$\boldsymbol{g}_{\mathrm{D}i} = \eta_{\mathrm{D}i\boldsymbol{\phi}'}. \tag{9.8.13}$$

In this book, the scalar product of $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{d \times d}$ and $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{d \times d}$, $\sum_{(i,j) \in \{1,\dots,d\}^2} a_{ij} b_{ij}$ is written as $\boldsymbol{A} \cdot \boldsymbol{B}$. Moreover, in deriving Eq. (9.8.5), the identity

$$\boldsymbol{a} \cdot (\boldsymbol{B} \boldsymbol{c}) = \left(\boldsymbol{B}^\top \boldsymbol{a} \right) \cdot \boldsymbol{c} = \left(\boldsymbol{a} \boldsymbol{c}^\top \right) \cdot \boldsymbol{B} \tag{9.8.14}$$

with respect to $\boldsymbol{a} \in \mathbb{R}^d$, $\boldsymbol{B} \in \mathbb{R}^{d \times d}$ and $\boldsymbol{c} \in \mathbb{R}^d$ was used. Hereinafter, these relationships will be used without explanation.

Using the results above, the following results regarding $\boldsymbol{g}_i$ of Eq. (9.8.5) can be obtained.

Theorem 9.8.2 (Shape derivative $\boldsymbol{g}_i$ of f_i) Let $\boldsymbol{\phi} \in \mathcal{D}$, b, p_{N}, u_{D}, ζ_i, $\eta_{\mathrm{N}i}$ and $\eta_{\mathrm{D}i}$ be given as functions fixed with the material satisfying Hypotheses 9.5.1, 9.5.3 and 9.6.1. Moreover, let u and v_i be the weak solutions of the state determination problem (Problem 9.5.4) and the adjoint problem (Problem 9.8.1) with respect to f_i, respectively, and are both in $\mathcal{S}$ of Eq. (9.5.2). When $\boldsymbol{g}_{\partial p i}$ and $\boldsymbol{g}_{\partial \eta i}$ in Eq. (9.8.10) and Eq. (9.8.12), respectively, are zero, the shape derivative of f_i becomes Eq. (9.8.5) and $\boldsymbol{g}_i$ is in X'. Furthermore, we have

$$
\begin{aligned}
\boldsymbol{G}_{\Omega i} &\in H^1 \cap L^\infty \left(\Omega\left(\boldsymbol{\phi}\right); \mathbb{R}^{d \times d}\right), \\
g_{\Omega i} &\in H^1 \cap L^\infty \left(\Omega\left(\boldsymbol{\phi}\right); \mathbb{R}\right), \\
\boldsymbol{g}_{\zeta b i} &\in H^1 \cap L^\infty \left(\Omega\left(\boldsymbol{\phi}\right); \mathbb{R}^{d}\right), \\
\boldsymbol{g}_{p i} &\in H^{1/2} \cap L^\infty \left(\Gamma_p\left(\boldsymbol{\phi}\right); \mathbb{R}^{d}\right), \\
\boldsymbol{g}_{\eta i} &\in H^{1/2} \cap L^\infty \left(\Gamma_{\eta i}\left(\boldsymbol{\phi}\right); \mathbb{R}^{d}\right), \\
\boldsymbol{g}_{\mathrm{D} i} &\in H^{1/2} \cap L^\infty \left(\Gamma_{\mathrm{D}}\left(\boldsymbol{\phi}\right); \mathbb{R}^{d}\right).
\end{aligned}
$$

□

Proof The fact that the shape derivative of f_i becomes $\boldsymbol{g}_i$ of Eq. (9.8.5) is as seen above. The following holds with respect to the regularity of $\boldsymbol{g}_i$. With respect to the first term on $\boldsymbol{G}_{\Omega i}$, from Hölder's inequality (Theorem A.9.1) and the corollary of Poincaré inequality (Corollary A.9.4), the inequalities

$$
\begin{aligned}
&\left\| \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^\top \right\} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right) \right\|_{L^1(\Omega(\boldsymbol{\phi}); \mathbb{R})} \\
&\quad \le \left\| \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^\top \right\|_{L^2\left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d \times d}\right)} \left\| \boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right\|_{L^2\left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d \times d}\right)} \\
&\quad \le \left\| \boldsymbol{\nabla} u \right\|_{L^4\left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d}\right)} \left\| \boldsymbol{\nabla} v_i \right\|_{L^4\left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d}\right)} \left\| \boldsymbol{\nabla} \boldsymbol{\varphi}^\top \right\|_{L^2\left(\Omega(\boldsymbol{\phi}); \mathbb{R}^{d \times d}\right)} \\
&\quad \le \left\| u \right\|_{W^{1,4}(D; \mathbb{R})} \left\| v_i \right\|_{W^{1,4}(D; \mathbb{R})} \left\| \boldsymbol{\varphi} \right\|_X \\
&\quad \le \left\| u \right\|_{W^{2,4}(D; \mathbb{R})} \left\| v_i \right\|_{W^{2,4}(D; \mathbb{R})} \left\| \boldsymbol{\varphi} \right\|_X
\end{aligned}
$$

hold. From the assumptions, the right-hand side is finite. Hence, $\boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^\top$ is in X'. Moreover, in view of the inequalities above, $\boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^\top$ is also contained in $H^1 \cap L^\infty \left(\Omega\left(\boldsymbol{\phi}\right); \mathbb{R}^{d \times d}\right)$. A similar result can be obtained with respect to other terms of $\boldsymbol{G}_{\Omega i}$. A similar result is also obtained with respect to $g_{\Omega i}$. The result for $\boldsymbol{g}_{\zeta b i}$ is obvious from Hypotheses 9.5.1 and 9.6.1.

The regularity of $\boldsymbol{g}_{p i}$ depends on the regularity of $\boldsymbol{\nu}$ and κ in addition to regularities of v_i and p_{N}. With respect to the first term on the right-hand side of Eq. (9.8.9), we have

$$
\left\| \kappa p_{\mathrm{N}} v_i \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \right\|_{L^1\left(\Gamma_p(\boldsymbol{\phi}); \mathbb{R}\right)}
$$

$$
\begin{aligned}
&\leq \|\kappa p_{\mathrm{N}} v_i \boldsymbol{\nu}\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \\
&\leq \|\kappa\|_{H^{1/2}\cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \|p_{\mathrm{N}}\|_{L^4(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \|v_i\|_{L^4(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \\
&\quad \times \|\boldsymbol{\nu}\|_{H^{3/2}\cap C^{0,1}(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \\
&\leq \|\gamma_{\partial\Omega}\|^3 \|p_{\mathrm{N}}\|_{W^{1,4}(\Omega(\boldsymbol{\phi});\mathbb{R})} \|v_i\|_{W^{1,4}(\Omega(\boldsymbol{\phi});\mathbb{R})} \\
&\quad \times \|\kappa\|_{H^{1/2}\cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \|\boldsymbol{\nu}\|_{H^{3/2}\cap C^{0,1}(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_X \\
&\leq \|\gamma_{\partial\Omega}\|^3 \|p_{\mathrm{N}}\|_{C^{1,1}(\Omega(\boldsymbol{\phi});\mathbb{R})} \|v_i\|_{W^{2,4}(\Omega(\boldsymbol{\phi});\mathbb{R})} \\
&\quad \times \|\kappa\|_{H^{1/2}\cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R})} \|\boldsymbol{\nu}\|_{H^{3/2}\cap C^{0,1}(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_X
\end{aligned}
$$

using Hölder's inequality (Theorem A.9.1) and the trace theorem (Theorem 4.4.2). Here,

$$
\gamma_{\partial\Omega} : H^1\left(\Omega(\boldsymbol{\phi});\mathbb{R}^d\right) \to H^{1/2}\left(\partial\Omega(\boldsymbol{\phi});\mathbb{R}^d\right)
$$

is a trace operator and its operator norm $\|\gamma_{\partial\Omega}\|$ is bounded from the fact that the boundary $\partial\Omega(\boldsymbol{\phi})$ is Lipschitz. Moreover, $\Gamma_p(\boldsymbol{\phi})$ is defined to be piecewise $H^3 \cap C^{1,1}$ in $\mathcal{D}$ of Eq. (9.1.3). Hence, $\boldsymbol{\nu}$ is in the class of $H^{3/2} \cap C^{0,1}$ and κ is in the class of $H^{1/2} \cap L^\infty$ on $\Gamma_p(\boldsymbol{\phi})$. Therefore, $\kappa p_{\mathrm{N}} v_i \boldsymbol{\nu}$ is an element of X' and is in $H^{1/2} \cap H^{1/2}(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d) \cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)$. The second term on the right-hand side of Eq. (9.8.9) belongs to $H^{1/2} \cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)$ because $\boldsymbol{\tau}_1(\boldsymbol{\phi}), \ldots, \boldsymbol{\tau}_{d-1}(\boldsymbol{\phi})$ is in the class $H^{3/2} \cap C^{0,1}$, $p_{\mathrm{N}} \in C^{1,1}(D;\mathbb{R})$ (Hypothesis 9.5.1) and $v_i \in W^{2,4}(D;\mathbb{R})$ on $\Gamma_p(\boldsymbol{\phi})$ (Practice **9.1**). The third term of Eq. (9.8.9) becomes $v_i p_{\mathrm{N}}' \in W^{2,4}(D;\mathbb{R})$. Hence, $\boldsymbol{g}_{pi} \in H^{1/2} \cap L^\infty(\Gamma_p(\boldsymbol{\phi});\mathbb{R}^d)$ is shown.

For the regularities of $\boldsymbol{g}_{\eta i}$, the same result as for $\boldsymbol{g}_{pi}$ can be obtained from $\boldsymbol{\nabla}\eta_{\mathrm{N}i} \in W^{1,q_{\mathrm{R}}}(D;\mathbb{R})$. Moreover, the result for $\boldsymbol{g}_{\mathrm{D}i}$ is obvious from Hypotheses 9.6.1. Therefore, the result of the theorem is established.

In addition, we assumed $\boldsymbol{g}_{\partial pi} = \mathbf{0}_{\mathbb{R}^d}$ because the trace of $\boldsymbol{\varphi} \in X$ on $\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta_p(\boldsymbol{\phi})$ can not be defined. Similarly, $\boldsymbol{g}_{\partial\eta i} = \mathbf{0}_{\mathbb{R}^d}$ was assumed. □

9.8.2 Second-Order Shape Derivative of f_i Using Formulae Based on Shape Derivative of a Function

Let us obtain the second-order shape derivative of the cost function based on the method shown in Section 7.5.3. Here, the formulae using the shape derivative of a function is used.

In order to obtain the second-order shape derivative of $\tilde{f}_i$, the following assumptions are established.

Hypothesis 9.8.3 (Second-order shape derivative of $\tilde{f}_i$) With respect to the state determination problem (Problem 9.5.4) and the cost function f_i defined in Eq. (9.6.1), the following assumptions are made, respectively:

(1) $b = 0$, $\zeta_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, \boldsymbol{\nabla}u)[\boldsymbol{\varphi}] = 0$.

(2) ζ_i is not a function of $\boldsymbol{\phi}$ and u, but is a bilinear form of $\boldsymbol{\nabla}u$.

(3) Equations (9.8.9) to (9.8.13) are zero, or $\tilde{\Gamma}_0 = \Gamma_{p0} \cup \Gamma_{\eta i0} \in \bar{\Omega}_{C0}$ in Eq. (9.1.1).

□

The Lagrange function $\mathscr{L}_i$ of f_i is defined by Eq. (9.6.3). Viewing $(\boldsymbol{\phi}, u)$ as a design variable and putting its admissible set and admissible direction set as

$$S = \{(\boldsymbol{\phi}, u) \in \mathcal{D} \times \mathcal{S} \mid \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, v) = 0 \text{ for all } v \in U\},$$
$$T_S(\boldsymbol{\phi}, u) = \{(\boldsymbol{\varphi}, \hat{v}) \in X \times U \mid \mathscr{L}_{\mathrm{S}\boldsymbol{\phi}u}(\boldsymbol{\phi}, u, v)[\boldsymbol{\varphi}, \hat{v}] = 0 \text{ for all } v \in U\},$$

the second-order Fréchet partial derivative of $\mathscr{L}_i$ with respect to arbitrary variations $(\boldsymbol{\varphi}_1, \hat{v}_1), (\boldsymbol{\varphi}_2, \hat{v}_2) \in T_S(\boldsymbol{\phi}, u)$ of $(\boldsymbol{\phi}, u) \in S$, similarly to Eq. (7.5.21), and considering Eq. (9.1.6), becomes

$$\begin{aligned}
&\mathscr{L}_{i(\boldsymbol{\phi}', u)(\boldsymbol{\phi}', u)}(\boldsymbol{\phi}, u, v_i)[(\boldsymbol{\varphi}_1, \hat{v}_1), (\boldsymbol{\varphi}_2, \hat{v}_2)] \\
&\quad = \left(\mathscr{L}_{0(\boldsymbol{\phi}', u)}\right)_{(\boldsymbol{\phi}', u)}(\boldsymbol{\phi}, u, v_i)[(\boldsymbol{\varphi}_1, \hat{v}_1), (\boldsymbol{\varphi}_2, \hat{v}_2)] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \left(\mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1] + \mathscr{L}_{iu}(\boldsymbol{\phi}, u, v_i)[\hat{v}_1]\right)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \left(\mathscr{L}_{i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1] + \mathscr{L}_{iu}(\boldsymbol{\phi}, u, v_i)[\hat{v}_1]\right)_{u}[\hat{v}_2] \\
&\qquad + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \left(\mathscr{L}_{i\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \mathscr{L}_{i\boldsymbol{\phi}'u}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1, \hat{v}_2] \\
&\qquad + \mathscr{L}_{i\boldsymbol{\phi}'u}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_2, \hat{v}_1] + \mathscr{L}_{iuu}(\boldsymbol{\phi}, u, v_i)[\hat{v}_1, \hat{v}_2] \\
&\qquad + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle,
\end{aligned} \tag{9.8.15}$$

where $\langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle$ follows the definition given in Eq. (9.1.8) (or Eq. (9.3.10)).

The first and fifth terms on the right-hand side of Eq. (9.8.15) become

$$\begin{aligned}
&\left(\mathscr{L}_{i\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\left\{\boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \left\{\boldsymbol{\nabla} v_i \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} u\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] - \left\{\zeta_{i(\boldsymbol{\nabla} u)^\top} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} u\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + (\zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \left\{\boldsymbol{\nabla} u (\boldsymbol{\nabla} v_i)^\top + \boldsymbol{\nabla} v_i (\boldsymbol{\nabla} u)^\top - \zeta_{i(\boldsymbol{\nabla} u)^\top}(\boldsymbol{\nabla} u)^\top\right\} \\
&\qquad\quad \cdot \left\{\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top - \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)\right\} \\
&\qquad + (\zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top - (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)\right\}\Big] \mathrm{d}x,
\end{aligned} \tag{9.8.16}$$

by using the first term on the right-hand side of Eq. (9.8.4) and Eq. (9.3.11). The first integrand on the right-hand side of Eq. (9.8.16) can be expressed as follows:

$$\left\{\boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2]$$

$$
\begin{aligned}
&= - \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \right\}_{\boldsymbol{\nabla} u} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} u \right) \\
&\quad - \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \right\}_{\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right) \\
&\quad - \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \right\}_{\boldsymbol{\nabla} v_i} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} v_i \right) \\
&\quad + \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \right\} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \\
&= - \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} u \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) - \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \\
&\quad - \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} v_i \right) + \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \\
&= - \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top \right\} \cdot \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right\} - \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top \right\} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right) \\
&\quad - \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top \right\} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right) + \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top \right\} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right).
\end{aligned}
\tag{9.8.17}
$$

In Eq. (9.8.17), the identities in Eq. (9.8.14) and

$$
\begin{aligned}
&\boldsymbol{A} \cdot (\boldsymbol{B}\boldsymbol{C}) = \left(\boldsymbol{B}^\top \boldsymbol{A} \right) \cdot \boldsymbol{C} = \left(\boldsymbol{A}\boldsymbol{C}^\top \right) \cdot \boldsymbol{B}, \\
&(\boldsymbol{A}\boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{B} \cdot \left(\boldsymbol{A}^\top \boldsymbol{C} \right) = \boldsymbol{A} \cdot \left(\boldsymbol{C}\boldsymbol{B}^\top \right),
\end{aligned}
\tag{9.8.18}
$$

where $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, $\boldsymbol{B} \in \mathbb{R}^{d \times d}$ and $\boldsymbol{C} \in \mathbb{R}^{d \times d}$, were used. In the remaining part, those identities will be used frequently.

Similarly, the second integrand on the right-hand side of Eq. (9.8.16) is similar to Eq. (9.8.17) with u and v_i interchanged. Meanwhile, the third integrand on the right-hand side of Eq. (9.8.16) becomes similar to Eq. (9.8.17) with u and v_i interchanged and v_i and $\zeta_{i(\boldsymbol{\nabla} u)^\top}$ interchanged. Lastly, the fourth integrand on the right-hand side of Eq. (9.8.16) becomes

$$
\begin{aligned}
&\left(\zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i \right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right)_{\boldsymbol{\phi}'} [\boldsymbol{\varphi}_2] \\
&\quad = \left(\zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i \right) \left\{ - \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right) \right\}.
\end{aligned}
$$

Hence, Eq. (9.8.16) becomes

$$
\begin{aligned}
&\left(\mathscr{L}_{i\boldsymbol{\phi}'} \right)_{\boldsymbol{\phi}'} (\boldsymbol{\phi}, u, v_i) [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0 (\boldsymbol{\phi}), \boldsymbol{t} (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[- \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i \right)^\top + \boldsymbol{\nabla} v_i \left(\boldsymbol{\nabla} u \right)^\top - \zeta_{i(\boldsymbol{\nabla} u)^\top} \left(\boldsymbol{\nabla} u \right)^\top \right\} \\
&\qquad \cdot \left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right\} \Big] \mathrm{d}x.
\end{aligned}
\tag{9.8.19}
$$

Next, we look at the second term on the right-hand side of Eq. (9.8.15). If the first term on the right-hand side of Eq. (9.8.4) is used, we get

$$
\begin{aligned}
&\mathscr{L}_{i\boldsymbol{\phi}' u} (\boldsymbol{\phi}, u, v_i) [\boldsymbol{\varphi}_1, \hat{v}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{\nabla} \hat{v}_2 \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} v_i \right) + \left(\boldsymbol{\nabla} v_i - \zeta_{i(\boldsymbol{\nabla} u)^\top} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \hat{v}_2 \right) \right.
\end{aligned}
$$

$$- \left(\boldsymbol{\nabla} \hat{v}_2 \cdot \boldsymbol{\nabla} v_i\right) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \Big\} \mathrm{d}x. \tag{9.8.20}$$

On the other hand, the variation of u satisfying the state determination problem with respect to an arbitrary domain variation $\boldsymbol{\varphi}_j \in Y$ for $j \in \{1, 2\}$ is given as $\hat{v}_j = v' (\boldsymbol{\phi}) \left[\boldsymbol{\varphi}_j\right]$. If the Fréchet partial derivative of the Lagrange function $\mathscr{L}_{\mathrm{S}}$ of the state determination problem defined by Eq. (9.5.3) is taken, we obtain

$$\begin{aligned}
&\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}' u} (\boldsymbol{\phi}, u, v) \left[\boldsymbol{\varphi}_j, \hat{v}_j\right] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left\{ \boldsymbol{\nabla} u \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top} \boldsymbol{\nabla} v\right) + \boldsymbol{\nabla} v \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top} \boldsymbol{\nabla} u\right) \right. \\
&\qquad \left. - \left(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v\right) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_j - \boldsymbol{\nabla} \hat{v}_j \cdot \boldsymbol{\nabla} v \right\} \mathrm{d}x \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left[\left\{ \left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top}\right)^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_j \right) \boldsymbol{\nabla} u - \boldsymbol{\nabla} \hat{v}_j \right\} \cdot \boldsymbol{\nabla} v \right] \mathrm{d}x \\
&\quad = 0
\end{aligned} \tag{9.8.21}$$

for all $\boldsymbol{v} \in U$. Here, Hypothesis 9.8.3 and the fact that v and $\hat{v}_j$ are both zero on Γ_{D} were used. From Eq. (9.8.21), we get

$$\boldsymbol{\nabla} \hat{v}_j = \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top}\right)^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_j^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_j \right\} \boldsymbol{\nabla} u. \tag{9.8.22}$$

This relation becomes possible by the following argument. Substituting Eq. (9.8.22) into Eq. (9.8.20), the second term on the right-hand side of Eq. (9.8.15) becomes

$$\begin{aligned}
&\mathscr{L}_{i\boldsymbol{\phi}' u} (\boldsymbol{\phi}, u, v_i) \left[\boldsymbol{\varphi}_1, \hat{v}_2\right] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left[\left\{ \left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top}\right)^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^{\top} \right. \right. \\
&\qquad \left. + \left(\boldsymbol{\nabla} v_i - \zeta_{i(\boldsymbol{\nabla} u)^{\top}} \right) \left(\boldsymbol{\nabla} u\right)^{\top} \left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top}\right)^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \right\} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^{\top} \\
&\qquad \left. - \left\{ \left(\left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top}\right)^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \boldsymbol{\nabla} u \right) \cdot \boldsymbol{\nabla} v_i \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right] \mathrm{d}x \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left[\left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^{\top} + \boldsymbol{\nabla} v_i \left(\boldsymbol{\nabla} u\right)^{\top} - \zeta_{i(\boldsymbol{\nabla} u)^{\top}} \left(\boldsymbol{\nabla} u\right)^{\top} \right\} \right. \\
&\qquad \cdot \left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}_1^{\top} \boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top} + \boldsymbol{\nabla} \boldsymbol{\varphi}_1^{\top} \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top}\right)^{\top} - \boldsymbol{\nabla} \boldsymbol{\varphi}_1^{\top} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2\right) \right\} \\
&\qquad - \left\{ \boldsymbol{\nabla} u \left(\boldsymbol{\nabla} v_i\right)^{\top} \right\} \cdot \left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\right) + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^{\top}\right)^{\top} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\right) \right\} \\
&\qquad \left. + \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2\right) \right] \mathrm{d}x.
\end{aligned} \tag{9.8.23}$$

Similarly, the third term on the right-hand side of Eq. (9.8.15) becomes $\mathscr{L}_{iu\boldsymbol{\phi}'} (\boldsymbol{\phi}, u, v_i) \left[\boldsymbol{\varphi}_2, \boldsymbol{\nabla} \hat{v}_1\right]$ similar to Eq. (9.8.23) with $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ interchanged. Lastly, the fourth term on the right-hand side of Eq. (9.8.15) vanishes.

Summarizing the results above, the second-order shape derivative of $\tilde{f}_i$ becomes

$$\begin{aligned} & h_i (\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{u}) [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] \\ & \quad = \int_{\Omega(\boldsymbol{\phi})} \Big[2 \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i (\boldsymbol{u}) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \\ & \qquad + \left\{ \boldsymbol{\nabla} u (\boldsymbol{\nabla} v_i)^\top + \boldsymbol{\nabla} v_i (\boldsymbol{\nabla} u)^\top - \zeta_{i(\boldsymbol{\nabla} u)^\top} (\boldsymbol{\nabla} u)^\top \right\} \\ & \qquad \cdot \left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top + \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top - \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 - \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \right\} \\ & \qquad - \left\{ \boldsymbol{\nabla} u (\boldsymbol{\nabla} v_i)^\top \right\} \\ & \qquad \cdot \left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \right. \\ & \qquad \left. + \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2) + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2) \right\} \Big] \mathrm{d}x. \end{aligned} \tag{9.8.24}$$

9.8.3 Second-Order Shape Derivative of Cost Function Using Lagrange Multiplier Method

When the Lagrange multiplier method is used to obtain the second-order shape derivative of a cost function, we use the same idea given in Section 7.5.4. Fixing $\boldsymbol{\varphi}_1$, we define the Lagrange function with respect to $\tilde{f}_i' (\boldsymbol{\phi}) [\boldsymbol{\varphi}_1] = \langle \boldsymbol{g}_i, \boldsymbol{\varphi}_1 \rangle$ in Eq. (9.8.5) by

$$\mathscr{L}_{\mathrm{I}i} (\boldsymbol{\phi}, u, v_i, w_i, z_i) = \langle \boldsymbol{g}_i, \boldsymbol{\varphi}_1 \rangle + \mathscr{L}_{\mathrm{S}} (\boldsymbol{\phi}, u, w_i) + \mathscr{L}_{\mathrm{A}i} (\boldsymbol{\phi}, v_i, z_i), \tag{9.8.25}$$

where $\mathscr{L}_{\mathrm{S}}$ is given by Eq. (9.5.3), and

$$\begin{aligned} & \mathscr{L}_{\mathrm{A}i} (\boldsymbol{\phi}, v_i, z_i) \\ & \quad = \int_{\Omega(\boldsymbol{\phi})} \left(-\boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} z_i + \zeta_{iu} z_i + \zeta_{i(\boldsymbol{\nabla} u)^\top} \cdot \boldsymbol{\nabla} z_i \right) \mathrm{d}x \\ & \qquad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}iu} z_i \,\mathrm{d}\gamma + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \left\{ z_i \partial_\nu v_i + (v_i - \eta_{\mathrm{D}i\partial_\nu u}) \partial_\nu z_i \right\} \mathrm{d}\gamma \end{aligned} \tag{9.8.26}$$

is the Lagrange function with respect to the adjoint problem (Problem 9.8.1). $w_i \in U$ and $z_i \in U$ are the adjoint variables provided for u and v_i in $\boldsymbol{g}_i$.

With respect to arbitrary variations $(\boldsymbol{\varphi}_2, \hat{u}, \hat{v}_i, \hat{w}_i, \hat{z}_i) \in \mathcal{D} \times U^4$ of $(\boldsymbol{\phi}, u, v_i, w_i, z_i)$, considering Eq. (9.1.6), the Fréchet derivative of $\mathscr{L}_{\mathrm{I}i}$ is written as

$$\begin{aligned} & \mathscr{L}_{\mathrm{I}i}' (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\boldsymbol{\varphi}_2, \hat{u}, \hat{v}_i, \hat{w}_i, \hat{z}_i] \\ & \quad = \mathscr{L}_{\mathrm{I}i\boldsymbol{\phi}'} (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0 (\boldsymbol{\phi}), \boldsymbol{t} (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\ & \qquad + \mathscr{L}_{\mathrm{I}iu} (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\hat{u}] + \mathscr{L}_{\mathrm{I}iv_i} (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\hat{v}_i] \\ & \qquad + \mathscr{L}_{\mathrm{I}iw_i} (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\hat{w}_i] + \mathscr{L}_{\mathrm{I}iz_i} (\boldsymbol{\phi}, u, v_i, w_i, z_i) [\hat{z}_i]. \end{aligned} \tag{9.8.27}$$

The fifth term on the right-hand side of Eq. (9.8.27) vanishes if u is the solution of the state determination problem. If v_i can be determined as the solution of the adjoint problem, the sixth term of Eq. (9.8.27) also vanishes.

Applying Proposition 9.3.7, the third term on the right-hand side of Eq. (9.8.27) is obtained as

$$\begin{aligned}
&\mathscr{L}_{\mathrm{I}iu}\left(\boldsymbol{\phi}, u, v_i, w_i, z_i\right)[\hat{u}] \\
&= \int_{\Omega(\boldsymbol{\phi})} \biggl[\biggl\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \right) \boldsymbol{\nabla} v_i - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \zeta_{i(\boldsymbol{\nabla}u)^{\top}} \\
&\qquad - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \boldsymbol{\nabla} v_i \biggr\} \cdot \boldsymbol{\nabla}\hat{u} \\
&\qquad - \left\{ \left(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\right)^{\top} \zeta_{i(\boldsymbol{\nabla}u)^{\top} u} \right) \cdot \boldsymbol{\nabla} u + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \zeta_{iu} \right\} \hat{u} - \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla}\hat{u} \biggr] \mathrm{d}x \\
&\quad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}iu} \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)_{\tau} \hat{u} \,\mathrm{d}\gamma. \qquad (9.8.28)
\end{aligned}$$

Here, the condition that Eq. (9.8.28) is zero for arbitrary $\hat{u} \in U$ is equivalent to setting w_i to be the solution of the following adjoint problem.

Problem 9.8.4 (Adjoint problem of w_i with respect to $\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_1 \rangle$) Under the assumption of Problem 9.6.3, letting $\boldsymbol{\varphi}_1 \in Y$ be given, find $w_i = w_i\left(\boldsymbol{\varphi}_1\right) \in U$ satisfying

$$\begin{aligned}
-\Delta w_i &= -\boldsymbol{\nabla}^{\top} \biggl\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \right) \boldsymbol{\nabla} v_i - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \zeta_{i(\boldsymbol{\nabla}u)^{\top}} \\
&\qquad\qquad - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \boldsymbol{\nabla} v_i \biggr\} \\
&\qquad - \left(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\right)^{\top} \zeta_{i(\boldsymbol{\nabla}u)^{\top} u} \right) \cdot \boldsymbol{\nabla} u - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \zeta_{iu} \quad \text{in } \Omega\left(\boldsymbol{\phi}\right), \\
\partial_{\nu} w_i &= \eta_{\mathrm{N}iu} \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)_{\tau} + \biggl\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \right) \boldsymbol{\nabla} v_i - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \zeta_{i(\boldsymbol{\nabla}u)^{\top}} \\
&\qquad\qquad - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \boldsymbol{\nabla} v_i \biggr\} \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_{\eta i}\left(\boldsymbol{\phi}\right), \\
\partial_{\nu} w_i &= 0 \quad \text{on } \Gamma_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \setminus \bar{\Gamma}_{\eta i}\left(\boldsymbol{\phi}\right), \\
w_i &= 0 \quad \text{on } \Gamma_{\mathrm{D}}\left(\boldsymbol{\phi}\right).
\end{aligned}$$

□

The fourth term on the right-hand side of Eq. (9.8.27) is

$$\begin{aligned}
&\mathscr{L}_{\mathrm{I}iv_i}\left(\boldsymbol{\phi}, u, v_i, w_i, z_i\right)[\hat{v}_i] \\
&= \int_{\Omega(\boldsymbol{\phi})} \biggl[\biggl\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top}\right)^{\top} \right) \boldsymbol{\nabla} u - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \boldsymbol{\nabla} u \biggr\} \cdot \boldsymbol{\nabla}\hat{v}_i \\
&\qquad + b\hat{v}_i - \boldsymbol{\nabla} z_i \cdot \boldsymbol{\nabla}\hat{v}_i \biggr] \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} p_{\mathrm{N}} \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)_{\tau} \hat{v}_i \,\mathrm{d}\gamma, \qquad (9.8.29)
\end{aligned}$$

where $\boldsymbol{\varphi}_1$ is assumed to be an H^2 class function in the neighborhood of $\Gamma_{\eta i}(\boldsymbol{\phi})$. Here, the condition that Eq. (9.8.29) is zero for arbitrary $\hat{v}_i \in U$ is equivalent to setting z_i to be the solution of the following adjoint problem.

Problem 9.8.5 (Adjoint problem of z_i with respect to $\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_1 \rangle$) Under the assumption of Problem 9.6.3, letting $\boldsymbol{\varphi}_1 \in Y$ be given, find $z_i = z_i(\boldsymbol{\varphi}_1) \in U$ satisfying

$$
\begin{aligned}
-\Delta z_i &= b - \boldsymbol{\nabla}^\top \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \right)^\top \right) \boldsymbol{\nabla} u - (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \boldsymbol{\nabla} u \right\} \quad \text{in } \Omega(\boldsymbol{\phi}), \\
\partial_\nu z_i &= p_{\mathrm{N}} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau \\
&\quad + \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \right)^\top \right) \boldsymbol{\nabla} u - (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \boldsymbol{\nabla} u \right\} \cdot \boldsymbol{\nu} \quad \text{on } \Gamma_p(\boldsymbol{\phi}), \\
\partial_\nu z_i &= 0 \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}) \setminus \bar{\Gamma}_p(\boldsymbol{\phi}), \\
z_i &= 0 \quad \text{on } \Gamma_{\mathrm{D}}.
\end{aligned}
$$

□

Finally, the first and second terms on the right-hand side of Eq. (9.8.27) become

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{I}i\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i, w_i(\boldsymbol{\varphi}_1), z_i(\boldsymbol{\varphi}_1))[\boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \mathscr{L}_{i\boldsymbol{\phi}'\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\qquad + \mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, w_i)[\boldsymbol{\varphi}_2] + \mathscr{L}_{\mathrm{A}i\boldsymbol{\phi}'}(\boldsymbol{\phi}, v_i, z_i)[\boldsymbol{\varphi}_2]. \qquad (9.8.30)
\end{aligned}
$$

The first and second terms of Eq. (9.8.30) are given by Eq. (9.8.19). The third and fourth terms become

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi}, u, w_i)[\boldsymbol{\varphi}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\boldsymbol{\nabla} u \cdot \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \right) \boldsymbol{\nabla} w_i(\boldsymbol{\varphi}_1) \right\} + \boldsymbol{\nabla} w_i(\boldsymbol{\varphi}_1) \cdot \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \right) \boldsymbol{\nabla} u \right\} \\
&\qquad\quad + (b w_i(\boldsymbol{\varphi}_1) - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w_i(\boldsymbol{\varphi}_1)) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Big] \mathrm{d}x \\
&\qquad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)_\tau \, \mathrm{d}\gamma \\
&\qquad + \int_{\Gamma_p(\boldsymbol{\phi})} p_{\mathrm{N}} w_i(\boldsymbol{\varphi}_1) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)_\tau \, \mathrm{d}\gamma, \\
&\mathscr{L}_{\mathrm{A}i\boldsymbol{\phi}'}(\boldsymbol{\phi}, v_i, z_i)[\boldsymbol{\varphi}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\left(\boldsymbol{\nabla} v_i - \zeta_{i(\boldsymbol{\nabla} u)^\top} \right) \cdot \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \right) \boldsymbol{\nabla} z_i(\boldsymbol{\varphi}_1) \right\} \\
&\qquad\quad + \boldsymbol{\nabla} z_i(\boldsymbol{\varphi}_1) \cdot \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \right) \boldsymbol{\nabla} v_i \right\} \\
&\qquad + (\zeta_{iu} z_i(\boldsymbol{\varphi}_1) - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} z_i(\boldsymbol{\varphi}_1)) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Big] \mathrm{d}x \\
&\qquad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}iu} z_i(\boldsymbol{\varphi}_1) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)_\tau \, \mathrm{d}\gamma
\end{aligned}
$$

$$+ \int_{\Gamma_p(\phi)} p_{\mathrm{N}} v_i \left(\nabla \cdot \varphi_1\right)_\tau \left(\nabla \cdot \varphi_2\right)_\tau \,\mathrm{d}\gamma$$

with respect to an arbitrary variation $\varphi_1 \in Y$.

Here, u, v_i, $w_i(\varphi_1)$ and $z_i(\varphi_1)$ are assumed to be the weak solutions of Problems 9.5.4, 9.8.1, 9.8.4 and 9.8.5, respectively. If we denote $f_i(\phi, u)$ by $\tilde{f}_i(\phi)$, then we obtain the relation

$$\begin{aligned}
&\mathscr{L}_{\mathrm{I}i\phi'}\left(\phi, u, v_i, w_i(\varphi_1), z_i(\varphi_1)\right)[\varphi_2] + \left\langle g_0(\phi), t(\varphi_1, \varphi_2)\right\rangle \\
&\quad = \tilde{f}_i''(\phi)[\varphi_1, \varphi_2] = \left\langle g_{\mathrm{H}i}(\phi, \varphi_1), \varphi_2\right\rangle \\
&\quad = \int_{\Omega(\phi)} \Big[-\left\{ \nabla u \left(\nabla v_i\right)^\top + \nabla v_i \left(\nabla u\right)^\top - \zeta_{i(\nabla u)^\top}\left(\nabla u\right)^\top \right\} \\
&\qquad\qquad \cdot \left\{ \nabla \varphi_1^\top \nabla \varphi_2^\top + \left(\nabla \varphi_2^\top\right)^\top \nabla \varphi_1^\top \right\} \\
&\qquad\quad + \left\{ \nabla u \left(\nabla w_i(\varphi_1)\right)^\top + \nabla w_i(\varphi_1) \left(\nabla u\right)^\top \right. \\
&\qquad\qquad \left. + \left(\nabla v_i - \zeta_{i(\nabla u)^\top}\right) \left(\nabla z_i(\varphi_1)\right)^\top \right\} \cdot \nabla \varphi_2^\top \\
&\qquad\quad + \left(b w_i(\varphi_1) + \zeta_{iu} z_i(\varphi_1) - \nabla u \cdot \nabla w_i(\varphi_1) - \nabla u \cdot \nabla z_i(\varphi_1)\right) \nabla \cdot \varphi_2 \\
&\qquad\quad + \zeta_{i\phi'\phi'}\left(\phi, u, \nabla u\right)[\varphi_1, \varphi_2] + u b''(\phi)[\varphi_1, \varphi_2] \Big] \,\mathrm{d}x \\
&\qquad + \int_{\Gamma_{\eta i}(\phi)} \left[\left\{ \eta_{\mathrm{N}i} \left(\nabla \cdot \varphi_1\right)_\tau + \eta_{\mathrm{N}iu} z_i(\varphi_1) \right\} \left(\nabla \cdot \varphi_2\right)_\tau \right. \\
&\qquad\quad \left. + \eta_{\mathrm{N}i\phi'\phi'}(\phi, u)[\varphi_1, \varphi_2] \right] \mathrm{d}\gamma \\
&\qquad + \int_{\Gamma_p(\phi)} \left[\left\{ p_{\mathrm{N}} v_i \left(\nabla \cdot \varphi_1\right)_\tau + p_{\mathrm{N}} w_i(\varphi_1) \right\} \left(\nabla \cdot \varphi_2\right)_\tau + p''(\phi)[\varphi_1, \varphi_2] \right] \mathrm{d}\gamma \\
&\qquad + \int_{\Gamma_{\mathrm{D}}(\phi)} \eta_{\mathrm{D}i\phi'\phi'}\left(\phi, \partial_\nu u\right)[\varphi_1, \varphi_2] \,\mathrm{d}\gamma,
\end{aligned} \tag{9.8.31}$$

where $g_{\mathrm{H}i}(\phi, \varphi_1)$ is the Hesse gradient of f_i.

9.8.4 Shape Derivative of f_i Using Formulae Based on Partial Shape Derivative of a Function

Next, we compute the shape derivative of f_i by computing the shape derivative of $\mathscr{L}_i$ using the formulae for the partial shape derivative of a function shown in Sect. 9.3.2.

Here, we assume that u and v_i are elements such that $u - u_{\mathrm{D}}$ and $v_i - \eta_{\mathrm{D}i\partial_\nu u}$ belong to $U(\phi) \cap W^{2,2q_{\mathrm{R}}}(D; \mathbb{R})$ ($q_{\mathrm{R}} > d$). Hypotheses 9.5.2 and 9.6.2 give the conditions for these.

Under these assumptions, the Fréchet derivative of $\mathscr{L}_i(\phi, u, v_i)$, with respect to an arbitrary $(\varphi, \hat{u}, \hat{v}_i) \in X \times U \times U$ can be written as

$$\begin{aligned}
&\mathscr{L}_i'(\phi, u, v_i)[\varphi, \hat{u}, \hat{v}_i] \\
&\quad = \mathscr{L}_{i\phi^*}(\phi, u, v_i)[\varphi] + \mathscr{L}_{iu}(\phi, u, v_i)[\hat{u}] + \mathscr{L}_{iv_i}(\phi, u, v_i)[\hat{v}_i]
\end{aligned} \tag{9.8.32}$$

using the notation of Eq. (9.3.21) and Eq. (9.3.27). Unlike Sect. 9.8.1, since Eq. (9.3.21) and Eq. (9.3.27) were used here, u^* was replaced by arbitrary $\hat{u} \in X$. Let us look at the detail of each term below.

The last term on the right-hand side of Eq. (9.8.32) becomes

$$\mathscr{L}_{iv_i}(\boldsymbol{\phi}, u, v_i)[\hat{v}_i] = \mathscr{L}_{\mathrm{S}v_i}(\boldsymbol{\phi}, u, v_i)[\hat{v}_i] = \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, u, \hat{v}_i). \tag{9.8.33}$$

Equation (9.8.33) is the Lagrange function of the state determination problem (Problem 9.5.4). Hence, if u is the weak solution of the state determination problem, this expression vanishes.

Meanwhile, the second term on the right-hand side of Eq. (9.8.32) is the same as Eq. (9.8.3). Hence, if v_i is such that Eq. (9.8.3) is zero with respect to an arbitrary $\hat{u} \in U$, the second term on the right-hand side of Eq. (9.8.32) also vanishes. This relationship holds when v_i is the weak solution of an adjoint problem (Problem 9.8.1) with respect to f_i. The regularities for ζ_{iu}, $\zeta_{i(\boldsymbol{\nabla} u)^\top}$, $\eta_{\mathrm{N}iu}$, $\eta_{\mathrm{D}i\partial_\nu u}$ in Hypothesis 9.6.2 give the conditions that $v_i - \eta_{\mathrm{D}i\partial_\nu u}$ belongs to $U(\boldsymbol{\phi}) \cap W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})$ $(q_{\mathrm{R}} > d)$.

Lastly, the first term on the right-hand side of Eq. (9.8.32) is manipulated as follows. Applying the formulae of Eq. (9.3.21), representing the result of Proposition 9.3.10, and Eq. (9.3.27), representing the result of Proposition 9.3.13, to the first term, we have

$$\begin{aligned}
&\mathscr{L}_{i\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} (\zeta_{i\boldsymbol{\phi}^*} + ub^*) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega(\boldsymbol{\phi})} (\zeta_i(u, \boldsymbol{\nabla} u) - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i + bv_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \,\mathrm{d}\gamma \\
&\qquad + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \{(\partial_\nu + \kappa)\, \eta_{\mathrm{N}i}(u)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} + \eta_{\mathrm{N}i\boldsymbol{\phi}^*} \cdot \boldsymbol{\varphi}\} \,\mathrm{d}\gamma \\
&\qquad + \int_{\partial\Gamma_{\eta i}(\boldsymbol{\phi}) \cup \Theta_{\eta i}(\boldsymbol{\phi})} \eta_{\mathrm{N}i}(u)\, \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \,\mathrm{d}\varsigma \\
&\qquad + \int_{\Gamma_p(\boldsymbol{\phi})} \{(\partial_\nu + \kappa)(p_{\mathrm{N}} v_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} + v_i p_{\mathrm{N}}^* \cdot \boldsymbol{\varphi}\} \,\mathrm{d}\gamma \\
&\qquad + \int_{\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta_p(\boldsymbol{\phi})} p_{\mathrm{N}} v_i \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \,\mathrm{d}\varsigma \\
&\qquad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \bigl[\{(u - u_{\mathrm{D}})\, \bar{w}(\boldsymbol{\varphi}, v_i) + (v_i - \eta_{\mathrm{D}i\partial_\nu u})\, \bar{w}(\boldsymbol{\varphi}, u)\} \\
&\qquad\quad + (\partial_\nu + \kappa)\{(u - u_{\mathrm{D}})\, \partial_\nu v_i + v_i \partial_\nu u - \eta_{\mathrm{D}i}\}\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} + \eta_{\mathrm{D}i\boldsymbol{\phi}^*} \cdot \boldsymbol{\varphi}\bigr] \,\mathrm{d}\gamma \\
&\qquad + \int_{\partial\Gamma_{\mathrm{D}}(\boldsymbol{\phi}) \cup \Theta_{\mathrm{D}}} \{(u - u_{\mathrm{D}})\, \partial_\nu v_i + v_i \partial_\nu u - \eta_{\mathrm{D}i}\}\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \,\mathrm{d}\varsigma,
\end{aligned} \tag{9.8.34}$$

where $\bar{w}(\boldsymbol{\varphi}, u)$ and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ obey Eq. (9.3.24) and Eq. (9.2.6), respectively.

With the above results in mind, we assume that u and v_i are the weak solutions to Problem 9.5.4 and Problem 9.8.1, respectively. In addition, we also assume that the condition for $\eta_{\mathrm{D}i}$ in Hypothesis 9.6.2 holds. Then, the notation in Eq. (7.5.15) for $\tilde{f}_i$ can be used to write

$$
\begin{aligned}
\tilde{f}_i'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \mathscr{L}_{i\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}] = \langle \bar{\boldsymbol{g}}_i, \boldsymbol{\varphi} \rangle \\
&= \int_{\Omega(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\zeta bi} \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial\Omega i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\Gamma_p(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{pi} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta_p(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial pi} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma + \int_{\Gamma_{\eta i}(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\eta i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_{\eta i}(\boldsymbol{\phi}) \cup \Theta_{\eta i}(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial\eta i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\mathrm{D}i} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma, \qquad (9.8.35)
\end{aligned}
$$

where

$$\bar{\boldsymbol{g}}_{\zeta bi} = \zeta_{i\boldsymbol{\phi}^*} + u b^*, \tag{9.8.36}$$

$$\bar{\boldsymbol{g}}_{\partial\Omega i} = (\zeta_i - \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i + b v_i)\, \boldsymbol{\nu}, \tag{9.8.37}$$

$$\bar{\boldsymbol{g}}_{pi} = \{\partial_\nu (p_{\mathrm{N}} v_i) + \kappa p_{\mathrm{N}} v_i\}\, \boldsymbol{\nu} + v_i p_{\mathrm{N}}^*, \tag{9.8.38}$$

$$\bar{\boldsymbol{g}}_{\partial pi} = p_{\mathrm{N}} v_i \boldsymbol{\tau}, \tag{9.8.39}$$

$$\bar{\boldsymbol{g}}_{\eta i} = (\partial_\nu \eta_{\mathrm{N}i} + \kappa \eta_{\mathrm{N}i})\, \boldsymbol{\nu} + \eta_{\mathrm{N}i\boldsymbol{\phi}^*}, \tag{9.8.40}$$

$$\bar{\boldsymbol{g}}_{\partial\eta i} = \eta_{\mathrm{N}i} \boldsymbol{\tau}, \tag{9.8.41}$$

$$\bar{\boldsymbol{g}}_{\mathrm{D}i} = \{\partial_\nu (u - u_{\mathrm{D}})\, \partial_\nu v_i + \partial_\nu (v_i - v_{\mathrm{D}i})\, \partial_\nu u\}\, \boldsymbol{\nu} + \eta_{\mathrm{D}i\boldsymbol{\phi}^*}. \tag{9.8.42}$$

If $\boldsymbol{g}_i$ of Eq. (9.8.5) and $\bar{\boldsymbol{g}}_i$ of Eq. (9.8.35) are compared in the case $\eta_{\mathrm{D}i\boldsymbol{\phi}'} = \boldsymbol{0}_{\mathbb{R}^d}$, although the term with $\bar{\boldsymbol{g}}_{\mathrm{D}}$ of Eq. (9.8.42) appears on $\Gamma_{\mathrm{D}}(\boldsymbol{\phi})$ in $\bar{\boldsymbol{g}}_i$, there is no such component in $\boldsymbol{g}_i$. This result shows that if $\boldsymbol{g}_i$ is used, and even when $\Gamma_{\mathrm{D}}(\boldsymbol{\phi})$ varies, no additional treatment is needed.

Based on the results above, the following results can be obtained with respect to the function space containing $\bar{\boldsymbol{g}}_i$ of Eq. (9.8.35).

Theorem 9.8.6 (Shape derivative $\bar{\boldsymbol{g}}_i$ of f_i) Let $\boldsymbol{\phi} \in \mathcal{D}$, b, p_{N}, u_{D}, ζ_i, $\eta_{\mathrm{N}i}$ and $\eta_{\mathrm{D}i}$ be given functions fixed in space satisfying Hypotheses 9.5.2 and 9.6.2, and $\partial\Omega(\boldsymbol{\phi})$ be in the class of $H^3 \cap C^{1,1}$. Moreover, let u and v_i be the weak solutions of the state determination problem (Problem 9.5.4) and the adjoint problem (Problem 9.8.1) with respect to f_i, respectively, such that $u - u_{\mathrm{D}}$ and $v_i - \eta_{\mathrm{D}i\partial_\nu u}$ belong to $U(\boldsymbol{\phi}) \cap W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})$ ($q_{\mathrm{R}} > d$). When $\bar{\boldsymbol{g}}_{\partial pi}$ and $\bar{\boldsymbol{g}}_{\partial\eta i}$ in Eq. (9.8.39) and Eq. (9.8.41), respectively, are zero, the shape derivative of f_i becomes $\bar{\boldsymbol{g}}_i$ in Eq. (9.8.35) and is an element of X'. Furthermore, we have

$$
\begin{aligned}
\bar{\boldsymbol{g}}_{\zeta bi}, \bar{\boldsymbol{g}}_{\partial\Omega i} &\in H^{1/2} \cap L^\infty\left(\partial\Omega(\boldsymbol{\phi}); \mathbb{R}^d\right), \\
\bar{\boldsymbol{g}}_{pi} &\in H^{1/2} \cap L^\infty\left(\Gamma_p(\boldsymbol{\phi}); \mathbb{R}^d\right), \\
\bar{\boldsymbol{g}}_{\eta i} &\in H^{1/2} \cap L^\infty\left(\Gamma_{\eta i}(\boldsymbol{\phi}); \mathbb{R}^d\right), \\
\bar{\boldsymbol{g}}_{\mathrm{D}} &\in H^{1/2} \cap L^\infty\left(\Gamma_{\mathrm{D}}(\boldsymbol{\phi}); \mathbb{R}^d\right).
\end{aligned}
$$

□

Proof The fact that the shape derivative of f_i becomes Eq. (9.8.35) is as shown above. The regularity of $\bar{\boldsymbol{g}}_i$ can be shown by using relationships similar to the proof of Theorem 9.8.2 using the fact that u and v_i are in $W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})$ and Hypothesis 9.6.2. □

From the results of Theorem 9.8.2 and Theorem 9.8.6 the following can be said about the regularity of the shape optimization problem.

Remark 9.8.7 (Irregularity of shape optimization problem) From Theorem 9.8.2 and Theorem 9.8.6, it was confirmed that $\boldsymbol{g}_i$ and $\bar{\boldsymbol{g}}_i$ are both in X' with respect to X defined in Eq. (9.1.1). In other words, it is possible to define the Fréchet derivatives of cost functions with respect to the domain variation. However, it is not necessarily the case that $\boldsymbol{g}_i$ and $\bar{\boldsymbol{g}}_i$ are in the linear space $H^2 \cap C^{0,1}\left(D;\mathbb{R}^d\right)$ containing the admissible set of design variables. This result indicates the fact that if $\boldsymbol{\varphi}$ is obtained by the gradient method substituting $-\boldsymbol{g}_i$ into $\boldsymbol{\varphi}$, $\boldsymbol{\phi}+\boldsymbol{\varphi}$ is not guaranteed to be contained in $H^2 \cap C^{0,1}\left(D;\mathbb{R}^d\right)$, which is the linear space for the admissible set of design variables. This is thought to be a reason for the numerically unstable phenomena such as the rippling shapes explained at the start of this chapter. □

9.9 Descent Directions of Cost Functions

Remark 9.8.7 points out the irregularity of the shape optimization problem of domain variation type. Hence, let us think about the gradient method and Newton method which both have the feature of regularizing the shape derivatives of cost functions in the framework of the abstract gradient and Newton methods on the linear space X of design variable. Here, let us assume that the gradient $\boldsymbol{g}_i \in X'$ of Eq. (9.8.5) and the Hessian $h_i \in \mathcal{L}^2(X \times X;\mathbb{R})$ of Eq. (9.8.24) with respect to the $i \in \{0,\dots,m\}$th cost function f_i are given and think about the way to obtain the descent direction of f_i using the gradient method and Newton method on the linear space X of design variables.

9.9.1 H^1 Gradient Method

Choose a cost function $f_i(\boldsymbol{\phi},u)$ among $i \in \{0,\dots,m\}$ and assume that the shape derivative $\boldsymbol{g}_i \in X'$ or $\bar{\boldsymbol{g}}_i \in X'$ at $\boldsymbol{\phi} \in \mathcal{D}^\circ$ is given. From now on, $\tilde{f}_i(\boldsymbol{\phi}) = f_i(\boldsymbol{\phi},u(\boldsymbol{\phi}))$ will be denoted as $f_i(\boldsymbol{\phi})$. The method for obtaining the decent direction vector $\boldsymbol{\varphi}_{gi} \in X$ (domain variation) of f_i as the solution to the next problem is called an H^1 gradient method of domain variation type.

Problem 9.9.1 (H^1 gradient method of domain variation type) Define X as in Eq. (9.1.1). Choose a coercive and bounded bilinear form $a_X : X \times X \to \mathbb{R}$ on X. In other words, suppose that there exist some positive constants α_X and β_X such that the inequalities

$$a_X(\boldsymbol{\varphi},\boldsymbol{\varphi}) \geq \alpha_X \|\boldsymbol{\varphi}\|_X^2, \quad |a_X(\boldsymbol{\varphi},\boldsymbol{\psi})| \leq \beta_X \|\boldsymbol{\varphi}\|_X \|\boldsymbol{\psi}\|_X \tag{9.9.1}$$

hold with respect to arbitrary $\boldsymbol{\varphi} \in X$ and $\boldsymbol{\psi} \in X$. Moreover, suppose $\boldsymbol{g}_i \in X'$ is given at $\boldsymbol{\phi} \in \mathcal{D}^{\circ}$. In this case, obtain $\boldsymbol{\varphi}_{gi} \in X$ which satisfies

$$a_X\left(\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right) = -\langle \boldsymbol{g}_i, \boldsymbol{\psi} \rangle \tag{9.9.2}$$

for any $\boldsymbol{\psi} \in X$. □

The way to choose $a_X : X \times X \to \mathbb{R}$ as in Problem 9.9.1 has arbitrary properties. Several specific examples will be shown in the section below.

Method Using the Inner Product in H^1 Space

Consider a method using the inner product on a real Hilbert space in a similar way to the H^1 gradient method of density variation type. In this case, it is allowed to assume that $\bar{\Omega}_{\mathrm{C0}} = \emptyset$ on Eq. (9.1.1).

The inner product on $X = H^1\left(D; \mathbb{R}^d\right)$ is defined as

$$(\boldsymbol{\varphi}, \boldsymbol{\psi})_X = \int_{\Omega(\boldsymbol{\phi})} \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\top}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\psi}^{\top}\right) + \boldsymbol{\varphi} \cdot \boldsymbol{\psi} \right\} \mathrm{d}x$$

with respect to $\boldsymbol{\varphi} \in X$ and $\boldsymbol{\psi} \in X$. Let c_Ω be some positive-valued function contained in $L^{\infty}(D; \mathbb{R})$ such that

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi})} \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\top}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\psi}^{\top}\right) + c_\Omega \boldsymbol{\varphi} \cdot \boldsymbol{\psi} \right\} \mathrm{d}x \tag{9.9.3}$$

is a bounded and coercive bilinear form on X. Here, c_Ω controls the weight of the first and second terms in the integrand. If c_Ω is taken to be small and the first term is made dominant, the smoothing function is prioritized. However, setting $c_\Omega = 0$ is not allowed, since the coercivity of the bilinear form will be lost, which is a requirement of the H^1 gradient method for it to hold. Moreover, if we write the symmetrical component of $\boldsymbol{\nabla} \boldsymbol{\varphi}^{\top}$ as

$$\boldsymbol{E}(\boldsymbol{\varphi}) = \left(e_{ij}(\boldsymbol{\varphi})\right)_{ij} = \frac{1}{2}\left\{ \boldsymbol{\nabla} \boldsymbol{\varphi}^{\top} + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^{\top}\right)^{\top} \right\},$$

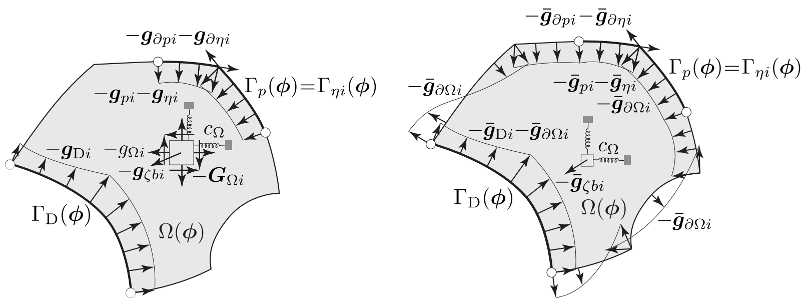
the following bilinear form on X:

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{E}(\boldsymbol{\varphi}) \cdot \boldsymbol{E}(\boldsymbol{\psi}) + c_\Omega \boldsymbol{\varphi} \cdot \boldsymbol{\psi} \right) \mathrm{d}x \tag{9.9.4}$$

is also bounded and coercive. Excluding antisymmetric components of $\boldsymbol{\nabla} \boldsymbol{\varphi}^{\top}$ indicates rotational motion which does not generate deformation.

Furthermore, $\boldsymbol{C} = \left(c_{ijkl}\right)_{ijkl} \in L^{\infty}\left(D; \mathbb{R}^{d \times d \times d \times d}\right)$ is taken to be a stiffness tensor used in a linear elastic problem. In other words, we assume that there exist positive constants α_X and β_X such that the bounds

$$\boldsymbol{A} \cdot (\boldsymbol{C} \boldsymbol{A}) \geq \alpha_X \|\boldsymbol{A}\|^2, \quad |\boldsymbol{A} \cdot (\boldsymbol{C} \boldsymbol{B})| \leq \beta_X \|\boldsymbol{A}\| \|\boldsymbol{B}\| \tag{9.9.5}$$



(a) When using $\boldsymbol{g}_i$ (b) When using $\bar{\boldsymbol{g}}_i$

Fig. 9.13: The H^1 gradient method using an inner product of $H^1\left(D;\mathbb{R}^d\right)$.

hold for any symmetric tensors $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ and $\boldsymbol{B} \in \mathbb{R}^{d\times d}$ and that the symmetricity condition $c_{ijkl} = c_{klij}$ also holds. Using these, let the stress tensor be

$$\boldsymbol{S}(\boldsymbol{\varphi}) = \boldsymbol{C}\boldsymbol{E}(\boldsymbol{\varphi}) = \left(\sum_{(k,l)\in\{1,\ldots,d\}^2} c_{ijkl}e_{kl}(\boldsymbol{\varphi})\right)_{ij}. \tag{9.9.6}$$

In this case, we have that

$$a_X(\boldsymbol{\varphi},\boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{S}(\boldsymbol{\varphi})\cdot\boldsymbol{E}(\boldsymbol{\psi}) + c_\Omega\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right)\mathrm{d}x \tag{9.9.7}$$

is a bounded and coercive bilinear form on X. $a_X(\boldsymbol{\varphi},\boldsymbol{\psi})$ of Eq. (9.9.7) is a bilinear form providing the variation of strain energy in a linear elastic problem when $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are viewed as the displacement and its variation. In this case, c_Ω indicates the spring constant of the distributed spring placed in D. Figure 9.13 provides an illustration of Problem 9.9.1 in this case.

Figure 9.13 (a) represents the case when $\boldsymbol{g}_i$ of Eq. (9.8.5) is used as the shape derivative of f_i. Problem 9.9.1 in this case is given by a weak-form equation. Hence, if we dare to rewrite this problem in its strong form, the following assumptions are needed. When u and v_i are elements of $W^{2,2q_{\mathrm{R}}}$, the first term on the right-hand side of Eq. (9.8.5) is written as

$$\begin{aligned}
&\int_{\Omega(\boldsymbol{\phi})} \left\{\boldsymbol{G}_{\Omega i}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right) + g_{\Omega i}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + \boldsymbol{g}_{\zeta b i}\cdot\boldsymbol{\varphi}\right\}\mathrm{d}x \\
&\quad= \int_{\Omega(\boldsymbol{\phi})} \Big\{\boldsymbol{\nabla}\cdot\left(\boldsymbol{G}_{\Omega i}\boldsymbol{\varphi}\right) - \left(\boldsymbol{\nabla}^\top\boldsymbol{G}_{\Omega i}\right)^\top\cdot\boldsymbol{\varphi} + \boldsymbol{\nabla}\cdot\left(g_{\Omega i}\boldsymbol{\varphi}\right) \\
&\qquad\quad - \left(\boldsymbol{\nabla}g_{\Omega i}\right)\cdot\boldsymbol{\varphi} + \boldsymbol{g}_{\zeta b i}\cdot\boldsymbol{\varphi}\Big\}\mathrm{d}x \\
&\quad= \int_{\Omega(\boldsymbol{\phi})} \tilde{\boldsymbol{g}}_{\Omega i}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \tilde{\boldsymbol{g}}_{\partial\Omega i}\cdot\boldsymbol{\varphi}\,\mathrm{d}\gamma,
\end{aligned} \tag{9.9.8}$$

where

$$\tilde{\boldsymbol{g}}_{\Omega i} = -\left(\boldsymbol{\nabla}^{\top}\boldsymbol{G}_{\Omega i}\right)^{\top} - \boldsymbol{\nabla} g_{\Omega i} + \boldsymbol{g}_{\zeta b i}, \tag{9.9.9}$$

$$\tilde{\boldsymbol{g}}_{\partial\Omega i} = \left(\boldsymbol{G}_{\Omega i} + g_{\Omega i}\right)\boldsymbol{\nu}. \tag{9.9.10}$$

Moreover, $\chi_{\Gamma_p(\boldsymbol{\phi})} : \partial\Omega(\boldsymbol{\phi}) \to \mathbb{R}$ represents the characteristic function which takes the value 1 on $\Gamma_p(\boldsymbol{\phi}) \subset \partial\Omega(\boldsymbol{\phi})$ and value 0 on $\partial\Omega(\boldsymbol{\phi}) \setminus \bar{\Gamma}_p(\boldsymbol{\phi})$. In this case, the strong form of Problem 9.9.1 using Eq. (9.9.7) for $a_X(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is given as follows.

Problem 9.9.2 (H^1 gradient method using H^1 inner product and $\boldsymbol{g}_i$) Let $\boldsymbol{g}_{pi}$, $\boldsymbol{g}_{\partial pi}$, $\boldsymbol{g}_{\eta i}$, $\boldsymbol{g}_{\partial\eta i}$ and $\boldsymbol{g}_{\mathrm{D}i}$ of Eq. (9.8.5) as well as $\tilde{\boldsymbol{g}}_{\Omega i}$ and $\tilde{\boldsymbol{g}}_{\partial\Omega i}$ of Eq. (9.9.9) and Eq. (9.9.10), respectively, be given at $\boldsymbol{\phi} \in \mathcal{D}^{\circ}$. Find $\boldsymbol{\varphi}_{gi} : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$-\boldsymbol{\nabla}^{\top}\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right) + c_{\Omega}\boldsymbol{\varphi}_{gi}^{\top} = -\tilde{\boldsymbol{g}}_{\Omega i}^{\top} \quad \text{in } \Omega(\boldsymbol{\phi}), \tag{9.9.11}$$

$$\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\nu} = -\chi_{\Gamma_p(\boldsymbol{\phi})}\boldsymbol{g}_{pi} - \chi_{\Gamma_{\eta i}(\boldsymbol{\phi})}\boldsymbol{g}_{\eta i} - \chi_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})}\boldsymbol{g}_{\mathrm{D}i} - \tilde{\boldsymbol{g}}_{\partial\Omega i} \quad \text{on } \partial\Omega(\boldsymbol{\phi}), \tag{9.9.12}$$

$$\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\tau} = -\chi_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta_p(\boldsymbol{\phi})}\boldsymbol{g}_{\partial pi} - \chi_{\partial\Gamma_{\eta i}(\boldsymbol{\phi})\cup\Theta_{\eta i}(\boldsymbol{\phi})}\boldsymbol{g}_{\partial\eta i} \quad \text{on } \partial\Omega(\boldsymbol{\phi}). \tag{9.9.13}$$

□

Figure 9.13 (b) shows the case when the shape gradient $\bar{\boldsymbol{g}}_i$ of f_i is given by Eq. (9.8.35). The strong form in this case is given as follows.

Problem 9.9.3 (H^1 gradient method using H^1 inner product and $\bar{\boldsymbol{g}}_i$) Let $\bar{\boldsymbol{g}}_{\zeta bi}$, $\bar{\boldsymbol{g}}_{\partial\Omega i}$, $\bar{\boldsymbol{g}}_{pi}$, $\bar{\boldsymbol{g}}_{\partial pi}$, $\bar{\boldsymbol{g}}_{\eta i}$, $\bar{\boldsymbol{g}}_{\partial\eta i}$ and $\bar{\boldsymbol{g}}_{\mathrm{D}i}$ as in Eq. (9.8.35) be given at $\boldsymbol{\phi} \in \mathcal{D}^{\circ}$, obtain $\boldsymbol{\varphi}_{gi}$ which satisfies

$$-\boldsymbol{\nabla}^{\top}\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right) + c_{\Omega}\boldsymbol{\varphi}_{gi}^{\top} = -\bar{\boldsymbol{g}}_{\zeta bi}^{\top} \quad \text{in } \Omega(\boldsymbol{\phi}),$$

$$\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\nu} = -\chi_{\Gamma_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{pi} - \chi_{\Gamma_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\eta i} - \chi_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\mathrm{D}i} - \bar{\boldsymbol{g}}_{\partial\Omega i} \quad \text{on } \partial\Omega(\boldsymbol{\phi}),$$

$$\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\tau} = -\chi_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial pi} - \chi_{\partial\Gamma_{\eta i}(\boldsymbol{\phi})\cup\Theta_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial\eta i} \quad \text{on } \partial\Omega(\boldsymbol{\phi}).$$

□

Method Using Boundary Condition

Moreover, as with the H^1 gradient method of varying density type, the bilinear form $a_X : X \times X \to \mathbb{R}$ can be made coercive by adding a boundary condition.

Firstly, think about using a Dirichlet boundary condition. In Eq. (9.1.1), $\bar{\Omega}_{\mathrm{C}0} \subset \bar{\Omega}_0$ was defined as a boundary or a closure of domain on which the domain variation is fixed as a design demand. Here, $\left|\bar{\Omega}_{\mathrm{C}0}\right| > 0$ is assumed. In this case,

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0}} \boldsymbol{S}(\boldsymbol{\varphi}) \cdot \boldsymbol{E}(\boldsymbol{\psi}) \,\mathrm{d}x \tag{9.9.14}$$

is a bounded and coercive bilinear form on X. This is because, when the measure of $\bar{\Omega}_{\mathrm{C}0}$ is positive and $\boldsymbol{\varphi} = \boldsymbol{0}_{\mathbb{R}^d}$ on $\bar{\Omega}_{\mathrm{C}0}$, Korn's inequality implies that there exists a positive constant c which depends only on $\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0}$ such that the inequality

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \geq \alpha_X \left\|\boldsymbol{E}(\boldsymbol{\varphi})\right\|^2_{L^2\left(\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0};\mathbb{R}^{d\times d}\right)} \geq c \left\|\boldsymbol{\varphi}\right\|^2_{H^1\left(\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0};\mathbb{R}^{d}\right)}$$

holds. Here α_X is a positive constant satisfying Eq. (9.9.5). The strong form in this case is shown as follows. Here, the situation when the shape gradient of f_i is given by $\bar{\boldsymbol{g}}_i$ in Eq. (9.8.35) is shown.

Problem 9.9.4 (H^1 gradient method using Dirichlet condition and $\bar{\boldsymbol{g}}_i$) Let $\bar{\boldsymbol{g}}_{\zeta bi}$, $\bar{\boldsymbol{g}}_{\partial\Omega i}$, $\bar{\boldsymbol{g}}_{pi}$, $\bar{\boldsymbol{g}}_{\partial pi}$, $\bar{\boldsymbol{g}}_{\eta i}$, $\bar{\boldsymbol{g}}_{\partial\eta i}$ and $\bar{\boldsymbol{g}}_{\mathrm{D}i}$ as in Eq. (9.8.35) be given at $\boldsymbol{\phi} \in \mathcal{D}^\circ$. Obtain $\boldsymbol{\varphi}_{gi} : \Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0} \to \mathbb{R}^d$ which satisfies

$$\begin{aligned}
-\boldsymbol{\nabla}^\top \boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right) + c_\Omega \boldsymbol{\varphi}_{gi}^\top &= -\bar{\boldsymbol{g}}_{\zeta bi}^\top \quad \text{in } \Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0},\\
\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\nu} &= -\chi_{\Gamma_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{pi} - \chi_{\Gamma_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\eta i} - \chi_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\mathrm{D}i} - \bar{\boldsymbol{g}}_{\partial\Omega i}\\
&\quad \text{on } \partial\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0},\\
\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\tau} &= -\chi_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial pi} - \chi_{\partial\Gamma_{\eta i}(\boldsymbol{\phi})\cup\Theta_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial\eta i}\\
&\quad \text{on } \partial\Omega(\boldsymbol{\phi}) \setminus \bar{\Omega}_{\mathrm{C}0},\\
\boldsymbol{\varphi}_{gi} &= \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \bar{\Omega}_{\mathrm{C}0}.
\end{aligned}$$

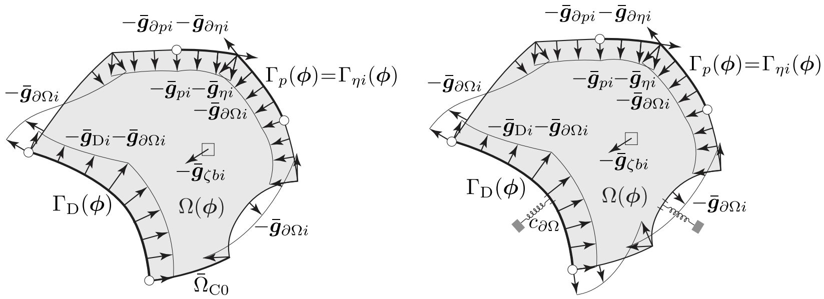
□

Figure 9.14 (a) illustrates Problem 9.9.4. This problem assumes that $\Omega(\boldsymbol{\phi})$ is a linear elastic body and obtains the displacement $\boldsymbol{\varphi}_{gi}$ when $\bar{\Omega}_{\mathrm{C}0}$ is fixed, the remaining boundaries are applied with the traction containing $-\bar{\boldsymbol{g}}_{\partial\Omega i}$, $-\bar{\boldsymbol{g}}_{pi}$, $-\bar{\boldsymbol{g}}_{\partial pi}$, $-\bar{\boldsymbol{g}}_{\eta i}$, $-\bar{\boldsymbol{g}}_{\partial\eta i}$ and $-\bar{\boldsymbol{g}}_{\mathrm{D}i}$, and the volume force $-\bar{\boldsymbol{g}}_{\zeta bi}$ is applied. From this sort of interpretation, Problem 9.9.4 is known as the traction method [2].

Furthermore, if the Robin condition is used, even if $\bar{\Omega}_{\mathrm{C}0} = \emptyset$ is assumed in Eq. (9.1.1), coerciveness of $a_X(\boldsymbol{\varphi}, \boldsymbol{\psi})$ can be obtained. Choose some positive-valued function $c_{\partial\Omega} \in L^\infty(\partial\Omega(\boldsymbol{\phi});\mathbb{R})$ and let

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{S}(\boldsymbol{\varphi}) \cdot \boldsymbol{E}(\boldsymbol{\psi}) \,\mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} c_{\partial\Omega} (\boldsymbol{\varphi} \cdot \boldsymbol{\nu})(\boldsymbol{\psi} \cdot \boldsymbol{\nu}) \,\mathrm{d}\gamma. \tag{9.9.15}$$

The strong form in this case is shown below. Here too, let us show only the case when the shape gradient of f_i is given by $\bar{\boldsymbol{g}}_i$ in Eq. (9.8.35).



(a) Dirichlet condition (b) Robin condition

Fig. 9.14: The H^1 gradient method (when $\Gamma_p(\boldsymbol{\phi}) = \Gamma_{\eta i}(\boldsymbol{\phi})$).

Problem 9.9.5 **(H^1 gradient method using Robin condition and $\bar{\boldsymbol{g}}_i$)** Let $\bar{\boldsymbol{g}}_{\zeta bi}$, $\bar{\boldsymbol{g}}_{\partial\Omega i}$, $\bar{\boldsymbol{g}}_{pi}$, $\bar{\boldsymbol{g}}_{\partial pi}$, $\bar{\boldsymbol{g}}_{\eta i}$, $\bar{\boldsymbol{g}}_{\partial\eta i}$ and $\bar{\boldsymbol{g}}_{\mathrm{D}i}$ as in Eq. (9.8.35) be given at $\boldsymbol{\phi} \in \mathcal{D}^\circ$. Find $\boldsymbol{\varphi}_{gi}$ which satisfies

$$
\begin{aligned}
-\boldsymbol{\nabla}^\top \boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right) &= -\bar{\boldsymbol{g}}_{\zeta bi}^\top \quad \text{in } \Omega(\boldsymbol{\phi}),\\
\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\nu} + c_{\partial\Omega}\left(\boldsymbol{\varphi}\cdot\boldsymbol{\nu}\right)\boldsymbol{\nu} &= -\chi_{\Gamma_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{pi} - \chi_{\Gamma_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\eta i}\\
&\quad - \chi_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\mathrm{D}i} - \bar{\boldsymbol{g}}_{\partial\Omega i} \quad \text{on } \partial\Omega(\boldsymbol{\phi}),\\
\boldsymbol{S}\left(\boldsymbol{\varphi}_{gi}\right)\boldsymbol{\tau} &= -\chi_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta_p(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial pi} - \chi_{\partial\Gamma_{\eta i}(\boldsymbol{\phi})\cup\Theta_{\eta i}(\boldsymbol{\phi})}\bar{\boldsymbol{g}}_{\partial\eta i}\\
&\quad \text{on } \partial\Omega(\boldsymbol{\phi}).
\end{aligned}
$$

□

Figure 9.14 (b) illustrates Problem 9.9.5. This problem assumes that $\Omega(\boldsymbol{\phi})$ is a linear elastic body and that a distribution spring with spring constant $c_{\partial\Omega}$ is placed on $\partial\Omega(\boldsymbol{\phi})$, then seeks the displacement $\boldsymbol{\varphi}_{gi}$ when $\bar{\Omega}_{\mathrm{C}0}$ is fixed, the remaining boundaries are applied with the traction containing $-\bar{\boldsymbol{g}}_{\partial\Omega i}$, $-\bar{\boldsymbol{g}}_{pi}$, $-\bar{\boldsymbol{g}}_{\partial pi}$, $-\bar{\boldsymbol{g}}_{\eta i}$, $-\bar{\boldsymbol{g}}_{\partial\eta i}$ and $-\bar{\boldsymbol{g}}_{\mathrm{D}i}$, and the volume force $-\bar{\boldsymbol{g}}_{\zeta bi}$ is applied. From this sort of interpretation, Problem 9.9.5 has been referred to as the traction method with spring, or traction method of Robin type [7].

Regularity of the H^1 Gradient Method

From the weak solutions of Problem 9.9.1 and its specific examples (Problems 9.9.3 to 9.9.5), the results below can be obtained. Here, we call the neighborhood of points or edges such that u does not belong to $\mathcal{S}$ (or $U(\boldsymbol{\phi}) \cap W^{2,2q_{\mathrm{R}}}(D;\mathbb{R})$ when $\bar{\boldsymbol{g}}_i$ in Theorem 9.8.6 is used) a neighborhood of singular points and will write it as $B(\boldsymbol{\phi})$ (refer to Hypothesis 9.5.3). Also, we let $f_i(\boldsymbol{\phi}, u)$ be denoted by $\tilde{f}_i(\boldsymbol{\phi})$ when u is the solution to Problem 9.5.4.

Theorem 9.9.6 (Regularity of the H^1 gradient method) There exists a unique weak solution $\boldsymbol{\varphi}_{gi} \in X$ for each Problem 9.9.2 to 9.9.5 using $\boldsymbol{g}_i$ of Theorem 9.8.2 or $\bar{\boldsymbol{g}}_i$ of Theorem 9.8.6. $\boldsymbol{\varphi}_{gi}$ is a function of class $H^2 \cap C^{0,1}$ on

$\Omega(\boldsymbol{\phi}) \setminus \bar{B}(\boldsymbol{\phi})$. Moreover, $\boldsymbol{\varphi}_{gi}$ points to the direction of the domain variation which decreases the value of $\tilde{f}_i(\boldsymbol{\phi})$. □

Proof Let us think about the weak solution $\boldsymbol{\varphi}_{gi}$ of Problem 9.9.2. Problem 9.9.2 is a boundary value problem of an elliptic partial differential equation with $\boldsymbol{G}_{\Omega i}$ and $g_{\Omega i}$ of class $H^1 \cap L^\infty$ in Theorem 9.8.2 given in the domain, and $\boldsymbol{g}_{pi}$, $\boldsymbol{g}_{\partial pi}$, $\boldsymbol{g}_{\eta i}$ and $\boldsymbol{g}_{\partial \eta i}$ of class $H^{1/2} \cap L^\infty$ in Theorem 9.8.2 given as Neumann boundary conditions. Hence, $\boldsymbol{\varphi}_{gi} \in X$ exists uniquely from the Lax-Milgram theorem. Moreover, $\boldsymbol{\varphi}_{gi}$ is of class $H^2 \cap C^{0,1}$ on $\Omega(\boldsymbol{\phi}) \setminus \bar{B}(\boldsymbol{\phi})$. This is due to the fact that $\boldsymbol{G}_{\Omega i}$ and $\boldsymbol{S}(\boldsymbol{\varphi}_{gi})$ in Eq. (9.9.11) have the same regularity, and thus $\boldsymbol{G}_{\Omega i}$ being of class $H^1 \cap L^\infty$ means $\boldsymbol{\varphi}_{gi}$ is of class $H^2 \cap C^{0,1}$. In fact, using Eq. (9.9.12) and Eq. (9.9.13), one can also infer that $\boldsymbol{\varphi}_{gi}$ is of class $H^2 \cap C^{0,1}$.

Similarly, the weak solution $\boldsymbol{\varphi}_{gi}$ of Problem 9.9.3 satisfies an elliptic partial differential equation with $\bar{\boldsymbol{g}}_{\partial\Omega i}$, $\bar{\boldsymbol{g}}_{pi}$, $\bar{\boldsymbol{g}}_{\partial pi}$, $\bar{\boldsymbol{g}}_{\eta i}$, $\bar{\boldsymbol{g}}_{\partial \eta i}$ and $\bar{\boldsymbol{g}}_{\mathrm{D}i}$ of class $H^{1/2} \cap L^\infty$ in Theorem 9.8.6 as Neumann boundary conditions. Here, there exists a unique weak solution $\boldsymbol{\varphi}_{gi} \in X$ from the Lax–Milgram theorem being of class $H^2 \cap C^{0,1}$ on $\Omega(\boldsymbol{\phi}) \setminus \bar{B}(\boldsymbol{\phi})$. Similar results can be obtained for the weak solutions of Problem 9.9.4 and Problem 9.9.5.

Furthermore, with respect to the weak solutions $\boldsymbol{\varphi}_{gi}$ of Problems 9.9.3 to 9.9.5, we have the estimate

$$\begin{aligned}\tilde{f}_i(\boldsymbol{\phi} + \bar{\epsilon}\boldsymbol{\varphi}_{gi}) - \tilde{f}_i(\boldsymbol{\phi}) &= \bar{\epsilon}\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_{gi}\rangle + o(|\bar{\epsilon}|) \\ &= -\bar{\epsilon} a_X(\boldsymbol{\varphi}_{gi}, \boldsymbol{\varphi}_{gi}) + o(|\bar{\epsilon}|) \leq -\bar{\epsilon}\alpha_X \|\boldsymbol{\varphi}_{gi}\|_X^2 + o(|\bar{\epsilon}|)\end{aligned}$$

for some positive constant $\bar{\epsilon}$. Hence, if $\|\boldsymbol{\varphi}_{gi}\|_X$ is taken to be sufficiently small, $\tilde{f}_i(\boldsymbol{\phi})$ is reduced. □

The following remark can be maid about the relationship between the result of Theorem 9.9.6 and the admissible set $\mathcal{D}$ of domain variations defined by Eq. (9.1.3).

Remark 9.9.7 (H^1 gradient method for shape optimization problem) From Theorem 9.9.6, it was confirmed that the domain variation $\boldsymbol{\varphi}_{gi}$ obtained by the H^1 gradient method with respect to the shape optimization problem is contained in the linear space $H^2 \cap C^{0,1}(D; \mathbb{R}^d)$ for the admissible set $\mathcal{D}$ of design variables excepting the neighborhood of singular points. From this, the domain can be moved via continuous mapping excluding the neighborhood of singular points. However, one cannot guarantee to have the bound $|\boldsymbol{\phi} + \boldsymbol{\varphi}_{gi}|_{C^{0,1}(D;\mathbb{R}^d)} \leq \sigma$ or the condition that $\tilde{\Gamma}(\boldsymbol{\phi} + \boldsymbol{\varphi}_{gi})$ ($\tilde{\Gamma}_0 = \Gamma_{p0} \cup \Gamma_{\eta 00} \cup \Gamma_{\eta 10} \cup \cdots \cup \Gamma_{\eta m0} \setminus \bar{\Omega}_{\mathrm{C}0}$) is of class $H^3 \cap C^{1,1}$ which are sufficient conditions for the inverse mapping of $\boldsymbol{\phi} + \boldsymbol{\varphi}_{gi}$ to become bijective. If a numerically unstable phenomenon caused by these conditions not being satisfied occurs, there is a need to consider additional requirements in order to satisfy the said conditions. □

As one of methods to improve the regularity of the boundary, an iterative method of the H^1 gradient method can be considered. This method is the following algorithm.

Algorithm 9.9.8 (Iterative method of the H^1 gradient method) Let a domain $\Omega(\boldsymbol{\phi})$ be given. Obtain a domain variation in the following way:

(1) Calculate a shape gradient $\boldsymbol{g}_i$ (or $\bar{\boldsymbol{g}}_i$).

(2) By the first H^1 gradient method, obtain $\boldsymbol{\varphi}_{gi} = \boldsymbol{\varphi}_{gi1}$ using $-\boldsymbol{g}_i$ (or $-\bar{\boldsymbol{g}}_i$). Here, $\boldsymbol{\varphi}_{gi1}$ is not used for domain variation.

(3) Using the trace $\boldsymbol{\varphi}_{gi1}|_{\partial\Omega(\boldsymbol{\phi})}$ of the solution $\boldsymbol{\varphi}_{gi1}$ of the first H^1 gradient method on $\partial\Omega(\boldsymbol{\phi})$ instead of $-\bar{\boldsymbol{g}}_i$, calculate $\boldsymbol{\varphi}_{gi2}$ by the second H^1 gradient method. Vary the domain $\Omega(\boldsymbol{\phi})$ with the $\boldsymbol{\varphi}_{gi2}$.

□

For the boundary of the new domain $\Omega\left(\boldsymbol{\phi}+\boldsymbol{\varphi}_{gi2}\right)$ obtained in the above way, it is expected that the differentiability improves by one order higher than $\Omega\left(\boldsymbol{\phi}+\boldsymbol{\varphi}_{gi1}\right)$.

9.9.2 H^1 Newton Method

Now, if the second-order derivative (Hessian) $h_i \in \mathcal{L}^2(X \times X; \mathbb{R})$ of the cost function f_i is computable, a Newton method on $X = H^1\left(D; \mathbb{R}^d\right)$ can be considered. This method is called an H^1 Newton method of domain variation type.

Problem 9.9.9 (H^1 Newton method of domain variation type) Let X and $\mathcal{D}$ be given by Eq. (9.1.1) and Eq. (9.1.3), respectively. Let the shape derivative and second-order shape derivative of $f_i \in C^2(\mathcal{D}; \mathbb{R})$ at $\boldsymbol{\phi}_k \in \mathcal{D}^\circ$ which is not a local minimizer be $\boldsymbol{g}_i(\boldsymbol{\phi}_k) \in X'$ and $h_i(\boldsymbol{\phi}_k) \in \mathcal{L}^2(X \times X; \mathbb{R})$, respectively. Moreover, assume that $a_X : X \times X \to \mathbb{R}$ is a bilinear form which assures coercivity and sufficient regularity of $h_i(\boldsymbol{\phi}_k)$ on X. In this case, obtain $\boldsymbol{\varphi}_{gi} \in X$ which satisfies

$$h_i(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right] + a_X\left(\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right) = -\left\langle \boldsymbol{g}_i(\boldsymbol{\phi}_k), \boldsymbol{\psi}\right\rangle \tag{9.9.16}$$

with respect to an arbitrary $\boldsymbol{\psi} \in X$. □

In Problem 9.9.9, if the Newton method is considered with only the expression for h_i appearing on the left-hand side of Eq. (9.9.16), there may be cases when the coerciveness of h_i on X may not be guaranteed. In reality, h_i calculated by Eq. (9.8.24) contains a negative term, hence, the addition of the bilinear form a_X which is bounded and coercive on X to the left-hand side of Eq. (9.9.16) in Problem 9.9.9. For instance, in the case using the inner product on X such as Eq. (9.9.3), we can assume

$$a_X(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_{\Omega(\boldsymbol{\phi})} \left\{ c_{\Omega 1} \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\psi}^\top\right) + c_{\Omega 0} \boldsymbol{\varphi} \cdot \boldsymbol{\psi} \right\} \mathrm{d}x. \tag{9.9.17}$$

Table 9.1: Correspondence between abstract optimal design problem and shape optimization problem of domain variation type.

	Abstract problem	Domain variation type problem
Design variable	$\phi \in X$	$\boldsymbol{\phi} \in X = H^1\left(D; \mathbb{R}^d\right)$
State variable	$u \in U$	$u \in U = H^1\left(D; \mathbb{R}\right)$
Fréchet derivative of f_i	$g_i \in X'$	$\boldsymbol{g}_i \in X' = H^{1\prime}\left(D; \mathbb{R}^d\right)$
Solution of gradient method	$\varphi_{gi} \in X$	$\boldsymbol{\varphi}_{gi} \in X = H^1\left(D; \mathbb{R}^d\right)$

Here, $c_{\Omega 0}$ and $c_{\Omega 1}$ are positive constants for achieving coercivity for a_X and desired regularity for $\boldsymbol{\varphi}_{gi}$ in Eq. (9.9.16), respectively. $c_{\Omega 0}$ has the same meaning as that explained after Eq. (8.6.3) in Chap. 8.

Furthermore, in the case of the Newton method when the second-order shape derivative of $f_i(\boldsymbol{\phi})$ is given by the Hesse gradient, Problem 9.9.9 is replaced with the following problem.

Problem 9.9.10 (Newton method using Hesse gradient) Under the assumption of Problem 9.9.9, the gradient of the shape derivative of f_i, a search vector and the Hesse gradient of f_i at a non-local minimum point $\boldsymbol{\phi}_k \in \mathcal{D}^\circ$ are denoted by $\boldsymbol{g}_i(\boldsymbol{\phi}_k) \in X'$, $\bar{\boldsymbol{\varphi}}_{gi} \in X$ and $\boldsymbol{g}_{\mathrm{H}i}\left(\boldsymbol{\phi}_k, \bar{\boldsymbol{\varphi}}_{gi}\right) \in X'$, respectively. Given a coercive and bounded bilinear form $a_X : X \times X \to \mathbb{R}$ on X, find a $\boldsymbol{\varphi}_{gi} \in X$ which satisfies

$$a_X\left(\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right) = -\left\langle \left(\boldsymbol{g}_i(\boldsymbol{\phi}_k) + \boldsymbol{g}_{\mathrm{H}i}\left(\boldsymbol{\phi}_k, \bar{\boldsymbol{\varphi}}_{gi}\right)\right), \boldsymbol{\psi} \right\rangle \tag{9.9.18}$$

with respect to an arbitrary $\boldsymbol{\psi} \in X$. □

9.10 Solution to Shape Optimization Problem of Domain Variation Type

The shape optimization problem (Problem 9.6.3) of domain variation type has a correspondence with the abstract optimal design problem, as shown in Table 9.1. Therefore, the gradient method with respect to constrained problems shown in Section 7.7.1 (Section 3.7) and the Newton method with respect to a constrained problem shown in Section 7.7.2 (Section 3.8) are applicable as similarly shown in Chap. 8.

9.10.1 Gradient Method for Constrained Problems

The gradient method with respect to constrained problems employs a simple numerical procedure such as that given in Algorithm 3.7.2 shown in Section 3.7.1 with only a few modifications as follows:

(1) The design variable $\boldsymbol{x}$ and its variation $\boldsymbol{y}$ are replaced by $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$, respectively.

(2) The equation (Eq. (3.7.10)) that describes the gradient method is replaced with a condition such that there holds the equation

$$c_a a_X \left(\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right) = -\left\langle \boldsymbol{g}_i, \boldsymbol{\psi} \right\rangle \tag{9.10.1}$$

for any $\boldsymbol{\psi} \in X$, where $a_X \left(\boldsymbol{\varphi}_{gi}, \boldsymbol{\psi}\right)$ is a bilinear form on X used in the weak form of one of Problems 9.9.2 to 9.9.5.

(3) The equation (Eq. (3.7.11)) used to seek for the search vector is replaced with

$$\boldsymbol{\varphi}_g = \boldsymbol{\varphi}_{g0} + \sum_{i \in I_{\mathrm{A}}} \lambda_i \boldsymbol{\varphi}_{gi}. \tag{9.10.2}$$

(4) The equation (Eq. (3.7.12)) used to seek for the Lagrange multiplier is replaced with

$$\left(\left\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_{gj} \right\rangle\right)_{(i,j) \in I_{\mathrm{A}}^2} \left(\lambda_j\right)_{j \in I_{\mathrm{A}}} = -\left(f_i + \left\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_{g0} \right\rangle\right)_{i \in I_{\mathrm{A}}}. \tag{9.10.3}$$

Furthermore, if instead a complicated numerical procedure such as that given by Algorithm 3.7.6 is used, the following changes are added in addition to (1) to (4) above:

(5) Replace the Armijo criteria Eq. (3.7.26) with

$$\mathscr{L}\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_g, \boldsymbol{\lambda}_{k+1}\right) - \mathscr{L}\left(\boldsymbol{\phi}, \boldsymbol{\lambda}\right) \le \xi \left\langle \boldsymbol{g}_0 + \sum_{i \in I_{\mathrm{A}}} \lambda_i \boldsymbol{g}_i, \boldsymbol{\varphi}_g \right\rangle, \tag{9.10.4}$$

where $\xi \in (0,1)$.

(6) Replace the Wolfe criteria Eq. (3.7.27) with

$$\begin{aligned} &\mu \left\langle \boldsymbol{g}_0 + \sum_{i \in I_{\mathrm{A}}} \lambda_i \boldsymbol{g}_i, \boldsymbol{\varphi}_g \right\rangle \\ &\quad \le \left\langle \boldsymbol{g}_0\left(\boldsymbol{\phi} + \boldsymbol{\varphi}_g\right) + \sum_{i \in I_{\mathrm{A}}} \lambda_{i\,k+1} \boldsymbol{g}_i \left(\boldsymbol{\phi} + \boldsymbol{\varphi}_g\right), \boldsymbol{\varphi}_g \right\rangle, \end{aligned} \tag{9.10.5}$$

where μ is such that $0 < \xi < \mu < 1$.

(7) Replace the update equation for $\boldsymbol{\lambda}$ from the Newton–Raphson method given in Eq. (3.7.21) with

$$\left(\delta \lambda_j\right)_{j \in I_{\mathrm{A}}} = -\left(\left\langle \boldsymbol{g}_i\left(\boldsymbol{\lambda}\right), \boldsymbol{\varphi}_{gj} \right\rangle\right)_{(i,j) \in I_{\mathrm{A}}^2}^{-1} \left(f_i\left(\boldsymbol{\lambda}\right)\right)_{i \in I_{\mathrm{A}}}. \tag{9.10.6}$$

9.10.2 Newton Method for Constrained Problems

If, in addition to the first-order shape derivative, the corresponding second-order shape derivative of a cost function is also computable, the gradient method can be improved to the Newton method to numerically solve the associated constrained problem. In this case, we substitute $h_i(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right]$ in Eq. (9.9.16) with the Hessian of the Lagrange function $\mathscr{L}$ with respect to the shape optimization problem (Problem 9.6.3) with

$$h_{\mathscr{L}}(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right] = h_0(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right] + \sum_{i\in I_{\mathrm{A}}(\boldsymbol{\phi}_k)} \lambda_{ik} h_i(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right]. \tag{9.10.7}$$

In other words, we let Eq. (9.9.16) be replaced with

$$c_h h_{\mathscr{L}}(\boldsymbol{\phi}_k)\left[\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right] + a_X\left(\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right) = -\left\langle \boldsymbol{g}_i(\boldsymbol{\phi}_k),\boldsymbol{\psi}\right\rangle, \tag{9.10.8}$$

where c_h and c_a are constants to control the step size. In this case, the simple Algorithm 3.8.4 shown in Section 3.8.1 can be used by applying the following substitution:

(1) Replace the design variable $\boldsymbol{x}$ and its variation $\boldsymbol{y}$ by $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$, respectively.

(2) Replace Eq. (3.7.10) with the solution of Eq. (9.10.8).

(3) Replace Eq. (3.7.11) with Eq. (9.10.2).

(4) Replace Eq. (3.7.12) with Eq. (9.10.3).

When the second-order shape derivative of $f_i(\boldsymbol{\phi})$ is obtained as a Hesse gradient, Eq. (9.10.7) and Eq. (9.10.8) are replaced with

$$\boldsymbol{g}_{\mathrm{H}\mathscr{L}}\left(\boldsymbol{\phi}_k,\bar{\boldsymbol{\varphi}}_g\right) = \boldsymbol{g}_{\mathrm{H}0}\left(\boldsymbol{\phi}_k,\bar{\boldsymbol{\varphi}}_g\right) + \sum_{i\in I_{\mathrm{A}}(\boldsymbol{\phi}_k)} \lambda_{ik}\boldsymbol{g}_{\mathrm{H}i}\left(\boldsymbol{\phi}_k,\bar{\boldsymbol{\varphi}}_g\right) \tag{9.10.9}$$

$$a_X\left(\boldsymbol{\varphi}_{gi},\boldsymbol{\psi}\right) = -\left\langle \left(\boldsymbol{g}_i(\boldsymbol{\phi}_k) + c_h\boldsymbol{g}_{\mathrm{H}\mathscr{L}}\left(\boldsymbol{\phi}_k,\bar{\boldsymbol{\varphi}}_g\right)\right),\boldsymbol{\psi}\right\rangle, \tag{9.10.10}$$

respectively. Using the definitions, the following step is added:

(5) Replace Eq. (3.8.11) with Eq. (9.10.10).

If instead one wishes to implement a more complicated numerical procedure such as that shown in Section 3.8.2, then several additional requirements are needed in response to the added functionality and characteristics of such problems as those examined in Chap. 8.

9.11 Error Estimation

When the shape optimization problem (Problem 9.6.3) of domain variation type is to be solved using an algorithm such as that shown in Sect. 9.10, the search vector $\boldsymbol{\varphi}_g$ can be obtained by Eq. (9.10.2). For this purpose, there is a need to seek the numerical solutions of u for the state determination problem (Problem 9.5.4), the numerical solutions of $v_0, v_{i_1}, \ldots, v_{i_{|I_A|}}$ for the adjoint problems with respect to $f_0, f_{i_1}, \ldots, f_{i_{|I_A|}}$ (Problem 9.8.1), as well as the numerical solutions of $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_{i_1}, \ldots, \boldsymbol{\varphi}_{i_{|I_A|}}$ for the H^1 gradient method (Problem 9.9.1). The Lagrange multipliers $\lambda_{i_1}, \ldots, \lambda_{i_{|I_A|}}$ are calculated using these numerical solutions. As in Chap. 8, we assume here too that a finite element method is used to obtain the numerical solutions for the three types of boundary value problems. We then use the estimated error from the numerical solutions via the finite element method seen in Section 6.6 in order to conduct an error estimation for the search vector $\boldsymbol{\varphi}_g$ [66,67].

In the case of the shape optimization problem of domain variation type, the defined domain of the boundary value problem is perturbed. Here, a situation is considered in which $\Omega(\boldsymbol{\phi})$ is assumed to be given and $\Omega(\boldsymbol{\phi}+\boldsymbol{\varphi})$ is sought. In this section, for simplicity, we write $\Omega(\boldsymbol{\phi})$ as Ω. Similarly, $(\,\cdot\,)(\boldsymbol{\phi})$ is denoted by $(\,\cdot\,)$. Assume that Ω is a polyhedron (Section 6.6.1) and consider a regular finite element division $\mathcal{T} = \{\Omega_i\}_{i\in\mathcal{E}}$ with respect to Ω. Moreover, define the diameter h of the finite element as $h(\mathcal{T})$ of Eq. (6.6.2) and consider the finite element division sequence $\{\mathcal{T}_h\}_{h\to 0}$. The notations we give below will be used in the rest of the discussion:

(1) The exact solution of the state determination problem (Problem 9.5.4) and the adjoint problems with respect to f_i (Problem 9.8.1) are written as u and $v_0, v_{i_1}, \ldots, v_{i_{|I_A|}}$, respectively. These numerical solutions from the finite element method are written as

$$u_h = u + \delta u_h, \tag{9.11.1}$$

$$v_{ih} = v_i + \delta v_{ih} \tag{9.11.2}$$

for all $i \in I_{\mathrm{A}} \cup \{0\}$.

(2) Regarding the shape derivatives of $f_0, f_{i_1}, \ldots, f_{i_{|I_A|}}$, we write the numerical solutions of $\boldsymbol{g}_i$ for each $i \in I_{\mathrm{A}} \cup \{0\}$ in Eq. (9.8.5) obtained using the formulae based on the shape derivative of a function as

$$\boldsymbol{g}_{ih} = \boldsymbol{g}_i + \delta \boldsymbol{g}_{ih}. \tag{9.11.3}$$

Moreover, the numerical solutions of $\bar{\boldsymbol{g}}_i$ for each $i \in I_{\mathrm{A}} \cup \{0\}$ in Eq. (9.8.35) obtained using the formulae based on the partial shape derivative of a function is written as

$$\bar{\boldsymbol{g}}_{ih} = \bar{\boldsymbol{g}}_i + \delta \bar{\boldsymbol{g}}_{ih}. \tag{9.11.4}$$

Here, $\boldsymbol{g}_i$ and $\bar{\boldsymbol{g}}_i$ are functions of $u, v_0, v_{i_1}, \ldots, v_{i_{|I_A|}}$ and $\boldsymbol{g}_{ih}$ and $\bar{\boldsymbol{g}}_{ih}$ are functions of $u_h, v_{0h}, v_{i_1 h}, \ldots, v_{i_{|I_A|}h}$, respectively.

(3) We write the exact solutions of the H^1 gradient method (for example, Problem 9.9.3) calculated using $\boldsymbol{g}_0$, $\boldsymbol{g}_{i_1}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}}$ as $\boldsymbol{\varphi}_{g0}$, $\boldsymbol{\varphi}_{gi_1}$, ... , $\boldsymbol{\varphi}_{gi_{|I_{\mathrm{A}}|}}$. Moreover, the exact solutions of the H^1 gradient method calculated using $\boldsymbol{g}_{0h}$, $\boldsymbol{g}_{i_1h}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}h}$ are written as

$$\hat{\boldsymbol{\varphi}}_{gi} = \boldsymbol{\varphi}_{gi} + \delta\hat{\boldsymbol{\varphi}}_{gi} \tag{9.11.5}$$

for all $i \in I_{\mathrm{A}} \cup \{0\}$. With respect to (2), the exact solutions and numerical solutions obtained via the formulae using the partial shape derivative of a function will have $(\bar{\cdot})$ attached. The exact solutions of the H^1 gradient method in this case will be written as

$$\hat{\bar{\boldsymbol{\varphi}}}_{gi} = \bar{\boldsymbol{\varphi}}_{gi} + \delta\hat{\bar{\boldsymbol{\varphi}}}_{gi}. \tag{9.11.6}$$

(4) The numerical solutions of the H^1 gradient method calculated using $\boldsymbol{g}_{0h}$, $\boldsymbol{g}_{i_1h}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}h}$ are written as

$$\boldsymbol{\varphi}_{gih} = \hat{\boldsymbol{\varphi}}_{gi} + \delta\hat{\boldsymbol{\varphi}}_{gih} = \boldsymbol{\varphi}_{gi} + \delta\boldsymbol{\varphi}_{gih} \tag{9.11.7}$$

for all $i \in I_{\mathrm{A}} \cup \{0\}$. Moreover, the numerical solutions of the H^1 gradient method obtained using the formulae based on the partial shape derivative of a function are written as

$$\bar{\boldsymbol{\varphi}}_{gih} = \hat{\bar{\boldsymbol{\varphi}}}_{gi} + \delta\hat{\bar{\boldsymbol{\varphi}}}_{gih} = \bar{\boldsymbol{\varphi}}_{gi} + \delta\bar{\boldsymbol{\varphi}}_{gih}. \tag{9.11.8}$$

(5) The coefficient matrix $\left(\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_{gj}\rangle\right)_{(i,j)\in I_{\mathrm{A}}^2}$ of Eq. (9.10.3) constructed from $\boldsymbol{g}_0$, $\boldsymbol{g}_{i_1}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}}$ and $\boldsymbol{\varphi}_{g0}$, $\boldsymbol{\varphi}_{gi_1}$, ... , $\boldsymbol{\varphi}_{gi_{|I_{\mathrm{A}}|}}$ is written as $\boldsymbol{A}$. Moreover, the coefficient matrix $\left(\langle \boldsymbol{g}_{ih}, \boldsymbol{\varphi}_{gjh}\rangle\right)_{(i,j)\in I_{\mathrm{A}}^2}$ of Eq. (9.10.3) constructed using $\boldsymbol{g}_{0h}$, $\boldsymbol{g}_{i_1h}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}h}$ and $\boldsymbol{\varphi}_{g0h}$, $\boldsymbol{\varphi}_{gi_1h}$, ... , $\boldsymbol{\varphi}_{gi_{|I_{\mathrm{A}}|}h}$ is written as $\boldsymbol{A}_h = \boldsymbol{A} + \delta\boldsymbol{A}_h$. Furthermore, assuming $f_i = 0$ for all $i \in I_{\mathrm{A}}$, we write $-\left(\langle \boldsymbol{g}_i, \boldsymbol{\varphi}_{g0}\rangle\right)_{i\in I_{\mathrm{A}}}$ as $\boldsymbol{b}$. Moreover, the expression $-\left(\langle \boldsymbol{g}_{ih}, \boldsymbol{\varphi}_{g0h}\rangle\right)_{i\in I_{\mathrm{A}}}$ is written as $\boldsymbol{b}_h = \boldsymbol{b} + \delta\boldsymbol{b}_h$. In addition, the exact solutions for the Lagrange multipliers are written as $\boldsymbol{\lambda} = \boldsymbol{A}^{-1}\boldsymbol{b}$. On the other hand, its numerical solution is written as

$$\boldsymbol{\lambda}_h = (\lambda_{ih})_{i\in I_{\mathrm{A}}} = \boldsymbol{A}_h^{-1}\boldsymbol{b}_h = \boldsymbol{\lambda} + \delta\boldsymbol{\lambda}_h. \tag{9.11.9}$$

Additionally, the exact solutions and numerical solutions obtained using the formulae based on the partial shape derivative of a function will have $(\bar{\cdot})$ attached. The numerical solutions for the Lagrange multipliers in this case are written as

$$\bar{\boldsymbol{\lambda}}_h = \left(\bar{\lambda}_{ih}\right)_{i\in I_{\mathrm{A}}} = \bar{\boldsymbol{A}}_h^{-1}\bar{\boldsymbol{b}}_h = \bar{\boldsymbol{\lambda}} + \delta\bar{\boldsymbol{\lambda}}_h. \tag{9.11.10}$$

(6) Equation (9.10.2) constructed from $\boldsymbol{\varphi}_{g0h}$, $\boldsymbol{\varphi}_{gi_1h}$, ... , $\boldsymbol{\varphi}_{gi_{|I_\mathrm{A}|}h}$ and λ_{i_1h}, ... , $\lambda_{i_{|I_\mathrm{A}|}h}$ is written as

$$\boldsymbol{\varphi}_{gh} = \boldsymbol{\varphi}_{g0h} + \sum_{i\in I_\mathrm{A}} \lambda_{ih}\boldsymbol{\varphi}_{gih} = \boldsymbol{\varphi}_g + \delta\boldsymbol{\varphi}_{gh}. \tag{9.11.11}$$

Moreover, Eq. (9.10.2) obtained using the formulae based on the partial shape derivative of a function is written as

$$\bar{\boldsymbol{\varphi}}_{gh} = \bar{\boldsymbol{\varphi}}_{g0h} + \sum_{i\in I_\mathrm{A}} \bar{\lambda}_{ih}\bar{\boldsymbol{\varphi}}_{gih} = \bar{\boldsymbol{\varphi}}_g + \delta\bar{\boldsymbol{\varphi}}_{gh}. \tag{9.11.12}$$

In the above definitions, the error for the search vector is given by $\delta\boldsymbol{\varphi}_{gh}$ and $\delta\bar{\boldsymbol{\varphi}}_{gh}$ of Eq. (9.11.11) and Eq. (9.11.12), respectively. Hence, the aim of this section is to conduct an order evaluation of h with respect to their norms. If such a result can be obtained, the way to select the order of the basis function such that the numerical solution for the search vector converging to the exact solution will be apparent. Here, the following assumptions are essential.

Hypothesis 9.11.1 (Error estimation of $\boldsymbol{\varphi}_g$ and $\bar{\boldsymbol{\varphi}}_g$) For $q_\mathrm{R} > d$ and $k_1, k_2, j \in \{1, 2, \dots\}$, we assume the following conditions hold:

(1) The homogeneous forms of the exact solutions u of the state determination problem and v_0, v_{i_1}, ... , $v_{i_{|I_\mathrm{A}|}}$ of the adjoint problem with respect to f_0, f_{i_1}, ... , $f_{i_{|I_\mathrm{A}|}}$ are elements of

$$\mathcal{S} = U \cap W^{\max\{k_1,k_2\}+1,2q_\mathrm{R}}(D;\mathbb{R}). \tag{9.11.13}$$

If necessary, Hypotheses 9.5.1, 9.6.1, 9.5.2, 9.6.2 and 9.5.3 will be amended so that this assumption holds. Also, we let $\partial\Omega$ be of class $H^2 \cap C^{0,1}$ where $\tilde{\Gamma}_0 = \Gamma_{p0} \cup \Gamma_{\eta 00} \cup \Gamma_{\eta 10} \cup \cdots \cup \Gamma_{\eta m0}$ belongs to a class of piecewise $H^3 \cap C^{1,1}$, and $X_1 = X \cap W^{1,q_\mathrm{R}}\left(D;\mathbb{R}^d\right)$ be the linear space of $\boldsymbol{\phi}$.

(2) If the formulae based on the shape derivative of a function are used, the integrands of the cost function f_i for each $i \in I_\mathrm{A} \cup \{0\}$ satisfy

$$\zeta_{iu\boldsymbol{\nabla}u} \in L^\infty\left(D;\mathbb{R}^d\right), \tag{9.11.14}$$
$$\zeta_{i\boldsymbol{\nabla}u(\boldsymbol{\nabla}u)^\top} \in L^\infty\left(D;\mathbb{R}^{d\times d}\right). \tag{9.11.15}$$

(3) There exist some positive constants c_1, c_2, c_3 and $\bar{c}_3$ which do not depend on h such that

$$\|\delta u_h\|_{W^{j,2q_\mathrm{R}}(\Omega;\mathbb{R})} \le c_1 h^{k_1+1-j} |u|_{W^{k_1+1,2q_\mathrm{R}}(\Omega;\mathbb{R})}, \tag{9.11.16}$$
$$\|\delta v_{ih}\|_{W^{j,2q_\mathrm{R}}(\Omega;\mathbb{R})} \le c_2 h^{k_1+1-j} |v_i|_{W^{k_1+1,2q_\mathrm{R}}(\Omega;\mathbb{R})}, \tag{9.11.17}$$
$$\left\|\delta\hat{\boldsymbol{\varphi}}_{gih}\right\|_{W^{j,2q_\mathrm{R}}(\Omega;\mathbb{R}^d)} \le c_3 h^{k_2+1-j} \left|\hat{\boldsymbol{\varphi}}_{gi}\right|_{W^{k_2+1,2q_\mathrm{R}}(\Omega;\mathbb{R}^d)}, \tag{9.11.18}$$
$$\left\|\delta\hat{\bar{\boldsymbol{\varphi}}}_{gih}\right\|_{W^{j,2q_\mathrm{R}}(\Omega;\mathbb{R}^d)} \le \bar{c}_3 h^{k_2+1-j} \left|\hat{\bar{\boldsymbol{\varphi}}}_{gi}\right|_{W^{k_2+1,2q_\mathrm{R}}(\Omega;\mathbb{R}^d)} \tag{9.11.19}$$

for all $i \in I_\mathrm{A} \cup \{0\}$.

(4) With respect to the coefficient matrices $\boldsymbol{A}_h$ and $\bar{\boldsymbol{A}}_h$ of Eq. (9.11.9) and Eq. (9.11.10), respectively, there exist positive constants c_4 and $\bar{c}_4$ that satisfy

$$\left\|\boldsymbol{A}_h^{-1}\right\|_{\mathbb{R}^{|I_{\mathrm{A}}|\times|I_{\mathrm{A}}|}} \leq c_4, \tag{9.11.20}$$

$$\left\|\bar{\boldsymbol{A}}_h^{-1}\right\|_{\mathbb{R}^{|I_{\mathrm{A}}|\times|I_{\mathrm{A}}|}} \leq \bar{c}_4, \tag{9.11.21}$$

where $\|\cdot\|_{\mathbb{R}^{|I_{\mathrm{A}}|\times|I_{\mathrm{A}}|}}$ represents the norm of the matrix (see Eq. (4.4.3)).

□

Since $k_1 \in \{1, 2, \ldots\}$, Hypothesis 9.11.1 (1) is a stronger condition than $\mathcal{S}$ defined in Eq. (9.5.2). The reason for this is because in Hypothesis 9.11.1 (3), the right-hand side of Eq. (9.11.17) and Eq. (9.11.16) require u and v_0, v_{i_1}, $\ldots$, $v_{i_{|I_{\mathrm{A}}|}}$ to be of class $W^{k_1+1,2q_{\mathrm{R}}}$. Hypothesis 9.11.1 (3) is based on Corollary 6.6.4. Hypothesis 9.11.1 (4) is a condition which holds when $\boldsymbol{g}_{i_1}$, $\ldots$, $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}}$ are linearly independent.

We shall give the result of the error estimation in Theorem 9.11.5 after we have proved the following lemmas.

Lemma 9.11.2 (Error estimation of $\boldsymbol{g}_i$ and $\bar{\boldsymbol{g}}_i$) Suppose Hypothesis 9.11.1 (1) and (2) as well as Eq. (9.11.16) and Eq. (9.11.17) are satisfied. Then, there exist positive constants c_5 and $\bar{c}_5$ which do not depend on h with respect to $\delta\boldsymbol{g}_{ih}$ and $\delta\bar{\boldsymbol{g}}_{ih}$ of Eq. (9.11.3) and Eq. (9.11.4), respectively, and the estimates

$$\langle \delta\boldsymbol{g}_{ih}, \boldsymbol{\varphi}\rangle \leq c_5 h^{k_1-1} \|\boldsymbol{\varphi}\|_{X_1}, \tag{9.11.22}$$

$$\langle \delta\bar{\boldsymbol{g}}_{ih}, \boldsymbol{\varphi}\rangle \leq \bar{c}_5 h^{k_1-1} \|\boldsymbol{\varphi}\|_{X_1} \tag{9.11.23}$$

hold for all $\boldsymbol{\varphi} \in X_1$. Furthermore, when Hypothesis 9.8.3 (3) is satisfied, we also have the estimate

$$\langle \delta\boldsymbol{g}_{ih}, \boldsymbol{\varphi}\rangle \leq c_5 h^{k_1} \|\boldsymbol{\varphi}\|_{X_1}. \tag{9.11.24}$$

□

Proof The numerical error $\delta\boldsymbol{g}_{ih}$ of $\boldsymbol{g}_i$ using the formulae based on the shape derivative of a function is a numerical error due to δu_h and δv_{ih}. Hence, from Eq. (9.8.5),

$$|\langle \delta\boldsymbol{g}_{ih}, \boldsymbol{\varphi}\rangle| \leq |\mathscr{L}_{i\boldsymbol{\phi}' u v_i}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}, \delta u_h, \delta v_{ih}]| \tag{9.11.25}$$

is established. If the Hölder inequality (Theorem A.9.1), the Poincaré inequality (Corollary A.9.4) and the trace theorem (Theorem 4.4.2) are used, the right-hand side of Eq. (9.11.25) is suppressed as

$$\begin{aligned}&|\mathscr{L}_{i\boldsymbol{\phi}' u v_i}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}, \delta u_h, \delta v_{ih}]| \\ &\quad \leq \|\delta\boldsymbol{G}_{\Omega ih}\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R}^{d\times d})} \left\|\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\right\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\end{aligned}$$

$$
\begin{aligned}
&+ \|\delta g_{\Omega ih}\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\nabla \cdot \varphi\|_{L^2(\Omega;\mathbb{R})} \\
&+ \left\|\delta \boldsymbol{g}_{\zeta bih}\right\|_{L^{q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\varphi\|_{L^2\left(\Omega;\mathbb{R}^d\right)} \\
&+ \left\|\delta \boldsymbol{g}_{pih}\right\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \|\varphi\|_{L^2\left(\Gamma_p;\mathbb{R}^d\right)} \\
&+ \left\|\delta \boldsymbol{g}_{\partial pih}\right\|_{L^\infty\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \|\varphi\|_{L^2\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \\
&+ \left\|\delta \boldsymbol{g}_{\eta ih}\right\|_{L^\infty\left(\Gamma_{\eta i};\mathbb{R}^d\right)} \|\varphi\|_{L^2\left(\Gamma_{\eta i};\mathbb{R}^d\right)} \\
&+ \left\|\delta \boldsymbol{g}_{\partial\eta ih}\right\|_{L^\infty\left(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d\right)} \|\varphi\|_{L^2\left(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d\right)} \\
\leq{}& \Big\{ \|\delta \boldsymbol{G}_{\Omega ih}\|_{L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)} + \|\delta g_{\Omega ih}\|_{L^\infty(\Omega;\mathbb{R})} + \left\|\delta \boldsymbol{g}_{\zeta bih}\right\|_{L^\infty\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \|\gamma_{\partial\Omega}\| \Big(\left\|\delta \boldsymbol{g}_{pih}\right\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} + \left\|\gamma_{\partial\Gamma_p}\right\| \left\|\delta \boldsymbol{g}_{\partial pih}\right\|_{L^\infty\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \\
&\qquad + \left\|\delta \boldsymbol{g}_{\eta ih}\right\|_{L^\infty\left(\Gamma_{\eta i};\mathbb{R}^d\right)} + \left\|\gamma_{\partial\Gamma_{\eta i}}\right\| \left\|\delta \boldsymbol{g}_{\partial\eta ih}\right\|_{L^\infty\left(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d\right)} \Big) \Big\} \|\varphi\|_{X_1} .
\end{aligned} \tag{9.11.26}
$$

Here, $\|\gamma_{\partial\Omega}\|$, $\left\|\gamma_{\partial\Gamma_p}\right\|$ and $\left\|\gamma_{\partial\Gamma_{\eta i}}\right\|$, respectively, represent the norms of the following trace operators for $\varphi \in X_1$:

$$
\begin{aligned}
\gamma_{\partial\Omega} &: W^{1,q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right) \to W^{1-1/q_{\mathrm{R}},q_{\mathrm{R}}}\left(\partial\Omega;\mathbb{R}^d\right), \\
\gamma_{\partial\Gamma_p} &: W^{1-1/q_{\mathrm{R}},q_{\mathrm{R}}}\left(\partial\Omega;\mathbb{R}^d\right) \to W^{1-2/q_{\mathrm{R}},q_{\mathrm{R}}}\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right), \\
\gamma_{\partial\Gamma_{\eta i}} &: W^{1-1/q_{\mathrm{R}},q_{\mathrm{R}}}\left(\partial\Omega;\mathbb{R}^d\right) \to W^{1-2/q_{\mathrm{R}},q_{\mathrm{R}}}\left(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d\right)
\end{aligned}
$$

and are bounded from the trace theorem because $\partial\Omega$ was assumed to be of class $H^2 \cap C^{0,1}$ in Hypothesis 9.11.1 (1). Moreover, we have the following estimates:

$$
\begin{aligned}
&\|\delta \boldsymbol{G}_{\Omega ih}\|_{L^{q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^{d\times d}\right)} \\
&\leq 2\Big(\|\nabla \delta u_h\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\nabla v_i\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \|\nabla u\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\nabla \delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \Big) \\
&\quad + \|\zeta_{iu\nabla u}\|_{L^\infty\left(\Omega;\mathbb{R}^d\right)} \|\delta u_h\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\nabla u\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \left\|\zeta_{i\nabla u(\nabla u)^\top}\right\|_{L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)} \|\nabla \delta u_h\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\nabla u\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \left\|\zeta_{i(\nabla u)^\top}\right\|_{W^{1,2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\nabla \delta u_h\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\leq 2\Big(\|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|v_i\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \|u\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \Big) \\
&\quad + \|\zeta_{iu\nabla u}\|_{L^\infty\left(\Omega;\mathbb{R}^d\right)} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|u\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \left\|\zeta_{i\nabla u(\nabla u)^\top}\right\|_{L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|u\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \left\|\zeta_{i(\nabla u)^\top}\right\|_{W^{1,2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} ,
\end{aligned} \tag{9.11.27}
$$

$$
\|\delta g_{\Omega ih}\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R})}
$$

$$
\begin{aligned}
&\le \|\zeta_{iu}\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta u_h\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \left\|\zeta_{i(\boldsymbol{\nabla} u)^\top}\right\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\boldsymbol{\nabla}\delta u_h\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \|\boldsymbol{\nabla}\delta u_h\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\boldsymbol{\nabla} v_i\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \|\boldsymbol{\nabla} u\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad + \|b\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\le \|\zeta_{iu}\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \left\|\zeta_{i(\boldsymbol{\nabla} u)^\top}\right\|_{L^{2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|v_i\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \|u\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\quad + \|b\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}, \qquad (9.11.28)
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\delta \boldsymbol{g}_{\zeta b i h}\right\|_{L^{q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)} \\
&\quad \le \left\|b'\right\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \le \left\|b'\right\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}, \\
&\qquad (9.11.29)
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\delta \boldsymbol{g}_{p i h}\right\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \\
&\quad \le \|\kappa\|_{C^0\left(\Gamma_p;\mathbb{R}\right)} \|\boldsymbol{\nu}\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \|p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \\
&\qquad + (d-1) \max_{i\in\{1,\dots,d-1\}} \|\boldsymbol{\tau}_i\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \Big(\|\boldsymbol{\nabla} p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}^d\right)} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \\
&\qquad\quad + \|p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}^d\right)} \Big) + \left\|p_{\mathrm{N}}'\right\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\Gamma_p;\mathbb{R}\right)} \\
&\quad \le \|\kappa\|_{C^0\left(\Gamma_p;\mathbb{R}\right)} \|\boldsymbol{\nu}\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \|\gamma_{\partial\Omega}\|^2 \|p_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\qquad + (d-1) \max_{i\in\{1,\dots,d-1\}} \|\boldsymbol{\tau}_i\|_{L^\infty\left(\Gamma_p;\mathbb{R}^d\right)} \|\gamma_{\partial\Omega}\|^2 \\
&\qquad\quad \times \Big(\|p_{\mathrm{N}}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} + \|p_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \Big) \\
&\qquad + \|\gamma_{\partial\Omega}\|^2 \left\|p_{\mathrm{N}}'\right\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}, \qquad (9.11.30)
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\delta \boldsymbol{g}_{\partial p i h}\right\|_{L^\infty\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \\
&\quad \le \|\boldsymbol{\tau}\|_{L^\infty\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \|p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}\left(\partial\Gamma_p\cup\Theta;\mathbb{R}\right)} \|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}\right)} \\
&\quad \le \|\boldsymbol{\tau}\|_{L^\infty\left(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d\right)} \|\gamma_{\partial\Omega}\|^2 \|\gamma_{\partial\Gamma}\|^2 \|p_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}. \\
&\qquad (9.11.31)
\end{aligned}
$$

A similar result is obtained for $\left\|\delta \boldsymbol{g}_{\eta i h}\right\|_{L^\infty\left(\Gamma_{\eta i};\mathbb{R}^d\right)}$ $\left\|\delta \boldsymbol{g}_{\partial\eta i h}\right\|_{L^\infty\left(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d\right)}$. Here, if Hypothesis 9.11.1 (1) and (2) are satisfied, all the expressions without the terms with δ are bounded. Moreover, if we focus on the terms with δ, there is a term containing $\|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}\left(\Omega;\mathbb{R}^d\right)}$ in Eq. (9.11.30). Similarly, $\|\delta u_h\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}$ is contained in the inequality equation for $\left\|\delta \boldsymbol{g}_{\eta i h}\right\|_{L^\infty\left(\Gamma_{\eta i};\mathbb{R}^d\right)}$. Hence, if Eq. (9.11.16) and Eq. (9.11.17) with $j = 2$ are substituted in for the terms with δ, these terms become bounded. Hence, we can obtain Eq. (9.11.22).

Furthermore, if Hypothesis 9.8.3 (3) (Eq. (9.8.9) to Eq. (9.8.13) are zero, or $\tilde{\Gamma}_0 =$

$\Gamma_{p0} \cup \Gamma_{\eta i0} \subset \bar{\Omega}_{\mathrm{C}0}$ in Eq. (9.1.1)) is satisfied, since the terms with $\boldsymbol{\tau}$ disappear, there are no terms which contain $\|\delta u_h\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}$ and $\|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R}^d)}$. Hence, Eq. (9.11.16) and Eq. (9.11.17) with $j = 1$ can be substituted into the terms with δ to obtain Eq. (9.11.24).

On the other hand, the numerical error $\delta\bar{\boldsymbol{g}}_{ih}$ using the formulae based on the partial shape derivative of a function satisfies

$$|\langle \delta\bar{\boldsymbol{g}}_{ih}, \boldsymbol{\varphi}\rangle| \le |\mathscr{L}_{i\boldsymbol{\phi}^* u v_i}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}, \delta u_h, \delta v_{ih}]| \tag{9.11.32}$$

from Eq. (9.8.35). If the Hölder inequality (Theorem A.9.1), the Poincaré inequality (Corollary A.9.4) and the trace theorem (Theorem 4.4.2) are used, the right-hand side of Eq. (9.11.32) is suppressed as

$$\begin{aligned}
&|\mathscr{L}_{i\boldsymbol{\phi}^* u v_i}(\boldsymbol{\phi}, u, v_i)[\boldsymbol{\varphi}, \delta u_h, \delta v_{ih}]| \\
&\quad \le \left\|\delta\bar{\boldsymbol{g}}_{\zeta b i h}\right\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\qquad + \|\delta\bar{\boldsymbol{g}}_{\partial\Omega i h}\|_{L^\infty(\partial\Omega;\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\partial\Omega;\mathbb{R}^d)} \\
&\qquad + \left\|\delta\bar{\boldsymbol{g}}_{p i h}\right\|_{L^\infty(\Gamma_p;\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Gamma_p;\mathbb{R}^d)} \\
&\qquad + \left\|\delta\bar{\boldsymbol{g}}_{\partial p i h}\right\|_{L^\infty(\partial\Gamma_p \cup \Theta_p;\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\partial\Gamma_p \cup \Theta_p;\mathbb{R}^d)} \\
&\qquad + \left\|\delta\bar{\boldsymbol{g}}_{\eta i h}\right\|_{L^\infty(\Gamma_{\eta i};\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Gamma_{\eta i};\mathbb{R}^d)} \\
&\qquad + \left\|\delta\bar{\boldsymbol{g}}_{\partial\eta i h}\right\|_{L^\infty(\partial\Gamma_{\eta i} \cup \Theta_{\eta i};\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\partial\Gamma_{\eta i} \cup \Theta_{\eta i};\mathbb{R}^d)} \\
&\qquad + \|\delta\bar{\boldsymbol{g}}_{\mathrm{D}h}\|_{L^\infty(\Gamma_{\mathrm{D}};\mathbb{R}^d)} \|\boldsymbol{\varphi}\|_{L^2(\Gamma_{\mathrm{D}};\mathbb{R}^d)} \\
&\quad \le \Big\{ \left\|\delta\bar{\boldsymbol{g}}_{\zeta b i h}\right\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R}^d)} + \|\gamma_{\partial\Omega}\|^2 \Big(\|\delta\bar{\boldsymbol{g}}_{\partial\Omega i h}\|_{L^\infty(\partial\Omega;\mathbb{R}^d)} + \left\|\delta\bar{\boldsymbol{g}}_{p i h}\right\|_{L^\infty(\Gamma_p;\mathbb{R}^d)} \\
&\qquad\quad + \|\gamma_{\partial\Gamma}\| \left\|\delta\bar{\boldsymbol{g}}_{\partial p i h}\right\|_{L^\infty(\partial\Gamma_p \cup \Theta_p;\mathbb{R}^d)} + \left\|\delta\bar{\boldsymbol{g}}_{\eta i h}\right\|_{L^\infty(\Gamma_{\eta i};\mathbb{R}^d)} \\
&\qquad\quad + \|\gamma_{\partial\Gamma}\| \left\|\delta\bar{\boldsymbol{g}}_{\partial\eta i h}\right\|_{L^\infty(\partial\Gamma_{\eta i} \cup \Theta_{\eta i};\mathbb{R}^d)} + \|\delta\bar{\boldsymbol{g}}_{\mathrm{D}h}\|_{L^\infty(\Gamma_{\mathrm{D}};\mathbb{R}^d)} \Big) \Big\} \|\boldsymbol{\varphi}\|_{X_1},
\end{aligned}$$

where

$$\begin{aligned}
&\left\|\delta\bar{\boldsymbol{g}}_{\zeta b i h}\right\|_{L^{q_{\mathrm{R}}}(\Omega;\mathbb{R}^d)} \le \|b\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\|\delta\bar{\boldsymbol{g}}_{\partial\Omega i h}\|_{L^\infty(\partial\Omega;\mathbb{R}^d)} \\
&\quad \le \Big(\|\zeta_{iu}\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R})} \|\delta u_h\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R})} \\
&\qquad\quad + \|\boldsymbol{\nabla}\delta u_h\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R}^d)} \|\boldsymbol{\nabla} v_i\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R}^d)} \\
&\qquad\quad + \|\boldsymbol{\nabla} u\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R}^d)} \|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R}^d)} \\
&\qquad\quad + \|b\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R})} \|\delta u_h\|_{L^{2q_{\mathrm{R}}}(\partial\Omega;\mathbb{R})} \Big) \|\boldsymbol{\nu}\|_{L^\infty(\partial\Omega;\mathbb{R}^d)} \\
&\quad \le \Big(\|\zeta_{iu}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\qquad\quad + \|\delta u_h\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|v_i\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\qquad\quad + \|u\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \\
&\qquad\quad + \|b\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \|\delta u_h\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})} \Big) \|\boldsymbol{\nu}\|_{L^\infty(\partial\Omega;\mathbb{R}^d)},
\end{aligned}$$

$$
\begin{aligned}
&\left\|\delta\bar{\boldsymbol{g}}_{pih}\right\|_{L^\infty(\Gamma_p;\mathbb{R}^d)}\\
&\quad\le \|\kappa\|_{C^0(\Gamma_p;\mathbb{R})}\|\boldsymbol{\nu}\|_{L^\infty(\Gamma_p;\mathbb{R}^d)}\|p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\\
&\qquad+\|\boldsymbol{\nu}\|^2_{L^\infty(\Gamma_p;\mathbb{R}^d)}\Big(\|\boldsymbol{\nabla} p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R}^d)}\|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\\
&\qquad\quad+\|p_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R}^d)}\Big)\\
&\qquad+\|p^*_{\mathrm{N}}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\|\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_p;\mathbb{R})}\\
&\quad\le \|\kappa\|_{C^0(\Gamma_p;\mathbb{R})}\|\boldsymbol{\nu}\|_{L^\infty(\Gamma_p;\mathbb{R}^d)}\|\gamma_{\partial\Omega}\|^2\|p_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\\
&\qquad+\|\boldsymbol{\nu}\|^2_{L^\infty(\Gamma_p;\mathbb{R}^d)}\|\gamma_{\partial\Omega}\|^2\Big(\|p_{\mathrm{N}}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\\
&\qquad\quad+\|p_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\Big)\\
&\qquad+\|\gamma_{\partial\Omega}\|^2\|p^*_{\mathrm{N}}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta v_{ih}\|_{W^{1,2q_{\mathrm{R}}}(\Omega;\mathbb{R})},\\
&\left\|\delta\bar{\boldsymbol{g}}_{\partial pih}\right\|_{L^\infty(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d)}=\left\|\delta\boldsymbol{g}_{\partial pih}\right\|_{L^\infty(\partial\Gamma_p\cup\Theta_p;\mathbb{R}^d)},\\
&\|\delta\bar{\boldsymbol{g}}_{\mathrm{D}h}\|_{L^\infty(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\\
&\quad\le\|\boldsymbol{\nu}\|^2_{L^\infty(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\Big(\|\boldsymbol{\nabla}\delta u_h\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\|\boldsymbol{\nabla} v_i\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\\
&\qquad\quad+\|\boldsymbol{\nabla}(u-u_{\mathrm{D}})\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\\
&\qquad\quad+\|\boldsymbol{\nabla}\delta v_{ih}\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\|\boldsymbol{\nabla} u\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\\
&\qquad\quad+\|\boldsymbol{\nabla}(v_i-v_{\mathrm{D}i})\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\|\boldsymbol{\nabla}\delta u_h\|_{L^{2q_{\mathrm{R}}}(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\Big)\\
&\quad\le\|\gamma_{\partial\Omega}\|^2\|\boldsymbol{\nu}\|^2_{L^\infty(\Gamma_{\mathrm{D}};\mathbb{R}^d)}\Big(\|\delta u_h\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|v_i\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\\
&\qquad\quad+\|(u-u_{\mathrm{D}})\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\\
&\qquad\quad+\|\delta v_{ih}\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|u\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\\
&\qquad\quad+\|(v_i-v_{\mathrm{D}i})\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\|\delta u_h\|_{W^{2,2q_{\mathrm{R}}}(\Omega;\mathbb{R})}\Big).
\end{aligned}
$$

Similar results can be obtained for $\left\|\delta\bar{\boldsymbol{g}}_{\eta ih}\right\|_{L^\infty(\Gamma_{\eta i};\mathbb{R}^d)}$ and $\left\|\delta\bar{\boldsymbol{g}}_{\partial\eta ih}\right\|_{L^\infty(\partial\Gamma_{\eta i}\cup\Theta_{\eta i};\mathbb{R}^d)}$. Here, if Hypothesis 9.11.1 (1) is satisfied, all expressions without the term with δ are bounded. Moreover, if Eq. (9.11.16) and Eq. (9.11.17) with $j=2$ are substituted into the terms with δ, Eq. (9.11.23) can be obtained, completing the proof of the lemma. □

Lemma 9.11.3 (Error estimation of $\boldsymbol{\varphi}_{gi}$ and $\bar{\boldsymbol{\varphi}}_{gi}$) Suppose Hypothesis 9.11.1 (1), (2) and Eq. (9.11.16) and Eq. (9.11.17) hold. Then, there exist positive constants c_6 and $\bar{c}_6$ which do not depend on h such that

$$\left\|\delta\boldsymbol{\varphi}_{gih}\right\|_{X_1}\le c_6 h^{\min\{k_1-1,k_2\}}, \tag{9.11.33}$$

$$\left\|\delta\bar{\boldsymbol{\varphi}}_{gih}\right\|_{X_1}\le \bar{c}_6 h^{\min\{k_1-1,k_2\}} \tag{9.11.34}$$

holds with respect to $\delta\boldsymbol{\varphi}_{gih}$ and $\delta\bar{\boldsymbol{\varphi}}_{gih}$ of Eq. (9.11.7) and Eq. (9.11.8), respectively. Furthermore, if Hypothesis 9.8.3 (3) is satisfied, then we also have

$$\left\|\delta\boldsymbol{\varphi}_{gih}\right\|_{X_1}\le c_6 h^{\min\{k_1,k_2\}}. \tag{9.11.35}$$

□

Proof When the formulae of the shape derivative of a function are used,

$$\left\| \delta \boldsymbol{\varphi}_{gih} \right\|_{X_1} \leq \left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1} + \left\| \delta \hat{\boldsymbol{\varphi}}_{gih} \right\|_{X_1} \tag{9.11.36}$$

holds because of Eq. (9.11.5) and Eq. (9.11.7). Here, $\left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1}$ shows the error in the exact solution of the H^1 gradient method (see, for example, Problem 9.9.3) caused by $\delta \boldsymbol{g}_{ih}$ of Lemma 9.11.2. $\left\| \delta \hat{\boldsymbol{\varphi}}_{gih} \right\|_{X_1}$ shows the error in the numerical solution of the H^1 gradient method. $\left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1}$ of Eq. (9.11.36) satisfies

$$a_X \left(\delta \hat{\boldsymbol{\varphi}}_{gi}, \boldsymbol{\varphi} \right) = - \left\langle \delta \boldsymbol{g}_{ih}, \boldsymbol{\varphi} \right\rangle$$

for all $\boldsymbol{\varphi} \in X_1$. Hence, if we let $\boldsymbol{\varphi} = \delta \hat{\boldsymbol{\varphi}}_{gi}$,

$$\alpha_X \left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1}^2 \leq \left| \left\langle \delta \boldsymbol{g}_{ih}, \delta \hat{\boldsymbol{\varphi}}_{gi} \right\rangle \right| \tag{9.11.37}$$

holds, where α_X is a positive constant used in Eq. (9.9.1). With respect to $\delta \boldsymbol{g}_{ih}$ of Eq. (9.11.37), if Eq. (9.11.22) of Lemma 9.11.2 is used,

$$\left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1} \leq \frac{c_5}{\alpha_X} h^{k_1 - 1} \tag{9.11.38}$$

is obtained. On the other hand, $\left\| \delta \hat{\boldsymbol{\varphi}}_{gih} \right\|_{H^1\left(\Omega; \mathbb{R}^d\right)}$ satisfies

$$\left\| \delta \hat{\boldsymbol{\varphi}}_{gih} \right\|_{X_1} \leq \left\| \delta \hat{\boldsymbol{\varphi}}_{gih} \right\|_{W^{1,2q_{\mathrm{R}}}\left(\Omega; \mathbb{R}^d\right)} \leq c_3 h^{k_2} \left\| \hat{\boldsymbol{\varphi}}_{gi} \right\|_{W^{k_2+1,2q_{\mathrm{R}}}\left(\Omega; \mathbb{R}^d\right)} \tag{9.11.39}$$

in view of Eq. (9.11.18) with $j = 1$. In Eq. (9.11.39), $\left\| \hat{\boldsymbol{\varphi}}_{gi} \right\|_{W^{k_2+1,2q_{\mathrm{R}}}\left(\Omega; \mathbb{R}^d\right)}$ is bounded. This is because if Hypothesis 9.11.1 (1) is used in the proof of Theorem 9.9.6, then we have that $\hat{\boldsymbol{\varphi}}_{gi} \in W^{k_2+1,\infty}\left(\Omega; \mathbb{R}^d\right)$. Hence, if Eq. (9.11.38) and Eq. (9.11.39) are substituted into Eq. (9.11.36), then we obtain Eq. (9.11.33).

Furthermore, if Hypothesis 9.8.3 (3) is satisfied, Eq. (9.11.24) of Lemma 9.11.2 can then be applied to $\delta \boldsymbol{g}_{ih}$ of Eq. (9.11.37) to get

$$\left\| \delta \hat{\boldsymbol{\varphi}}_{gi} \right\|_{X_1} \leq \frac{c_5}{\alpha_X} h^{k_1}. \tag{9.11.40}$$

Here, if Eq. (9.11.40) and Eq. (9.11.39) are substituted into Eq. (9.11.36), then we arrive at Eq. (9.11.35) of the lemma.

If $\delta \boldsymbol{g}_{ih}$ of Eq. (9.11.37) is changed to $\delta \bar{\boldsymbol{g}}_{ih}$ and Eq. (9.11.23) of Lemma 9.11.2 is used with respect to $\delta \bar{\boldsymbol{g}}_{ih}$, then we get Eq. (9.11.34), which finishes the proof of the lemma. □

Lemma 9.11.4 (Error estimation of $\boldsymbol{\lambda}_h$ and $\bar{\boldsymbol{\lambda}}_h$) Suppose Hypothesis 9.11.1 holds. Then, there exist positive constants c_7 $\bar{c}_7$ which do not depend on h such that

$$\left\| \delta \boldsymbol{\lambda}_h \right\|_{\mathbb{R}^{|I_{\mathrm{A}}|}} \leq c_7 h^{\min\{k_1 - 1, k_2\}}, \tag{9.11.41}$$

$$\left\| \delta \bar{\boldsymbol{\lambda}}_h \right\|_{\mathbb{R}^{|I_{\mathrm{A}}|}} \leq \bar{c}_7 h^{\min\{k_1 - 1, k_2\}} \tag{9.11.42}$$

hold with respect to $\boldsymbol{\lambda}_h$ of Eq. (9.11.9) and $\bar{\boldsymbol{\lambda}}_h$ of Eq. (9.11.10), respectively. Furthermore, if Hypothesis 9.8.3 (3) is satisfied, then we also have

$$\|\delta\boldsymbol{\lambda}_h\|_{\mathbb{R}^{|I_{\mathrm{A}}|}} \le c_7 h^{\min\{k_1,k_2\}}. \tag{9.11.43}$$

□

Proof When the formulae based on the shape derivative of a function are used, one has

$$\begin{aligned}\delta\boldsymbol{\lambda}_h &= \boldsymbol{A}_h^{-1}\left(-\delta\boldsymbol{A}_h\boldsymbol{\lambda} + \delta\boldsymbol{b}_h\right)\\ &= \boldsymbol{A}_h^{-1}\Big\{-\Big(\left(\langle\delta\boldsymbol{g}_{ih},\boldsymbol{\varphi}_{gj}\rangle\right)_{(i,j)\in I_{\mathrm{A}}^2} - \left(\langle\boldsymbol{g}_i,\delta\boldsymbol{\varphi}_{gjh}\rangle\right)_{(i,j)\in I_{\mathrm{A}}^2}\Big)\boldsymbol{\lambda}\\ &\quad + \left(\langle\delta\boldsymbol{g}_{ih},\boldsymbol{\varphi}_{g0}\rangle\right)_{i\in I_{\mathrm{A}}} + \left(\langle\boldsymbol{g}_i,\delta\boldsymbol{\varphi}_{g0h}\rangle\right)_{i\in I_{\mathrm{A}}}\Big\}\end{aligned}$$

with respect to $\boldsymbol{\lambda}_h$ of Eq. (9.11.9). Hence, if Eq. (9.11.20) is used, then we get the bound

$$\begin{aligned}\|\delta\boldsymbol{\lambda}_h\|_{\mathbb{R}^{|I_{\mathrm{A}}|}} &\le c_4\left(1 + |I_{\mathrm{A}}|\max_{i\in I_{\mathrm{A}}}|\lambda_i|\right)\\ &\quad\times\max_{(i,j)\in I_{\mathrm{A}}\times(I_{\mathrm{A}}\cup\{0\})}\left(\left|\langle\delta\boldsymbol{g}_{ih},\boldsymbol{\varphi}_{gj}\rangle\right| + \left|\langle\boldsymbol{g}_i,\delta\boldsymbol{\varphi}_{gjh}\rangle\right|\right).\end{aligned} \tag{9.11.44}$$

For $\left|\langle\delta\boldsymbol{g}_{ih},\boldsymbol{\varphi}_{gj}\rangle\right|$ of Eq. (9.11.44), if Eq. (9.11.22) of Lemma 9.11.2 is used,

$$\left|\langle\delta\boldsymbol{g}_{ih},\boldsymbol{\varphi}_{gj}\rangle\right| \le c_5 h^{k_1-1}\left\|\boldsymbol{\varphi}_{gj}\right\|_X \tag{9.11.45}$$

holds. Moreover, we have

$$\left|\langle\boldsymbol{g}_i,\delta\boldsymbol{\varphi}_{gjh}\rangle\right| \le c_6 h^{k_1-1}\|\boldsymbol{g}_i\|_{X_1'} \tag{9.11.46}$$

from Eq. (9.11.33) of Lemma 9.11.3. In Eq. (9.11.46), $\|\boldsymbol{g}_i\|_{X_1'}$ is bounded. This is because from Theorem 9.8.2, $\boldsymbol{g}_i \in X'$ holds, and using $X' \subset X_1'$, $\|\boldsymbol{g}_i\|_{X_1'} \le \|\boldsymbol{g}_i\|_{X'} < \infty$ is obtained. Here, if Eq. (9.11.45) and Eq. (9.11.46) are substituted into Eq. (9.11.44), then we obtain Eq. (9.11.41) of the lemma.

Furthermore, if Hypothesis 9.8.3 (3) is satisfied, by applying Eq. (9.11.24) of Lemma 9.11.2 to $\delta\boldsymbol{g}_{ih}$ of Eq. (9.11.45), then we get Eq. (9.11.43) of the lemma.

If $\delta\boldsymbol{g}_{ih}$ and $\delta\boldsymbol{\varphi}_{gjh}$ of Eq. (9.11.45) and Eq. (9.11.46) are replaced by $\delta\bar{\boldsymbol{g}}_{ih}$ and $\delta\bar{\boldsymbol{\varphi}}_{gjh}$, respectively, then we can apply Theorem 9.8.2 in place of Theorem 9.8.6, and, in addition, applying Eq. (9.11.33) of Lemma 9.11.3 in place of Eq. (9.11.34), we eventually obtain Eq. (9.11.42), completing the proof of the lemma. □

The following results can be obtained based on these lemmas.

Theorem 9.11.5 (Error estimation of $\boldsymbol{\varphi}_g$ and $\bar{\boldsymbol{\varphi}}_g$) Suppose Hypothesis 9.11.1 holds. Then, there exist positive constants c and $\bar{c}$ which do not depend on h such that

$$\left\|\delta\boldsymbol{\varphi}_{gh}\right\|_{X_1} \le c h^{\min\{k_1-1,k_2\}}, \tag{9.11.47}$$

$$\left\|\delta\bar{\boldsymbol{\varphi}}_{gh}\right\|_{X_1} \le \bar{c} h^{\min\{k_1-1,k_2\}} \tag{9.11.48}$$

hold with respect to $\delta\boldsymbol{\varphi}_{gh}$ and $\delta\bar{\boldsymbol{\varphi}}_{gh}$ of Eq. (9.11.11) and Eq. (9.11.12), respectively. Furthermore, if Hypothesis 9.8.3 (3) holds, then we also have

$$\left\|\delta\boldsymbol{\varphi}_{gh}\right\|_{X_1} \leq ch^{\min\{k_1,k_2\}}. \tag{9.11.49}$$

□

Proof From Eq. (9.11.11), we have

$$\delta\boldsymbol{\varphi}_{gh} = \delta\boldsymbol{\varphi}_{g0h} + \sum_{i\in I_{\mathrm{A}}} \left(\delta\lambda_{ih}\boldsymbol{\varphi}_{gi} + \lambda_i\delta\boldsymbol{\varphi}_{gih}\right) \tag{9.11.50}$$

from which we get

$$\begin{aligned}\left\|\delta\boldsymbol{\varphi}_{gh}\right\|_{X_1} \leq & \left(1 + |I_{\mathrm{A}}| \max_{i\in I_{\mathrm{A}}} |\lambda_i|\right) \max_{i\in I_{\mathrm{A}}\cup\{0\}} \left\|\delta\boldsymbol{\varphi}_{gih}\right\|_{X_1} \\ & + \|\delta\boldsymbol{\lambda}_h\|_{\mathbb{R}^{|I_{\mathrm{A}}|}} \max_{i\in I_{\mathrm{A}}} \left\|\boldsymbol{\varphi}_{gi}\right\|_{X_1}.\end{aligned} \tag{9.11.51}$$

If Eq. (9.11.33) of Lemma 9.11.3 and Eq. (9.11.41) of Lemma 9.11.4 are substituted into Eq. (9.11.51), Eq. (9.11.47) of the theorem can be obtained.

Furthermore, if Hypothesis 9.8.3 (3) holds, then, by substituting Eq. (9.11.35) of Lemma 9.11.3 and Eq. (9.11.43) of Lemma 9.11.4 into Eq. (9.11.51), we obtain Eq. (9.11.49) of the theorem.

If $\delta\boldsymbol{\varphi}_{gih}$ and $\delta\boldsymbol{\lambda}_h$ of Eq. (9.11.51) are replaced by $\delta\bar{\boldsymbol{\varphi}}_{gih}$ and $\delta\bar{\boldsymbol{\lambda}}_h$, respectively, and Eq. (9.11.34) of Lemma 9.11.3 and Eq. (9.11.42) of Lemma 9.11.4 are substituted into Eq. (9.11.51), then we obtain Eq. (9.11.48) of the theorem, which finishes the proof. □

Theorem 9.11.5 allows us to infer the following remark about the error estimation of the finite element solution with respect to the shape optimization problem of domain variation type.

Remark 9.11.6 (Error estimation of $\boldsymbol{\varphi}_g$ and $\bar{\boldsymbol{\varphi}}_g$) From Theorem 9.11.5, in order to reduce the error $\left\|\delta\boldsymbol{\varphi}_{gh}\right\|_{X_1}$ of the search vector $\boldsymbol{\varphi}_{gh}$ with respect to h of the finite element division sequence $\{\mathcal{T}_h\}_{h\to 0}$ linearly, the following conditions need to be satisfied.

When Hypothesis 9.11.1 is satisfied:

(1) use the finite element solutions of the state determination problem and the adjoint problems for f_0, f_{i_1}, … , $f_{i_{|I_{\mathrm{A}}|}}$ based on a $k_1 = 2$-order basis function, and

(2) use finite element solutions with respect to $\boldsymbol{g}_0$, $\boldsymbol{g}_{i_1}$, … , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}}$ of Eq. (9.8.5) (formulae based on the shape derivative of a function) or $\bar{\boldsymbol{g}}_0$, $\bar{\boldsymbol{g}}_{i_1}$, … , $\bar{\boldsymbol{g}}_{i_{|I_{\mathrm{A}}|}}$ of Eq. (9.8.35) (formulae based on the partial shape derivative of a function) in the H^1 gradient method based on a $k_2 = 1$-order basis function.

Furthermore, if Hypothesis 9.8.3 (3) is satisfied:

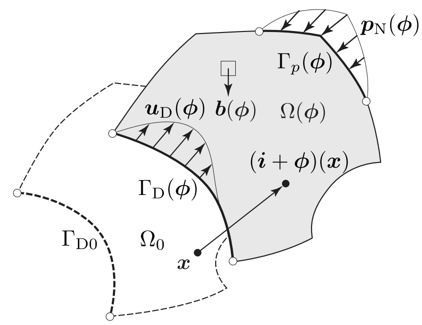


Fig. 9.15: Initial domain $\Omega_0 \subset D$ and domain variation (displacement) $\boldsymbol{\phi}$ in a linear elastic body.

(1) use the finite element solutions of the state determination problem and the adjoint problems for f_0, f_{i_1}, ... , $f_{i_{|I_{\mathrm{A}}|}}$ based on a $k_1 = 1$-order basis function, and

(2) use the finite element solutions with respect to $\boldsymbol{g}_0$, $\boldsymbol{g}_{i_1}$, ... , $\boldsymbol{g}_{i_{|I_{\mathrm{A}}|}}$ of Eq. (9.8.5) (formulae based on the shape derivative of a function) in the H^1 gradient method based on a $k_2 = 1$-order basis function.

□

9.12 Shape Optimization Problem of Linear Elastic Body

As an application of the shape optimization problem, let us consider a mean compliance minimization problem of a linear elastic body, and compute the shape derivatives of cost functions associated with the problem. Here too, the conditions with respect to the initial domain Ω_0, the definitions of Γ_{D0}, Γ_{N0} and Γ_{p0} as well as the definitions of X and $\mathcal{D}$ are taken to be the same as in Sect. 9.1 (Fig. 9.15). However, we describe $\mathcal{D}$ more specifically as follows:

$$\mathcal{D} = \left\{ \boldsymbol{\phi} \in Y \;\middle|\; \begin{cases} |\boldsymbol{\phi}|_{C^{0,1}(D;\mathbb{R}^d)} \le \sigma, \\ \|\boldsymbol{\phi}\|_{H^2\cap C^{0,1}(D;\mathbb{R}^d)} \le \beta \quad (\Gamma_{p0} = \emptyset \text{ or } \Gamma_{p0} \subset \bar{\Omega}_{\mathrm{C0}}), \\ \|\boldsymbol{\phi}\|_{H^3\cap C^{1,1}(D;\mathbb{R}^d)} \le \beta \quad (\Gamma_{p0} \not\subset \bar{\Omega}_{\mathrm{C0}}) \end{cases} \right\}. \tag{9.12.1}$$

9.12.1 State Determination Problem

Define a linear elastic problem as a state determination problem. In the sequel, the notation of Problem 5.4.2 will be used, and in addition, the precise shape optimization problem will be presented. For a given $\boldsymbol{\phi} \in \mathcal{D}$, let the linear space U of state variable (solution of state determination problem) $\boldsymbol{u}$ be

$$U = \left\{ \boldsymbol{u} \in H^1\left(D; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}}(\boldsymbol{\phi}) \right\}. \tag{9.12.2}$$

Notice that the range of U in Eq. (9.5.1) is $\mathbb{R}$, but it is $\mathbb{R}^d$ in Eq. (9.12.2). Moreover, the admissible set containing $\boldsymbol{u}$ is taken to be

$$\mathcal{S} = U \cap W^{2,4}\left(D; \mathbb{R}^d\right). \tag{9.12.3}$$

In this section too, the required regularity conditions will be specified when necessary. In order to satisfy the regularity requirements, we assume the same set of hypotheses (Hypotheses 9.5.1 and 9.5.2) with respect to the regularities of known functions, where the domain is changed to D and the functions are now denoted by bold letters. In addition, for the linear elastic problems, we let $\boldsymbol{E}(\boldsymbol{u})$ and $\boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{u}) = \boldsymbol{C}(\boldsymbol{\phi}) \boldsymbol{E}(\boldsymbol{u})$ be the linear strain and stress, respectively, that were defined in Eq. (5.4.2) and Eq. (5.4.6). Also, we assume that the stiffness $\boldsymbol{C}$ is elliptic (Eq. (5.4.8))) and bounded (Eq. (5.4.9)). Suppose that in the modified hypotheses, Hypotheses 9.5.1 and 9.5.2 shown above, the condition

$$\boldsymbol{C} \in C^1_{\mathrm{S}'}\left(B; C^{0,1}\left(D; \mathbb{R}^{d\times d\times d\times d}\right)\right) \tag{9.12.4}$$

is added. For the regularity of the boundary, Hypothesis 9.5.3 is used.

Using the above assumptions, a linear elastic problem of domain variation type is defined as follows.

Problem 9.12.1 (Linear elastic problem of domain variation type) For a $\boldsymbol{\phi} \in \mathcal{D}$, let $\boldsymbol{b}(\boldsymbol{\phi})$, $\boldsymbol{p}_{\mathrm{N}}(\boldsymbol{\phi})$, $\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi})$ and $\boldsymbol{C}(\boldsymbol{\phi})$ be given. Find the $\boldsymbol{u} : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ which satisfies

$$-\boldsymbol{\nabla}^{\top} \boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{u}) = \boldsymbol{b}^{\top}(\boldsymbol{\phi}) \quad \text{in } \Omega(\boldsymbol{\phi}), \tag{9.12.5}$$

$$\boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{u}) \boldsymbol{\nu} = \boldsymbol{p}_{\mathrm{N}}(\boldsymbol{\phi}) \quad \text{on } \Gamma_p(\boldsymbol{\phi}), \tag{9.12.6}$$

$$\boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{u}) \boldsymbol{\nu} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}) \setminus \bar{\Gamma}_p(\boldsymbol{\phi}), \tag{9.12.7}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi}) \quad \text{on } \Gamma_{\mathrm{D}}(\boldsymbol{\phi}). \tag{9.12.8}$$

□

From now on, we write $\boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{u})$ as $\boldsymbol{S}(\boldsymbol{u})$ for simplicity. For later use, referring to the weak form (Problem 5.4.3) of a linear elastic problem and the Dirichlet boundary condition, we define the Lagrange function with respect to Problem 9.12.1 as

$$\begin{aligned} \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}) &= \int_{\Omega(\boldsymbol{\phi})} \left(-\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}) + \boldsymbol{b} \cdot \boldsymbol{v}\right) \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v} \, \mathrm{d}\gamma \\ &\quad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \left\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{v}) \boldsymbol{\nu}) + \boldsymbol{v} \cdot (\boldsymbol{S}(\boldsymbol{u}) \boldsymbol{\nu})\right\} \mathrm{d}\gamma, \end{aligned} \tag{9.12.9}$$

where $\boldsymbol{u}$ is not necessarily the solution of Problem 9.12.1, and $\boldsymbol{v}$ is an element of U introduced as a Lagrange multiplier. In this case, if $\boldsymbol{u}$ is the solution of Problem 9.12.1, then

$$\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}) = 0$$

holds for any $\boldsymbol{v} \in U$. This equation is equivalent to the weak form of Problem 9.12.1.

9.12.2 Mean Compliance Minimization Problem

Let us define a shape optimization problem of linear elastic body. The cost functions we consider here is defined as follows. With respect to the solution $\boldsymbol{u}$ of Problem 9.12.1, the functional

$$\begin{aligned} f_0(\boldsymbol{\phi}, \boldsymbol{u}) &= \hat{l}(\boldsymbol{\phi})(\boldsymbol{u}) \\ &= \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{b} \cdot \boldsymbol{u} \, \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u} \, \mathrm{d}\gamma \\ &\quad - \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \boldsymbol{u}_{\mathrm{D}} \cdot (\boldsymbol{S}(\boldsymbol{u}) \boldsymbol{\nu}) \, \mathrm{d}\gamma \end{aligned} \tag{9.12.10}$$

is referred to as the mean compliance. The reason for such use of the terminology is given in Section 8.9.2. Here, $\hat{l}(\boldsymbol{\phi})(\boldsymbol{u})$ shows that $\hat{l}(\boldsymbol{u})$ defined in Eq. (5.2.3) also depends on $\boldsymbol{\phi}$. Moreover,

$$f_1(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})} \mathrm{d}x - c_1 \tag{9.12.11}$$

is called a constraint function with respect to the domain measure. Here, c_1 is a positive constant such that $f_1(\boldsymbol{\phi}) \leq 0$ holds with respect to some $\boldsymbol{\phi} \in \mathcal{D}$.

Here, a mean compliance minimization problem is defined as follows.

Problem 9.12.2 (Mean compliance minimization problem) Suppose $\mathcal{D}$ and $\mathcal{S}$ is defined as Eq. (9.12.1) and Eq. (9.12.3), respectively. Let f_0 and f_1 be Eq. (9.12.10) and Eq. (9.12.11). In this case, find $\Omega(\boldsymbol{\phi})$ such that

$$\min_{(\boldsymbol{\phi}, \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \in \mathcal{D} \times \mathcal{S}} \{ f_0(\boldsymbol{\phi}, \boldsymbol{u}) \mid f_1(\boldsymbol{\phi}) \leq 0, \text{ Problem 9.12.1} \}.$$

□

9.12.3 Shape Derivatives of Cost Functions

Let us obtain the shape derivatives of $f_0(\boldsymbol{\phi}, \boldsymbol{u})$ and $f_1(\boldsymbol{\phi})$. Here, we will look separately at the case when the formulae based on the shape derivative of a function is used and the case when the formulae based on the partial shape derivative of a function is utilized. When the formulae based on the shape derivative of a function is used, the corresponding expression up to the second-order shape derivative will be established. As preparation for this, let the Lagrange function of $f_0(\boldsymbol{\phi}, \boldsymbol{u})$ be

$$\begin{aligned}
&\mathscr{L}_0(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0) \\
&\quad = f_0(\boldsymbol{\phi}, \boldsymbol{u}) + \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0) \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left(-\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0) + \boldsymbol{b} \cdot (\boldsymbol{u} + \boldsymbol{v}_0)\right) \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{p}_{\mathrm{N}} \cdot (\boldsymbol{u} + \boldsymbol{v}_0)\ \mathrm{d}\gamma \\
&\qquad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \left\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu})\right\} \mathrm{d}\gamma.
\end{aligned} \tag{9.12.12}$$

Here, $\mathscr{L}_{\mathrm{S}}$ is the Lagrange function of the state determination problem defined by Eq. (9.12.9). Moreover, $\boldsymbol{v}_0$ is the Lagrange multiplier with respect to the state determination problem prepared for f_0, and $\tilde{\boldsymbol{v}}_0 = \boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}$ is assumed to be an element of U.

Shape Derivatives of f_0 and f_1 Using Formulae Based on Shape Derivative of a Function

Let us obtain the shape derivative of f_0 using the formulae based on the shape derivative of a function. Here, $\boldsymbol{b}(\boldsymbol{\phi})$, $\boldsymbol{p}_{\mathrm{N}}(\boldsymbol{\phi})$, $\boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi})$ and $\boldsymbol{C}(\boldsymbol{\phi})$ are assumed to be fixed with the material. Here, if $\boldsymbol{b}(\boldsymbol{\phi})$ is written as $\boldsymbol{b}$, $\boldsymbol{\phi}$ is also omitted in other equations.

Here, the Fréchet derivative of $\mathscr{L}_0$ can be written as

$$\begin{aligned}
\mathscr{L}_0'(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}_0] = \mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{u}}] \\
+ \mathscr{L}_{0\boldsymbol{v}_0}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_0]
\end{aligned} \tag{9.12.13}$$

with respect to an arbitrary variation $(\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}_0) \in X \times U \times U$. Here, it will go along with the notations of Eq. (9.3.5) and Eq. (9.3.15). Each term is considered below.

The third term on the right-hand side of Eq. (9.12.13) can be rewritten as

$$\mathscr{L}_{0\boldsymbol{v}_0}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_0] = \mathscr{L}_{\mathrm{S}\boldsymbol{v}_0}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_0] = \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, \hat{\boldsymbol{v}}_0). \tag{9.12.14}$$

Equation (9.12.14) is the Lagrange function of the state determination problem (Problem 9.12.1). Hence, if $\boldsymbol{u}$ is the weak solution of the state determination problem, the third term on the right-hand side of Eq. (9.12.13) equates to zero.

Moreover, the second term on the right-hand side of Eq. (9.12.13) can be written as

$$\begin{aligned}
\mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{u}}] &= \int_{\Omega(\boldsymbol{\phi})} \left(-\boldsymbol{S}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\boldsymbol{v}_0) + \boldsymbol{b} \cdot \hat{\boldsymbol{u}}\right) \mathrm{d}x + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{p}_{\mathrm{N}} \cdot \hat{\boldsymbol{u}}\ \mathrm{d}\gamma \\
&\quad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \left\{\hat{\boldsymbol{u}} \cdot (\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) + (\boldsymbol{v}_0 - \boldsymbol{u}) \cdot (\boldsymbol{S}(\hat{\boldsymbol{u}})\boldsymbol{\nu})\right\} \mathrm{d}\gamma \\
&= \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{v}_0, \hat{\boldsymbol{u}}).
\end{aligned} \tag{9.12.15}$$

If Eq. (9.12.15) and Eq. (9.12.14) are compared, it is clear that it is a relationship whereby $\boldsymbol{v}_0$ and $\boldsymbol{u}$ are swapped over. Hence, if the self-adjoint relationship

$$\boldsymbol{v}_0 = \boldsymbol{u} \tag{9.12.16}$$

holds, the second term on the right-hand side of Eq. (9.12.13) vanishes.

Furthermore, the first term on the right-hand side of Eq. (9.12.13) becomes

$$\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] \\
&\quad= \int_{\Omega(\boldsymbol{\phi})} \left[\left(\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top + \boldsymbol{S}(\boldsymbol{v}_0)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right)\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right. \\
&\qquad\quad \left. + \left\{-\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0) + \boldsymbol{b}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\right]\mathrm{d}x \\
&\qquad + \int_{\Gamma_p(\boldsymbol{\phi})}\left[\kappa\left\{\boldsymbol{p}_\mathrm{N}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\nu}\cdot\boldsymbol{\varphi} - \boldsymbol{\nabla}_\tau\left\{\boldsymbol{p}_\mathrm{N}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\cdot\boldsymbol{\varphi}_\tau\right]\mathrm{d}\gamma \\
&\qquad + \int_{\partial\Gamma_p(\boldsymbol{\phi})\cup\Theta(\boldsymbol{\phi})}\left\{\boldsymbol{p}_\mathrm{N}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\mathrm{d}\varsigma \\
&\qquad + \int_{\Gamma_\mathrm{D}(\boldsymbol{\phi})}\left[\left\{(\boldsymbol{u}-\boldsymbol{u}_\mathrm{D})\cdot\boldsymbol{w}(\boldsymbol{\varphi},\boldsymbol{v}_0) + (\boldsymbol{v}_0-\boldsymbol{u}_\mathrm{D})\cdot\boldsymbol{w}(\boldsymbol{\varphi},\boldsymbol{u})\right\}\right. \\
&\qquad\quad \left. + \left\{(\boldsymbol{u}-\boldsymbol{u}_\mathrm{D})\cdot(\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) + (\boldsymbol{v}_0-\boldsymbol{u}_\mathrm{D})\cdot(\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu})\right\}(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau\right]\mathrm{d}\gamma
\end{aligned} \tag{9.12.17}$$

using Eq. (9.3.5), representing the result of Proposition 9.3.4, and Eq. (9.3.15) of Proposition 9.3.7, where

$$\boldsymbol{w}(\boldsymbol{\varphi},\boldsymbol{u}) = \boldsymbol{S}(\boldsymbol{u})\left[\left\{\boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nu}\right)\right\}\boldsymbol{\nu} - \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\right)^\top\right\}\boldsymbol{\nu}\right] \tag{9.12.18}$$

and $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau$ follows Eq. (9.2.6). In order to obtain Eq. (9.12.17), the following identity:

$$\begin{aligned}
&-(\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0))_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}] \\
&\quad= -(\boldsymbol{E}(\boldsymbol{u})\cdot\boldsymbol{S}(\boldsymbol{v}_0))_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}] \\
&\quad= (\boldsymbol{E}(\boldsymbol{u})\cdot\boldsymbol{S}(\boldsymbol{v}_0))_{\boldsymbol{\nabla}\boldsymbol{u}^\top}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right) \\
&\qquad + (\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0))_{\boldsymbol{\nabla}\boldsymbol{v}_0^\top}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) - \boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}) \\
&\quad= \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\mathrm{s}\cdot\boldsymbol{S}(\boldsymbol{v}_0) + \boldsymbol{S}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\mathrm{s} \\
&\qquad - \boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}) \\
&\quad= \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\cdot\boldsymbol{S}(\boldsymbol{v}_0) + \boldsymbol{S}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) \\
&\qquad - \boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}) \\
&\quad= \left(\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right)\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}^\top + \left(\boldsymbol{S}(\boldsymbol{v}_0)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right)\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}^\top \\
&\qquad - \boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})
\end{aligned}$$

is used, which is derived using Eq. (9.8.18). The notation $(\,\cdot\,)^\mathrm{s}$ represents $\left((\,\cdot\,)^\top + (\,\cdot\,)\right)/2$.

With the above results in mind, assume that $\boldsymbol{u}$ is the weak solution of Problem 9.12.1 and that the self-adjoint relationship (Eq. (9.12.16)) holds. In this case, from the fact that Dirichlet condition holds for Problem 9.12.1, Eq. (9.12.17) can be written as

$$\begin{aligned}\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle \\ &= \int_{\Omega(\boldsymbol{\phi})} \left(\boldsymbol{G}_{\Omega 0} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}^{\top} + g_{\Omega 0} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \right) \mathrm{d}x \\ &\quad + \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{g}_{p0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\partial \Gamma_p(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} \boldsymbol{g}_{\partial p0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma \end{aligned} \tag{9.12.19}$$

using the notation of Eq. (7.5.15) for $\tilde{f}_0$, where

$$\boldsymbol{G}_{\Omega 0} = 2\boldsymbol{S}(\boldsymbol{u}) \left(\boldsymbol{\nabla} \boldsymbol{u}^{\top} \right)^{\top}, \tag{9.12.20}$$

$$g_{\Omega 0} = -\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{u}) + 2\boldsymbol{b} \cdot \boldsymbol{u}, \tag{9.12.21}$$

$$\boldsymbol{g}_{p0} = 2\kappa (\boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u}) \boldsymbol{\nu}, \tag{9.12.22}$$

$$\boldsymbol{g}_{\partial p0} = 2 (\boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u}) \boldsymbol{\tau}. \tag{9.12.23}$$

From the results above, similar conclusions with Theorem 9.8.2 can be obtained for $\boldsymbol{g}_0$ of Eq. (9.12.19).

On the other hand, the shape derivative of $f_1(\boldsymbol{\phi})$ is obtained as

$$f_1'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle = \int_{\Omega(\boldsymbol{\phi})} g_{\Omega 1} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi} \, \mathrm{d}x, \tag{9.12.24}$$

where

$$g_{\Omega 1} = 1. \tag{9.12.25}$$

This is established by letting $u = 1$ in Proposition 9.3.1 without using $\mathscr{L}_{\mathrm{S}}$, which, on the other hand, is due to the fact that the solution to the state determination problem is not used.

Second-Order Shape Derivatives of f_0 and f_1 Using Formulae Based on Shape Derivative of a Function

Now, let us obtain the second-order shape derivatives of the mean compliance f_0 and the constraint cost function f_1 with respect to the domain measure of linear elastic body. Here, the formulae based on the shape derivative of a function is used following the procedures shown in Sect. 9.8.2.

Firstly, let us think about the second-order shape derivative of f_0. We assume that $\boldsymbol{b} = \boldsymbol{0}_{\mathbb{R}^d}$ corresponding to Hypothesis 9.8.3 (1). The relationship corresponding to Hypothesis 9.8.3 (2) is satisfied here. Moreover, assume (3) in Hypothesis 9.8.3.

The Lagrange function $\mathscr{L}_0$ of f_0 is defined by Eq. (9.12.12). Viewing $(\boldsymbol{\phi}, \boldsymbol{u})$ as a design variable, we define its corresponding admissible set and admissible direction set as

$$S = \{(\boldsymbol{\phi}, \boldsymbol{u}) \in \mathcal{D} \times \mathcal{S} \mid \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in U\},$$
$$T_S(\boldsymbol{\phi}, \boldsymbol{u}) = \{(\boldsymbol{\varphi}, \hat{\boldsymbol{v}}) \in X \times U \mid \mathscr{L}_{\mathrm{S}\boldsymbol{\phi}\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v})[\boldsymbol{\varphi}, \hat{\boldsymbol{v}}] = 0 \text{ for all } \boldsymbol{v} \in U\}.$$

Considering Eq. (9.1.6), the second-order Fréchet partial derivative of $\mathscr{L}_0$ of Eq. (9.12.12) with respect to arbitrary variations $(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2) \in T_S(\boldsymbol{\phi}, \boldsymbol{u})$ of the design variable $(\boldsymbol{\phi}, \boldsymbol{u}) \in S$ becomes

$$\begin{aligned}
&\mathscr{L}_{0(\boldsymbol{\phi}',\boldsymbol{u})(\boldsymbol{\phi}',\boldsymbol{u})}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2)] \\
&\quad = \left(\mathscr{L}_{0(\boldsymbol{\phi}',\boldsymbol{u})}\right)_{(\boldsymbol{\phi}',\boldsymbol{u})}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2)] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \left(\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_1]\right)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \left(\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_1]\right)_{\boldsymbol{u}}[\hat{\boldsymbol{v}}_2] \\
&\qquad + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \left(\mathscr{L}_{0\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_2] \\
&\qquad + \mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_1] + \mathscr{L}_{0\boldsymbol{u}\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_1, \hat{\boldsymbol{v}}_2] \\
&\qquad + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle,
\end{aligned} \tag{9.12.26}$$

where $\langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle$ follows the definition given in Eq. (9.1.8).

Here, the first and fifth terms in Eq. (9.12.26) becomes

$$\begin{aligned}
&\left(\mathscr{L}_{0\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left[\left\{\left(\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right) \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2]\right. \\
&\qquad + \left\{\left(\boldsymbol{S}(\boldsymbol{v}_0)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right) \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad - (\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0))(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + 2\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top - \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)\right) \\
&\qquad \left. - \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{u})\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top - (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)\right\}\right] \mathrm{d}x.
\end{aligned} \tag{9.12.27}$$

Here, Eq. (9.3.11) was used. The first term of the integrand on the right-hand side of Eq. (9.12.27) becomes

$$\begin{aligned}
&\left[\left\{\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right\} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right]_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\quad = \left\{\boldsymbol{S}(\boldsymbol{u}) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\quad = \left\{\boldsymbol{E}(\boldsymbol{u}) \cdot \left(\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^{\mathrm{s}}\right)\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\quad = \left[\boldsymbol{\nabla}\boldsymbol{v}_0^\top \cdot \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \boldsymbol{S}(\boldsymbol{u})\right\}\right]_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2]
\end{aligned}$$

$$
\begin{aligned}
&= -\left\{\boldsymbol{E}(\boldsymbol{u}) \cdot \left(\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^{\mathrm{s}}\right)\right\}_{\boldsymbol{\nabla}\boldsymbol{u}^\top} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{u}^\top\right) \\
&\quad - \left[\left\{\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right\} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right]_{\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right) \\
&\quad - \left[\boldsymbol{\nabla}\boldsymbol{v}_0^\top \cdot \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \boldsymbol{S}(\boldsymbol{u})\right\}\right]_{\boldsymbol{\nabla}\boldsymbol{v}_0^\top} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) \\
&\quad + \left[\left\{\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right\} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right] \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \\
&= -\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{u}^\top\right)^{\mathrm{s}} \cdot \left(\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^{\mathrm{s}}\right) \\
&\quad - \left\{\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right\} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right) \\
&\quad - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) \cdot \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \boldsymbol{S}(\boldsymbol{u})\right\} \\
&\quad + \left[\left\{\boldsymbol{S}(\boldsymbol{u})\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\right\} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right] \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2.
\end{aligned} \tag{9.12.28}
$$

Similarly, the second term of the integrand on the right-hand side of Eq. (9.12.27) is similar to Eq. (9.12.28) with $\boldsymbol{u}$ and $\boldsymbol{v}_0$ interchanged. The third term of the integrand on the right-hand side of Eq. (9.12.27) is

$$
\begin{aligned}
&-\left(\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0)\right)\left\{\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\quad = -\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0)\left\{-\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)\right\}.
\end{aligned} \tag{9.12.29}
$$

Hence, noting the self-adjoint relationship, Eq. (9.12.27) becomes

$$
\begin{aligned}
&\left(\mathscr{L}_{0\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[-\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{u}^\top\right)^{\mathrm{s}} \cdot \left(\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^{\mathrm{s}}\right) \\
&\qquad - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) \cdot \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \boldsymbol{S}(\boldsymbol{u})\right\} \\
&\qquad - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^{\mathrm{s}} \cdot \left(\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{u}^\top\right)^{\mathrm{s}}\right) \\
&\qquad - \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{u}^\top\right) \cdot \left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \boldsymbol{S}(\boldsymbol{v}_0)\right\}\Big] \mathrm{d}x
\end{aligned} \tag{9.12.30}
$$

Next, consider the second term on the right-hand side of Eq. (9.12.26). If Eq. (9.12.17) with the Dirichlet condition of the state determination problem substituted in is used, we get

$$
\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\left\{\boldsymbol{S}(\hat{\boldsymbol{v}}_2)\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top + \boldsymbol{S}(\boldsymbol{v}_0)\left(\boldsymbol{\nabla}\hat{\boldsymbol{v}}_2^\top\right)^\top\right\} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \\
&\qquad - \left(\boldsymbol{S}(\hat{\boldsymbol{v}}_2) \cdot \boldsymbol{E}(\boldsymbol{v}_0)\right) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\Big] \mathrm{d}x.
\end{aligned} \tag{9.12.31}
$$

On the other hand, the variation of $\boldsymbol{u}$ satisfying the state determination problem with respect to an arbitrary domain variation $\boldsymbol{\varphi}_j \in Y$ for $j \in \{1, 2\}$ is written as $\hat{\boldsymbol{v}}_j = \boldsymbol{v}'(\boldsymbol{\phi})\left[\boldsymbol{\varphi}_j\right]$. If the Fréchet partial derivative of the Lagrange function $\mathscr{L}_{\mathrm{S}}$ of the state determination problem is taken, we obtain

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'\boldsymbol{u}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v})\left[\boldsymbol{\varphi}_j,\hat{\boldsymbol{v}}_j\right] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})}\Big\{\boldsymbol{S}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{v}^{\top}\right)^{\mathrm{s}}+\boldsymbol{S}(\boldsymbol{v})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}} \\
&\qquad -\left(\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v})\right)\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j-\boldsymbol{S}(\hat{\boldsymbol{v}}_j)\cdot\boldsymbol{E}(\boldsymbol{v})\Big\}\mathrm{d}x \\
&\quad = \int_{\Omega(\boldsymbol{\phi})}\Big[\Big\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\right)^{\top}\boldsymbol{S}(\boldsymbol{u})+\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}}-\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j \\
&\qquad -\boldsymbol{S}(\hat{\boldsymbol{v}}_j)\Big\}\left(\boldsymbol{\nabla}\boldsymbol{v}^{\top}\right)^{\top}\Big]\cdot\boldsymbol{I}\,\mathrm{d}x \\
&\quad = \int_{\Omega(\boldsymbol{\phi})}\Big[\boldsymbol{\nabla}\boldsymbol{v}^{\top}\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top} \\
&\qquad +\boldsymbol{S}(\boldsymbol{v})\left\{\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\left(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\right)^{\top}-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right)-\left(\boldsymbol{\nabla}\hat{\boldsymbol{v}}_j^{\top}\right)^{\top}\right\}\Big]\cdot\boldsymbol{I}\,\mathrm{d}x \\
&\quad = 0
\end{aligned}
\tag{9.12.32}
$$

for any $\boldsymbol{v}\in U$. Here, the Dirichlet boundary conditions of $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}_j$ were used. From the fact that Eq. (9.12.32) holds with respect to an arbitrary $\boldsymbol{v}\in U$, the identities

$$
\begin{aligned}
&\boldsymbol{S}(\hat{\boldsymbol{v}}_j)\cdot\boldsymbol{E}(\boldsymbol{v}) \\
&\quad =\boldsymbol{S}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{v}^{\top}\right)^{\mathrm{s}}+\boldsymbol{S}(\boldsymbol{v})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}}-\left(\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v})\right)\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j,
\end{aligned}
\tag{9.12.33}
$$

$$
\begin{aligned}
&\boldsymbol{S}(\hat{\boldsymbol{v}}_j)\left(\boldsymbol{\nabla}\boldsymbol{v}^{\top}\right)^{\top} \\
&\quad =\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\right)^{\top}\boldsymbol{S}(\boldsymbol{u})+\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}}-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\boldsymbol{S}(\boldsymbol{u})\right\}\left(\boldsymbol{\nabla}\boldsymbol{v}^{\top}\right)^{\top}
\end{aligned}
\tag{9.12.34}
$$

$$
\begin{aligned}
&\boldsymbol{S}(\boldsymbol{v})\left(\boldsymbol{\nabla}\hat{\boldsymbol{v}}_j^{\top}\right)^{\top} \\
&\quad =\boldsymbol{\nabla}\boldsymbol{v}^{\top}\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}+\boldsymbol{S}(\boldsymbol{v})\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\left\{\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^{\top}\right)^{\top}-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right\}
\end{aligned}
\tag{9.12.35}
$$

are obtained. Substituting from Eq. (9.12.33) to Eq. (9.12.35) into Eq. (9.12.31), we get

$$
\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}_0)\left[\boldsymbol{\varphi}_1,\hat{\boldsymbol{v}}_2\right] \\
&\quad =\int_{\Omega(\boldsymbol{\phi})}\Big[\Big\{\Big(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}\right)^{\top}\boldsymbol{S}(\boldsymbol{u})+\boldsymbol{C}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}}-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\boldsymbol{S}(\boldsymbol{u})\Big)\left(\boldsymbol{\nabla}\boldsymbol{v}_0^{\top}\right)^{\top} \\
&\qquad +\boldsymbol{\nabla}\boldsymbol{v}_0^{\top}\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}+\boldsymbol{S}(\boldsymbol{v}_0)\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\left(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}\right)^{\top}-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right)\Big\}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_1^{\top} \\
&\qquad -\Big\{\boldsymbol{S}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}\boldsymbol{\nabla}\boldsymbol{v}_0^{\top}\right)^{\mathrm{s}}+\boldsymbol{S}(\boldsymbol{v}_0)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\mathrm{s}} \\
&\qquad -\left(\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v}_0)\right)\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\Big\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\Big]\mathrm{d}x.
\end{aligned}
\tag{9.12.36}
$$

Similarly, the third term on the right-hand side of Eq. (9.12.26) takes the form as in Eq. (9.12.36) with $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ interchanged. Lastly, the fourth term on the right-hand side of Eq. (9.12.26) is actually equal to zero.

Summarizing the above results, the second-order shape derivative of $\tilde{f}_0$ becomes

$$\begin{aligned} & h_0\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{u}\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \\ &\quad = \int_{\Omega(\boldsymbol{\phi})} \Bigl[2\boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{u}\right)\left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2\right)\left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1\right) \\ &\qquad + \left(\boldsymbol{S}\left(\boldsymbol{u}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right) \cdot \Bigl\{ \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top \\ &\qquad\quad + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \\ &\qquad\quad - 4\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 - 4\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Bigr\} \Bigr] \mathrm{d}x. \end{aligned} \tag{9.12.37}$$

On the other hand, the second-order shape derivative of $f_1\left(\boldsymbol{\phi}\right)$ becomes

$$h_1\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] = \left(f_1'\right)'\left(\boldsymbol{\phi}\right) + \left\langle \boldsymbol{g}_1\left(\boldsymbol{\phi}\right), \boldsymbol{t}\left(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right)\right\rangle = 0 \tag{9.12.38}$$

with respect to arbitrary variations $\boldsymbol{\varphi}_1 \in Y$ and $\boldsymbol{\varphi}_2 \in Y$. Here, Eq. (9.3.11) was used.

Second-Order Shape Derivative of Cost Function Using Lagrange Multiplier Method

The application of the Lagrange multiplier method in obtaining the second-order shape derivative of the mean compliance f_0 is described as follows. Fixing $\boldsymbol{\varphi}_1$, we define the Lagrange function for $\tilde{f}_0'\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}_1\right] = \left\langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1\right\rangle$ in Eq. (9.12.19) by

$$\mathscr{L}_{\mathrm{I}0}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right) = \left\langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1\right\rangle + \mathscr{L}_{\mathrm{S}}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right), \tag{9.12.39}$$

where $\mathscr{L}_{\mathrm{S}}$ is given by Eq. (9.12.9), and $\boldsymbol{w}_0 \in U$ is the adjoint variable provided for $\boldsymbol{u}$ in $\boldsymbol{g}_0$.

Considering Eq. (9.1.6), with respect to arbitrary variations $\left(\boldsymbol{\varphi}_2, \hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}_0\right) \in \mathcal{D} \times U^2$ of $\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)$, the Fréchet derivative of $\mathscr{L}_{\mathrm{I}0}$ is written as

$$\begin{aligned} & \mathscr{L}_{\mathrm{I}0}'\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)\left[\boldsymbol{\varphi}_2, \hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}_0\right] \\ &\quad = \mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)\left[\boldsymbol{\varphi}_2\right] + \left\langle \boldsymbol{g}_0\left(\boldsymbol{\phi}\right), \boldsymbol{t}\left(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right)\right\rangle + \mathscr{L}_{\mathrm{I}0\boldsymbol{u}}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)\left[\hat{\boldsymbol{u}}\right] \\ &\qquad + \mathscr{L}_{\mathrm{I}0\boldsymbol{w}_0}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)\left[\hat{\boldsymbol{w}}_0\right]. \end{aligned} \tag{9.12.40}$$

The fourth term on the right-hand side of Eq. (9.12.40) vanishes if $\boldsymbol{u}$ is the solution of the state determination problem.

Assuming that $\boldsymbol{\varphi}_1$ is an H^2 class function in the neighborhood of $\Gamma_p\left(\boldsymbol{\phi}\right)$ and then applying Proposition 9.3.7, the third term on the right-hand side of Eq. (9.12.40) is obtained as

$$\mathscr{L}_{\mathrm{I}0\boldsymbol{u}}\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0\right)\left[\hat{\boldsymbol{u}}\right]$$

$$
\begin{aligned}
&= \int_{\Omega(\boldsymbol{\phi})} \Big[2 \Big\{ \boldsymbol{C} \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right)^{\mathrm{s}} + \Big(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^\top - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \Big) \boldsymbol{S}(\boldsymbol{v}_0) \Big\} \cdot \boldsymbol{\nabla} \hat{\boldsymbol{u}}^\top \\
&\qquad + 2 (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \boldsymbol{b} \cdot \hat{\boldsymbol{u}} - \boldsymbol{S}(\boldsymbol{w}_0) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) \Big] \mathrm{d}x \\
&\quad + \int_{\Gamma_p(\boldsymbol{\phi})} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau \boldsymbol{p}_{\mathrm{N}} \cdot \hat{\boldsymbol{u}} \, \mathrm{d}\gamma. \qquad (9.12.41)
\end{aligned}
$$

Here, the condition that Eq. (9.12.41) is zero for arbitrary $\hat{\boldsymbol{u}} \in U$ is equivalent to setting $\boldsymbol{w}_0$ to be the solution of the following adjoint problem.

Problem 9.12.3 (Adjoint problem of $\boldsymbol{w}_0$ with respect to $\langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1 \rangle$) Under the assumption of Problem 9.12.1, let $\boldsymbol{\varphi}_1 \in Y$ be given. Find $\boldsymbol{w}_0 = \boldsymbol{w}_0(\boldsymbol{\varphi}_1) \in U$ satisfying

$$
\begin{aligned}
-\boldsymbol{\nabla}^\top \boldsymbol{S}(\boldsymbol{w}_0) &= -2 \boldsymbol{\nabla}^\top \Big\{ \boldsymbol{C} \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right)^{\mathrm{s}} + \Big(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^\top - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \Big) \boldsymbol{S}(\boldsymbol{v}_0) \Big\} \\
&\quad + 2 \boldsymbol{b}^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \quad \text{in } \Omega(\boldsymbol{\phi}), \\
\boldsymbol{S}(\boldsymbol{w}_0) \boldsymbol{\nu} &= (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau \boldsymbol{p}_{\mathrm{N}} \quad \text{on } \Gamma_p(\boldsymbol{\phi}), \\
\boldsymbol{S}(\boldsymbol{w}_0) \boldsymbol{\nu} &= \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}) \setminus \bar{\Gamma}_p(\boldsymbol{\phi}), \\
\boldsymbol{w}_0 &= \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}}.
\end{aligned}
$$

□

Finally, the first and second terms on the right-hand side of Eq. (9.12.40) become

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0, \boldsymbol{w}_0(\boldsymbol{\varphi}_1), \boldsymbol{z}_0(\boldsymbol{\varphi}_1))[\boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\qquad + \mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0)[\boldsymbol{\varphi}_2] \qquad (9.12.42)
\end{aligned}
$$

with respect an arbitrary $\boldsymbol{\varphi}_1 \in Y$. The first and second terms on the right-hand side of Eq. (9.12.42) are given by Eq. (9.12.30). The third term is given by

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{w}_0)[\boldsymbol{\varphi}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big\{ \boldsymbol{S}(\boldsymbol{u}) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{w}_0^\top \right)^{\mathrm{s}} + \boldsymbol{S}(\boldsymbol{w}_0) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right)^{\mathrm{s}} \\
&\qquad + (\boldsymbol{b} \cdot \boldsymbol{w}_0 - \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{w}_0)) \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Big\} \mathrm{d}x \\
&\qquad + \int_{\Gamma_p(\boldsymbol{\phi})} \{ \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau + \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{w}_0(\boldsymbol{\varphi}_1)(\boldsymbol{\varphi}_1) \} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)_\tau \, \mathrm{d}\gamma.
\end{aligned}
$$

Here, $\boldsymbol{u}$ and $\boldsymbol{w}_0(\boldsymbol{\varphi}_1)$ are assumed to be the weak solutions of Problems 9.12.1 and 9.12.3, respectively. If we denote $f_0(\boldsymbol{\phi}, \boldsymbol{u})$ by $\tilde{f}_0(\boldsymbol{\phi})$, then we arrive at the relation

$$
\mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{\phi}_1, \boldsymbol{u}, \boldsymbol{w}_0(\boldsymbol{\varphi}_1))[\boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle
$$

$$
\begin{aligned}
&= \tilde{f}_0''(\boldsymbol{\phi})[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = \langle \boldsymbol{g}_{\mathrm{H}0}(\boldsymbol{\phi}, \boldsymbol{\varphi}_1), \boldsymbol{\varphi}_2 \rangle \\
&= \int_{\Omega(\boldsymbol{\phi})} \Big[-2 \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right)^{\mathrm{s}} \cdot \left(\boldsymbol{C} \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right)^{\mathrm{s}} \right) \\
&\quad - 2 \left\{ \boldsymbol{S}(\boldsymbol{u}) \left(\boldsymbol{\nabla} \boldsymbol{u}^\top \right)^\top \right\} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right) \\
&\quad + \left\{ \boldsymbol{S}(\boldsymbol{u}) \left(\boldsymbol{\nabla} \boldsymbol{w}_0^\top (\boldsymbol{\varphi}_1) \right)^\top + \boldsymbol{S}(\boldsymbol{w}_0(\boldsymbol{\varphi}_1)) \left(\boldsymbol{\nabla} \boldsymbol{u}^\top \right)^\top \right\} \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \\
&\quad - \left\{ \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{w}_0(\boldsymbol{\varphi}_1)) - \boldsymbol{b} \cdot \boldsymbol{w}_0(\boldsymbol{\varphi}_1) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Big] \mathrm{d}x \\
&\quad + \int_{\Gamma_p(\boldsymbol{\phi})} 2 \boldsymbol{p}_{\mathrm{N}} \cdot (\boldsymbol{u} + \boldsymbol{w}_0(\boldsymbol{\varphi}_1)) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_\tau (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)_\tau \, \mathrm{d}\gamma, \qquad (9.12.43)
\end{aligned}
$$

where $\boldsymbol{g}_{\mathrm{H}0}$ is the Hesse gradient of the mean compliance.

If $\boldsymbol{b} = \boldsymbol{0}_{\mathbb{R}^d}$ and (3) in Hypothesis 9.8.3 are satisfied, with respect to the solution $\boldsymbol{w}_0$ of Problem 9.12.3,

$$
\begin{aligned}
\boldsymbol{S}(\boldsymbol{w}_0) &= 2 \Big\{ \boldsymbol{C} \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right)^{\mathrm{s}} \\
&\qquad + \left(\left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \right)^{\mathrm{s}} - \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right) \boldsymbol{S}(\boldsymbol{v}_0) \Big\}, \qquad (9.12.44) \\
\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{w}_0) &= \boldsymbol{E}(\boldsymbol{u}) \cdot \boldsymbol{S}(\boldsymbol{w}_0), \qquad (9.12.45) \\
\boldsymbol{S}(\boldsymbol{u}) \left(\boldsymbol{\nabla} \boldsymbol{w}_0^\top \right)^\top \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top &= \boldsymbol{S}(\boldsymbol{u}) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{w}_0^\top \right)^{\mathrm{s}} \\
&= \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \right)^\top \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{w}_0) \\
&= \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \right)^\top \boldsymbol{E}(\boldsymbol{u}) \cdot \boldsymbol{S}(\boldsymbol{w}_0) \qquad (9.12.46)
\end{aligned}
$$

holds. Here, we used

$$
\boldsymbol{S}(\boldsymbol{u}) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right)^{\mathrm{s}} = \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \right)^\top \boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0).
$$

Indeed, this relation is obtained from the fact that the inner product of $\boldsymbol{S}(\hat{\boldsymbol{v}}_j)$ obtained from Eq. (9.12.34) and $\boldsymbol{E}(\boldsymbol{v})$ accords with Eq. (9.12.33). Substituting Eq. (9.12.44) to Eq. (9.12.46) into Eq. (9.12.43), it can be confirmed that Eq. (9.12.43) accords with Eq. (9.12.37).

Shape Derivatives of f_0 and f_1 Using Formulae Based on Partial Shape Derivative of a Function

Next, let us compute the shape derivative of f_0 using the formulae based on the partial shape derivative of a function. Here, it is assumed that $\boldsymbol{b}$, $\boldsymbol{p}_{\mathrm{N}}$, $\boldsymbol{u}_{\mathrm{D}}$ and $\boldsymbol{C}$ are functions fixed in space. Moreover, we assume that $\boldsymbol{u}$ and $\boldsymbol{v}_0$ are elements of $W^{2,2q_{\mathrm{R}}}\left(D;\mathbb{R}^d\right)$ where $q_{\mathrm{R}} > d$.

Under these assumptions, the Fréchet derivative of $\mathscr{L}_0(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)$ can be written as

$$
\begin{aligned}
\mathscr{L}_0'(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}_0] &= \mathscr{L}_{0\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] + \mathscr{L}_{0\boldsymbol{u}}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{u}}] \\
&\quad + \mathscr{L}_{0\boldsymbol{v}_0}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_0] \qquad (9.12.47)
\end{aligned}
$$

for any $(\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}_0) \in X \times U \times U$. Here, the notations of Eq. (9.3.21) and Eq. (9.3.27) were used. Each term is considered below.

The third term on the right-hand side of Eq. (9.12.47) is given by Eq. (9.12.14). Hence, if $\boldsymbol{u}$ is the weak solution of the state determination problem, the said expression equates to zero. Similarly, the second term on the right-hand side of Eq. (9.12.47) is the same as Eq. (9.12.15). Hence, when the self-adjoint relationship (Eq. (9.12.16)) holds, the term also vanishes.

Furthermore, the first term on the right-hand side of Eq. (9.12.47) becomes

$$
\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] \\
&= \int_{\partial\Omega(\boldsymbol{\phi})} \{-\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{v}_0) + \boldsymbol{b} \cdot (\boldsymbol{u} + \boldsymbol{v}_0)\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\Gamma_p(\boldsymbol{\phi})} (\partial_\nu + \kappa) \{\boldsymbol{p}_{\mathrm{N}} \cdot (\boldsymbol{u} + \boldsymbol{v}_0)\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} \{\boldsymbol{p}_{\mathrm{N}} \cdot (\boldsymbol{u} + \boldsymbol{v}_0)\} \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma \\
&\quad + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \big[\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot \bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{v}_0) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}) \cdot \bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{u})\} \\
&\qquad + \{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu})\} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau \\
&\qquad + (\partial_\nu + \kappa) \{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu})\} \boldsymbol{\nu} \cdot \boldsymbol{\varphi}\big] \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_{\mathrm{D}}(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} \big\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{v}_0)\boldsymbol{\nu}) \\
&\qquad + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}) \cdot (\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu})\big\} \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma
\end{aligned}
$$

using Eq. (9.3.21), representing the result of Proposition 9.3.10, and Eq. (9.3.27) of Proposition 9.3.13. Here, we denote $(\boldsymbol{\nu} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = \left(\boldsymbol{\nabla} \boldsymbol{u}^\top\right)^\top \boldsymbol{\nu}$ as $\partial_\nu \boldsymbol{u}$,

$$
\begin{aligned}
\bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{u}) = -\boldsymbol{S}(\boldsymbol{u}) &\left[\sum_{i \in \{1, \ldots, d-1\}} \left\{\boldsymbol{\tau}_i \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}^\top \boldsymbol{\nu}\right)\right\} \boldsymbol{\tau}_i\right] \\
&+ (\boldsymbol{\nu} \cdot \boldsymbol{\varphi}) \left(\boldsymbol{\nabla}^\top \boldsymbol{S}(\boldsymbol{u})\right)^\top,
\end{aligned} \tag{9.12.48}
$$

and $(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi})_\tau$ as Eq. (9.2.6).

With the above results in mind, assume that $\boldsymbol{u}$ is a weak solution of Problem 9.12.1 and that the self-adjoint relationship (Eq. (9.12.16)) holds. In this case, we can write Eq. (9.12.48) as

$$
\begin{aligned}
\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \mathscr{L}_{0\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] = \langle \bar{\boldsymbol{g}}_0, \boldsymbol{\varphi} \rangle \\
&= \int_{\partial\Omega(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial\Omega 0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma + \int_{\Gamma_p(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{p0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma \\
&\quad + \int_{\partial\Gamma_p(\boldsymbol{\phi}) \cup \Theta(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial p0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\varsigma + \int_{\Gamma_{\mathrm{D}}(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\mathrm{D}0} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma
\end{aligned} \tag{9.12.49}
$$

using the notation of Eq. (7.5.15) for $\tilde{f}_0$ and the Dirichlet condition in Problem 9.12.1, where

$$\bar{\boldsymbol{g}}_{\partial\Omega 0} = \left(-\boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{u}\right) + 2\boldsymbol{b} \cdot \boldsymbol{u}\right) \boldsymbol{\nu}, \tag{9.12.50}$$

$$\bar{\boldsymbol{g}}_{p0} = 2\left(\partial_\nu + \kappa\right)\left(\boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u}\right) \boldsymbol{\nu}, \tag{9.12.51}$$

$$\bar{\boldsymbol{g}}_{\partial p0} = 2\left(\boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u}\right) \boldsymbol{\tau}, \tag{9.12.52}$$

$$\bar{\boldsymbol{g}}_{\mathrm{D}0} = 2\left\{\partial_\nu\left(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}\right) \cdot \left(\boldsymbol{S}\left(\boldsymbol{u}\right)\boldsymbol{\nu}\right)\right\} \boldsymbol{\nu}. \tag{9.12.53}$$

Furthermore, on a homogeneous Dirichlet boundary, since there is a strain component only in the normal direction,

$$\partial_\nu \boldsymbol{u} = \boldsymbol{E}\left(\boldsymbol{u}\right)\boldsymbol{\nu} \tag{9.12.54}$$

holds. Hence, Eq. (9.12.53) can be written as

$$\bar{\boldsymbol{g}}_{\mathrm{D}0} = 2\left\{\left(\boldsymbol{E}\left(\boldsymbol{u}\right)\boldsymbol{\nu}\right) \cdot \left(\boldsymbol{S}\left(\boldsymbol{u}\right)\boldsymbol{\nu}\right)\right\} \boldsymbol{\nu} = 2\left(\boldsymbol{E}\left(\boldsymbol{u}\right) \cdot \boldsymbol{S}\left(\boldsymbol{u}\right)\right) \boldsymbol{\nu}. \tag{9.12.55}$$

Here, if $\bar{\boldsymbol{g}}_0$ is written on the homogeneous Dirichlet boundary and homogeneous Neumann boundary, we get

$$\bar{\boldsymbol{g}}_0 = \left(-\boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{u}\right) + 2\boldsymbol{b} \cdot \boldsymbol{u}\right) \boldsymbol{\nu} \quad \text{on } \Gamma_{\mathrm{N}}\left(\boldsymbol{\phi}\right) \setminus \bar{\Gamma}_p\left(\boldsymbol{\phi}\right), \tag{9.12.56}$$

$$\bar{\boldsymbol{g}}_0 = \left(\boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{u}\right) + 2\boldsymbol{b} \cdot \boldsymbol{u}\right) \boldsymbol{\nu} \quad \text{on } \Gamma_{\mathrm{D}}\left(\boldsymbol{\phi}\right). \tag{9.12.57}$$

From these results, it is evident that the sign of strain energy density $\boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{u}\right)/2$ swaps between the homogeneous Dirichlet boundary and homogeneous Neumann boundary.

From the above results, conclusions similar to Theorem 9.8.6 can be obtained with respect to the function space containing $\bar{\boldsymbol{g}}_0$ of Eq. (9.12.49).

On the other hand, the shape derivative of $f_1\left(\boldsymbol{\phi}\right)$ can be written as

$$f_1'\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}\right] = \left\langle \bar{\boldsymbol{g}}_1, \boldsymbol{\varphi} \right\rangle = \int_{\partial\Omega(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial\Omega 1} \cdot \boldsymbol{\varphi} \, \mathrm{d}\gamma, \tag{9.12.58}$$

where

$$\bar{\boldsymbol{g}}_{\partial\Omega 1} = \boldsymbol{\nu}. \tag{9.12.59}$$

This can be obtained by letting $u = 1$ in Proposition 9.3.9 and is actually due to the fact that the solution for the state determination problem is not used.

9.12.4 Relation with Optimal Design Problem of Stepped One-Dimensional Linear Elastic Body

Let us think about the relationship between the shape derivative of the cost function in the mean compliance minimization problem (Problem 9.12.2) of a $d \in \{2, 3\}$-dimensional linear elastic body and the cross-sectional derivative of the cost function in the mean compliance minimization problem (Problem 1.1.4)

Table 9.2: Correspondence between cross-sectional optimization problem and shape optimization problem.

Comparison item	Cross-sectional optimization	Shape optimization
Design variable	$\boldsymbol{a} \in X = \mathbb{R}^2$	$\boldsymbol{\phi} \in X = H^1\left(D;\mathbb{R}^d\right)$
State variable	$\boldsymbol{u} \in U = \mathbb{R}^2$	$\boldsymbol{u} \in U = H^1\left(D;\mathbb{R}^d\right)$
State determination	$\mathscr{L}_{\mathrm{S}}\left(\boldsymbol{a},\boldsymbol{u},\boldsymbol{v}\right) = 0\ \forall \boldsymbol{v} \in U$	$\mathscr{L}_{\mathrm{S}}\left(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}\right) = 0\ \forall \boldsymbol{v} \in U$
Object function	$f_0 = \boldsymbol{p} \cdot \boldsymbol{u}$	$f_0 = \hat{l}\left(\boldsymbol{\phi}\right)\left(\boldsymbol{u}\right)$
Constraint function	$f_1 = (\text{volume}) - c_1$	$f_1 = (\text{domain measure}) - c_1$
Gradient	$\boldsymbol{g}_i \in X' = \mathbb{R}^2$	$\boldsymbol{g}_i, \bar{\boldsymbol{g}}_i \in X' = H^{1\prime}\left(D;\mathbb{R}^d\right)$
Gradient method	$\boldsymbol{y}_{gi} \cdot \boldsymbol{A}\boldsymbol{z} = -\boldsymbol{g} \cdot \boldsymbol{z}\ \forall \boldsymbol{z} \in X$	$a\left(\boldsymbol{\varphi}_{gi}, \boldsymbol{z}\right) = -\left\langle \boldsymbol{g}_i, \boldsymbol{z}\right\rangle\ \forall \boldsymbol{z} \in X$

of the stepped one-dimensional linear elastic body seen in Chap. 1. Table 9.2 shows some comparisons between the two problems.

In Problem 1.1.4, the body force and known displacement were not used. If this assumption is applied to Problem 9.12.2, it corresponds to putting the cost function as

$$f_0\left(\boldsymbol{\phi},\boldsymbol{u}\right) = \hat{l}\left(\boldsymbol{\phi}\right)\left(\boldsymbol{u}\right) = \int_{\Gamma_p(\boldsymbol{\phi})} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{u}\ \mathrm{d}\gamma = \sum_{i\in\{1,2\}} \int_{\Gamma_i} \frac{p_i}{a_i} u_i\ \mathrm{d}\gamma, \tag{9.12.60}$$

where p_i, u_i, a_i with respect to $i \in \{1,2\}$ follow the respective definitions in Problem 1.1.4. Moreover, in Problem 1.1.4, external forces p_1 and p_2 were fixed with respect to the variation of the cross-sectional area. In other words, p_1 and p_2 are assumed to vary with boundary measures (Definition 9.4.4). In this case, the shape derivative of $f_{0\boldsymbol{\phi}}\left(\boldsymbol{\phi},\boldsymbol{u}\right)[\boldsymbol{\varphi}]$ becomes zero. On the other hand, $\boldsymbol{p}_{\mathrm{N}}$ was assumed to be fixed with the material (Definition 9.4.1) in Eq. (9.12.19) which gives the shape derivative of f_0. Considering their differences, the shape derivative of f_0 defined by Eq. (9.12.60) can be written as

$$\tilde{f}_0'\left(\boldsymbol{\phi}\right)[\boldsymbol{\varphi}] = \left\langle \boldsymbol{g}_0, \boldsymbol{\varphi}\right\rangle = \sum_{i\in\{1,2\}} \int_0^l \left\{ \boldsymbol{G}_{\Omega 0 i} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_i^{\top}\right) + g_{\Omega 0 i} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i \right\} a_i\ \mathrm{d}x. \tag{9.12.61}$$

Here, we assume that the cross-section of the stepped one-dimensional linear elastic body is a rectangle with unit depth. In addition, the x-coordinate is viewed as the x_1-coordinate, and the height direction is viewed as the x_2-coordinate. For each $i \in \{1,2\}$, a_i represents the cross-section and b_i represents its variation. Moreover, σ_1, ε_1, σ_2, ε_2 denotes $\sigma(u_1)$, $\varepsilon(u_1)$, $\sigma(u_2 - u_1)$, $\varepsilon(u_2 - u_1)$, respectively. In this case, the following relationships:

$$\boldsymbol{G}_{\Omega 0 i} = 2\boldsymbol{S}\left(\boldsymbol{u}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top} = 2\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} \sigma_i\varepsilon_i & 0 \\ 0 & 0 \end{pmatrix},$$

$$g_{\Omega 0i} = -\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{u}) = -\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix} = -\sigma_i \varepsilon_i,$$
$$\boldsymbol{\nabla}\boldsymbol{\varphi}_i^\top = \begin{pmatrix} 0 & 0 \\ 0 & b_i/a_i \end{pmatrix}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i = (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau = \frac{b_i}{a_i}$$

hold. Using these relationships, we get

$$\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle = l \begin{pmatrix} -\sigma_1 \varepsilon_1 \\ -\sigma_2 \varepsilon_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{g}_0 \cdot \boldsymbol{b}. \tag{9.12.62}$$

Here, $\boldsymbol{g}_0$ on the right-hand side of Eq. (9.12.62) matches the cross-sectional gradient of Eq. (1.1.28).

Moreover, the shape derivative of $f_1(\boldsymbol{\phi})$ becomes

$$\begin{aligned} f_1'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \sum_{i \in \{1,2\}} \langle \boldsymbol{g}_1, \boldsymbol{\varphi} \rangle = \int_0^l (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)\, a_i \, \mathrm{d}x \\ &= l \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{g}_1 \cdot \boldsymbol{b}. \end{aligned} \tag{9.12.63}$$

$\boldsymbol{g}_1$ on the right-hand side of Eq. (9.12.63) matches the cross-sectional gradient of Eq. (1.1.17).

Furthermore, the Hessian matrix of f_0 defined by Eq. (9.12.60) can be obtained as follows. For each $j \in \{1,2\}$, the following hold:

$$\boldsymbol{\nabla}\boldsymbol{\varphi}_{ji}^\top = \begin{pmatrix} 0 & 0 \\ 0 & \frac{b_{ji}}{a_i} \end{pmatrix}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_{ji} = \frac{b_{ji}}{a_i},$$
$$\boldsymbol{E}(\boldsymbol{u}) = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{S}(\boldsymbol{u}) = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}.$$

The shape derivative of the first term on the right-hand side of Eq. (9.12.61) is calculated by Eq. (9.12.37). Hence, we get

$$\begin{aligned} &h_0(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] \\ &= \sum_{i \in \{1,2\}} \int_0^l \Big[2\boldsymbol{S}(\boldsymbol{u}) \cdot \boldsymbol{E}(\boldsymbol{u}) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \\ &\quad + \left(\boldsymbol{S}(\boldsymbol{u}) \left(\boldsymbol{\nabla}\boldsymbol{u}^\top \right)^\top \right) \cdot \Big\{ \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \right)^\top \\ &\quad + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \right)^\top - 4\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 - 4\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \Big\} \Big] a_i \, \mathrm{d}x \\ &= \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \cdot \left(2l \begin{pmatrix} \frac{\sigma_1 \varepsilon_1}{a_1} & 0 \\ 0 & \frac{\sigma_2 \varepsilon_2}{a_2} \end{pmatrix} \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix} \right) = \boldsymbol{b}_1 \cdot (\boldsymbol{H}_0 \boldsymbol{b}_2). \end{aligned} \tag{9.12.64}$$

The $\boldsymbol{H}_0$ on the right-hand side of Eq. (9.12.64) matches the Hessian matrix of Eq. (1.1.29).

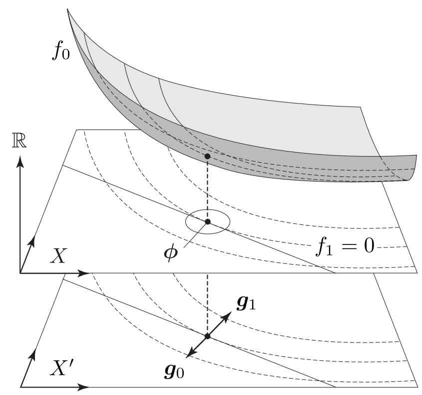


Fig. 9.16: The image of a minimizer $\boldsymbol{\phi}$ of the mean compliance minimization problem (Problem 9.12.2).

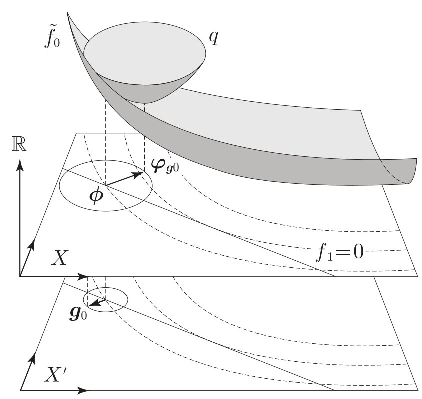


Fig. 9.17: The image of the H^1 gradient method with respect to $\tilde{f}_0$.

Based on these comparisons, the image of the minimum point of Problem 9.12.2 is thought to be as that depicted in Fig. 9.16 using Fig. 1.1.4 of Exercise 1.1.7.

Figures 9.17 and 9.18 show the images of the H^1 gradient method for obtaining the domain variations $\boldsymbol{\varphi}_{g0}$ and $\boldsymbol{\varphi}_{g1}$ that decreases $\tilde{f}_0$ and f_1, respectively. Figure 9.19 shows the image of the Lagrange multiplier λ_1 such that the constraint concerning the domain measure is satisfied. In these figures, it is assumed that although the domain measure constraint is satisfied at $\Omega(\boldsymbol{\phi})$, $\boldsymbol{\phi}$ is not a minimizer. The search direction $\boldsymbol{\varphi}_g = \boldsymbol{\varphi}_{g0} + \lambda_1 \boldsymbol{\varphi}_{g1}$ in Fig. 9.19 is orthogonal to $\boldsymbol{g}_1$ in Fig. 9.18. In other words, the search direction is in the direction that the constraint is satisfied. This is due to the fact that Eq. (9.10.3) which determines the Lagrange multiplier in the gradient method with respect

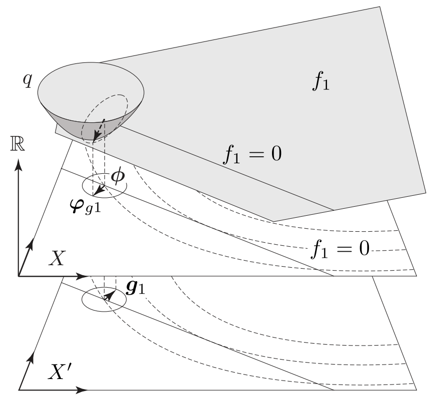


Fig. 9.18: The image of the H^1 gradient method with respect to f_1.

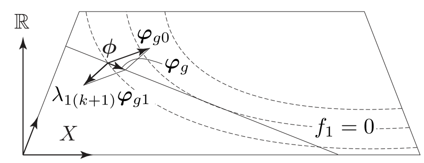


Fig. 9.19: Image of Lagrange multiplier λ_1.

to a constrained problem is actually given by

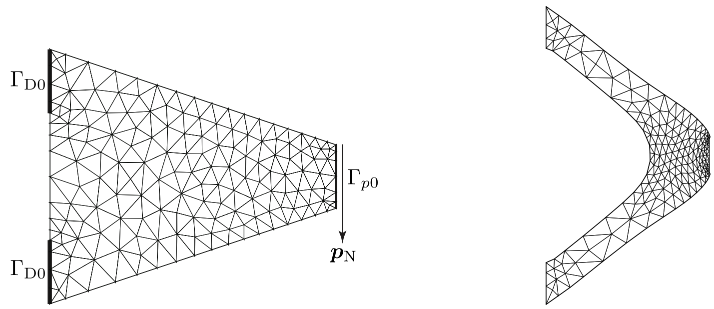
$$\lambda_1 = -\frac{\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} \rangle}{\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g1} \rangle} \tag{9.12.65}$$

in Problem 9.12.2 and can be written as

$$\langle \boldsymbol{g}_1, \boldsymbol{\varphi}_{g0} + \lambda_{1(k+1)} \boldsymbol{\varphi}_{g1} \rangle = 0. \tag{9.12.66}$$

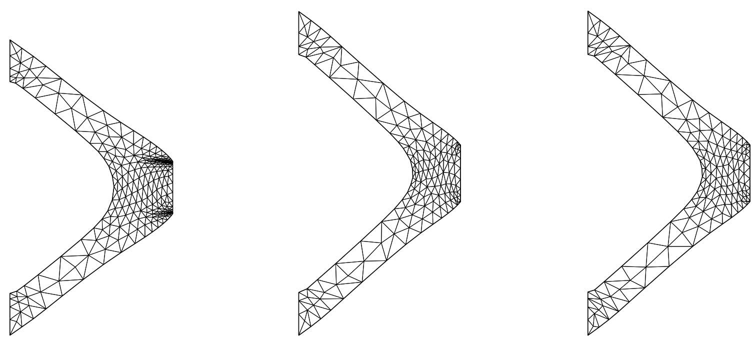
9.12.5 Numerical Example

Let us show a numerical example. In Figs. 9.20 to 9.22, the results of the mean compliance minimization with respect to a two-dimensional linear elastic body with a boundary condition referred to as the coat-hanging problem are shown. Figure 9.20 (a) shows the initial shape and the boundary conditions of the state determination problem. The boundary condition with respect to the domain variation is assumed to be $\bar{\Omega}_{\mathrm{C0}} = \Gamma_{\mathrm{D0}} \cup \Gamma_{p0}$ in Eq. (9.1.1). Here, it is assumed that these boundaries deform in the tangential direction. In addition, $\boldsymbol{p}_{\mathrm{N}}$ is assumed to vary with boundary measure. The program is written using the programming language FreeFEM (`https://freefem.org/`) [33] by the finite element method with reference to Example 37 in the book [70]. In the finite element analyses of the linear elastic problem and the H^1 gradient method or



(a) Initial shape and boundary conditions

(b) H^1 gradient method ($\bar{\boldsymbol{g}}_{\mathscr{L}}$)



(c) H^1 gradient method ($\boldsymbol{g}_{\mathscr{L}}$)

(d) H^1 Newton method ($h_0, \boldsymbol{g}_{\mathscr{L}}$)

(e) H^1 Newton method ($\boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{g}_{\mathscr{L}}$)

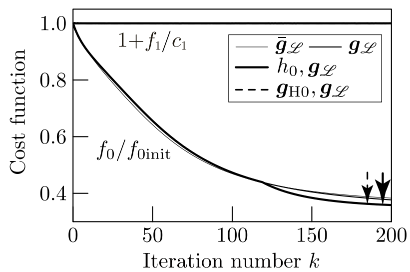
Fig. 9.20: Numerical example of mean compliance minimization problem: shape ($k = 200$).

the H^1 Newton method, second-order triangular elements were used. In the case using the H^1 Newton method, the routine of the H^1 Newton method was started at $k_{\mathrm{N}} = 120$. The parameters (c_a in Eq. (9.10.1), c_Ω in Eq. (9.9.3), k_{N}, $c_{\Omega 1}$ and $c_{\Omega 0}$ in Eq. (9.9.17), c_h in Eq. (9.10.8) and the parameter (*errelas*) that controls the error level in the adaptive mesh) affect the result. The details are described in the programs.[4]

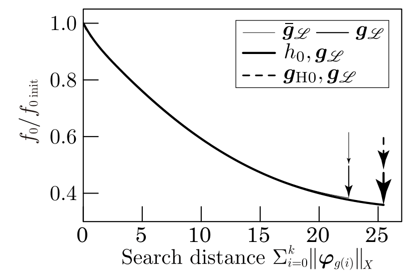
Figures 9.20 (b) to (e) show the shapes obtained by the four methods (H^1 gradient method using $\bar{\boldsymbol{g}}_{\mathscr{L}} = \bar{\boldsymbol{g}}_0 + \lambda_1 \bar{\boldsymbol{g}}_1$ of the boundary integral type, H^1 gradient method using $\boldsymbol{g}_{\mathscr{L}} = \boldsymbol{g}_0 + \lambda_1 \boldsymbol{g}_1$ of the domain integral type, H^1 Newton method using $h_{\mathscr{L}} = h_0 + \lambda_1 h_1$ and $\boldsymbol{g}_{\mathscr{L}}$, and H^1 Newton method using $\boldsymbol{g}_{\mathrm{H}0}$, h_1 and $\boldsymbol{g}_{\mathscr{L}}$).

Figure 9.21 (a) shows the cost functions $f_0/f_{0\mathrm{init}}$ and $1 + f_1/c_1$ normalized with f_0 at the initial shape denoted by $f_{0\mathrm{init}}$ and c_1 set with the initial volume, respectively, with respect to the iteration number k. Figure 9.21 (b) shows those values with respect to the distance $\sum_{i=0}^{k-1} \left\| \boldsymbol{\varphi}_{g(i)} \right\|_X$ on the search path in

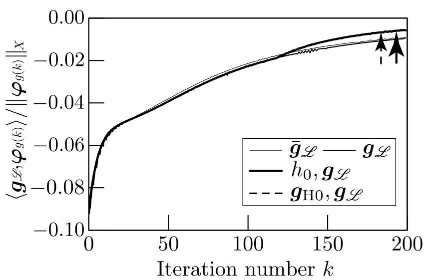
[4]See Electronic supplementary material.



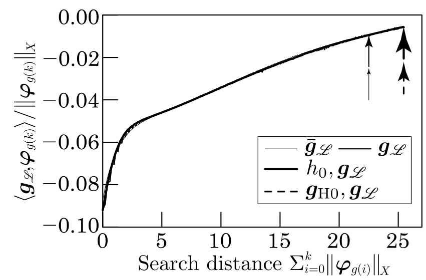
(a) Cost functions



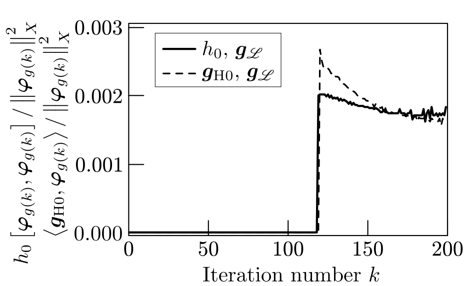
(b) Cost functions
(search distance)



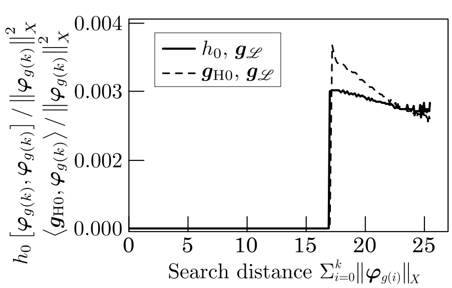
(c) Gradient of f_0 on search path



(d) Gradient of f_0 on search path
(search distance)

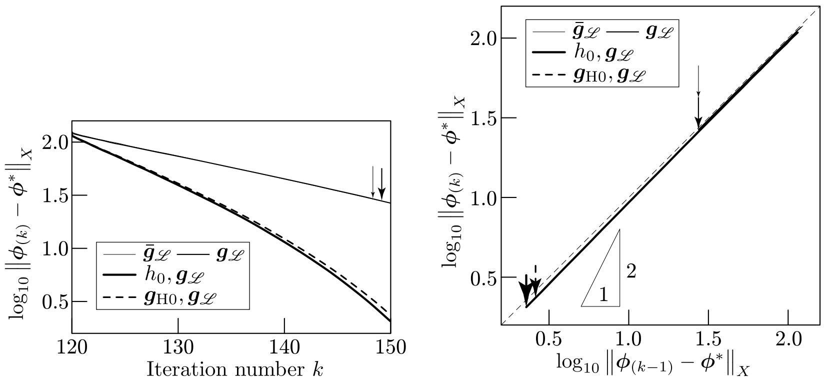


(e) Hessian of f_0 on search path



(f) Hessian of f_0 on search path
(search distance)

Fig. 9.21: Numerical example of mean compliance minimization problem: cost functions, their gradients and Hessians on the search path ($\bar{\boldsymbol{g}}_{\mathscr{L}}$: H^1 gradient method using $\bar{\boldsymbol{g}}_{\mathscr{L}}$, $\boldsymbol{g}_{\mathscr{L}}$: H^1 gradient method using $\boldsymbol{g}_{\mathscr{L}}$, $h_0, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method, $\boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method using Hesse gradient).



(a) Iteration history (b) $(k-1)$-th vs. k-th plot

Fig. 9.22: Numerical example of mean compliance minimization problem: distance $\left\|\boldsymbol{\phi}_{(k)}-\boldsymbol{\phi}^*\right\|_X$ from an approximate minimum point $\boldsymbol{\phi}^*$ ($\bar{\boldsymbol{g}}_{\mathscr{L}}$: H^1 gradient method using $\bar{\boldsymbol{g}}_{\mathscr{L}}$, $\boldsymbol{g}_{\mathscr{L}}$: H^1 gradient method using $\boldsymbol{g}_{\mathscr{L}}$, $h_0, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method, $\boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method using Hesse gradient).

X. The graphs of f_0's gradient (the gradient of the Lagrange function $\mathscr{L} = \mathscr{L}_0 + \lambda_1 f_1$) calculated as $\left\langle \boldsymbol{g}_{\mathscr{L}}, \boldsymbol{\varphi}_{g(k)} \right\rangle / \left\|\boldsymbol{\varphi}_{g(k)}\right\|_X$ are shown in Fig. 9.21 (c) and (d) with respect to the iteration number and the search distance, respectively. Moreover, Fig. 9.21 (e) and (f) shows the graphs of f_0's second-order derivative $h_0\left[\boldsymbol{\varphi}_{g(k)}, \boldsymbol{\varphi}_{g(k)}\right] / \left\|\boldsymbol{\varphi}_{g(k)}\right\|_X^2$ (in the case of the Newton method using Hesse gradient, $\left\langle \boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{\varphi}_{g(k)} \right\rangle / \left\|\boldsymbol{\varphi}_{g(k)}\right\|_X^2$) with respect to the iteration number and the search distance, respectively. In these notations, the norm of the i-th search vector is defined by

$$\left\|\boldsymbol{\varphi}_{g(i)}\right\|_X = \left(\int_{\Omega(\boldsymbol{\phi})} \left\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_{g(i)}^{\top}\right) \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_{g(i)}^{\top}\right) + \boldsymbol{\varphi}_{g(i)} \cdot \boldsymbol{\varphi}_{g(i)} \right\} \mathrm{d}x \right)^{1/2}. \tag{9.12.67}$$

The computational times until $k = 200$ by PC were 24.443, 37.132, 46.026, 59.312 sec when the H^1 gradient method of the boundary integral type, the H^1 gradient method of the domain integral type, the H^1 Newton method and the H^1 Newton method using the Hesse gradient were used, respectively.

Regarding the computational results obtained from the above-mentioned methods, we give the the following explanations and provide some considerations. The graphs in Fig. 9.21 (a) show that the convergence speed with respect to the iteration number k is faster when using the H^1 Newton method than when applying the H^1 gradient method. However, when the H^1 Newton method started, $c_{\Omega 1}$ and $c_{\Omega 0}$ in Eq. (9.9.17) were replaced with smaller values (the step size was enlarged) within the area where numerical instability did not happen. As a result, it can be considered that the convergence speed was increased. In this problem, when c_h is set to zero (that is, the H^1 gradient method), it was observed that the convergence speed was increased. The reason we consider to be behind the increase in convergence speed is the increase in the magnitude of the step size which was due to the exclusion of the term $h_{\mathscr{L}}$. In most cases, it is observed that when the step size is taken bigger, the H^1 Newton method keeps the computation until termination, but the H^1 gradient method fails to continue after a number of iterations. Moreover, the aspect around the minimum point can be observed in Fig. 9.21 (d) and (f). From these graphs, based on the observation that the Hessian of f_0 on the search path is positive valued, we infer that the point of convergence is a local minimum point.

In addition, Fig. 9.22 (a) shows the graphs of the distance $\left\|\boldsymbol{\phi}_{(k)} - \boldsymbol{\phi}^*\right\|_X$ from the k-th approximation $\boldsymbol{\phi}_{(k)}$ to an approximate minimum point $\boldsymbol{\phi}^*$ obtained by the four methods with respect to the iteration number k. The approximate minimum point $\boldsymbol{\phi}^*$ is given as the numerical solution of $\boldsymbol{\phi}$ when the iteration time is taken larger than the given value in the H^1 Newton method. From this figure, it can be confirmed that the convergence orders for the results obtained through the H^1 Newton method are more than the first order. However, Fig. 9.22 (b), plotting the k-th distance $\|\boldsymbol{\phi}_k - \boldsymbol{\phi}^*\|_X$ with respect to the $(k-1)$-th distance (the gradient of the graph shows the order of convergence as explained by using Eq. (3.8.13)) shows that the convergence order of the H^1 Newton method is less than the second order but is more than the first order. The reason behind this finding is provided at the end of Section 8.9.6. Namely, the addition of the bilinear form a_X in X to the original Hessian in order to ensure coercivity and regularity of the left-hand side of Eq. (9.9.16) makes the H^1 Newton method different from the original Newton method.

9.13 Shape Optimization Problem of Stokes Flow Field

As an example of an application in flow field problems, let us consider a mean flow resistance minimization problem of a Stokes flow field and look at the process for obtaining the shape derivatives of cost functions. The image of the initial domain Ω_0 is shown in Fig. 9.23. The linear space X with respect to domain movement and its admissible set $\mathcal{D}$ are defined as in Sect. 9.1.

9.13.1 State Determination Problem

Let us consider a Stokes problem as a state determination problem. Here, in addition to the symbols used in Problem 5.5.1, the Stokes problem will be written in the following way for the shape optimization problem. Here, to guarantee the unique existence of the solution, $\partial\Omega(\boldsymbol{\phi})$ is taken to be a Dirichlet boundary with respect to $\boldsymbol{\phi} \in \mathcal{D}$ and $\boldsymbol{u}_{\mathrm{D}} : \partial\Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ is taken to be a known flow velocity. Detailed conditions will be shown in Eq. (9.13.5) and Eq. (9.13.6)

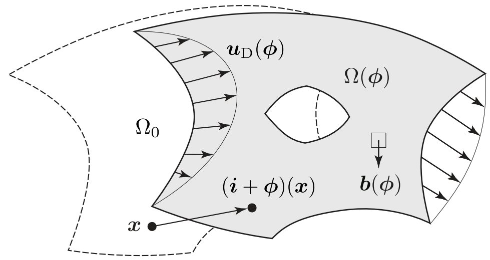


Fig. 9.23: The initial domain $\Omega_0 \subset D$ and the domain variation (displacement) $\boldsymbol{\phi}$ with respect to a Stokes flow field.

later. μ is a positive constant expressing the coefficient of viscosity. With respect to the flow velocity $\boldsymbol{u}$, which is the solution to the state determination problem shown later, let $\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}$ be denoted as $\tilde{\boldsymbol{u}}$. Here, let the admissible set and the Hilbert space containing $\tilde{\boldsymbol{u}}$ be defined as

$$U = \left\{ \boldsymbol{u} \in H^1\left(D;\mathbb{R}^d\right) \;\middle|\; \boldsymbol{u} = \mathbf{0}_{\mathbb{R}^d} \text{ on } \partial\Omega\left(\boldsymbol{\phi}\right) \right\}, \tag{9.13.1}$$

$$\mathcal{S} = U \cap W^{2,4}\left(D;\mathbb{R}^d\right), \tag{9.13.2}$$

respectively. Moreover, the admissible set and the real Hilbert space containing the pressure p are taken to be

$$P = \left\{ q \in L^2\left(D;\mathbb{R}\right) \;\middle|\; \int_{\Omega(\boldsymbol{\phi})} q \,\mathrm{d}x = 0 \right\}, \tag{9.13.3}$$

$$\mathcal{Q} = P \cap W^{1,4}\left(D;\mathbb{R}\right), \tag{9.13.4}$$

respectively. For known functions, in conjunction with Hypothesis 9.5.1, it is assumed that

$$\boldsymbol{b} \in C^1_{\mathrm{S}'}\left(B; L^\infty\left(D;\mathbb{R}^d\right)\right), \quad \boldsymbol{u}_{\mathrm{D}} \in C^1_{\mathrm{S}'}\left(B; U_{\mathrm{div}} \cap C^{0,1}\left(D;\mathbb{R}^d\right)\right) \tag{9.13.5}$$

and these are fixed with the material. Moreover, with respect to Hypothesis 9.5.2,

$$\boldsymbol{b} \in C^1_{\mathrm{S}^*}\left(B; W^{1,2q_{\mathrm{R}}}\left(D;\mathbb{R}^d\right)\right), \quad \boldsymbol{u}_{\mathrm{D}} \in C^1_{\mathrm{S}^*}\left(B; U_{\mathrm{div}} \cap W^{2,2q_{\mathrm{R}}}\left(D;\mathbb{R}^d\right)\right) \tag{9.13.6}$$

and these are assumed to be fixed in space, where $q_{\mathrm{R}} > d$. Here, let

$$U_{\mathrm{div}} = \left\{ \boldsymbol{u} \in H^1\left(D;\mathbb{R}^d\right) \;\middle|\; \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \text{ in } D \right\}.$$

Here too, $\left(\boldsymbol{\nu} \cdot \boldsymbol{\nabla}\right)\boldsymbol{u} = \left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top \boldsymbol{\nu}$ is written as $\partial_\nu \boldsymbol{u}$. Given these definitions, we define a state determination problem as follows.

Problem 9.13.1 (Stokes problem) For $\boldsymbol{\phi} \in \mathcal{D}$, let $\boldsymbol{b}$, $\boldsymbol{u}_{\mathrm{D}}$ and μ be given. Find $(\boldsymbol{u}, p) : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^{d+1}$ which satisfies

$$
\begin{aligned}
-\boldsymbol{\nabla}^{\top}\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top}\right) + \boldsymbol{\nabla}^{\top} p = \boldsymbol{b}^{\top}(\boldsymbol{\phi}) \quad & \text{in } \Omega(\boldsymbol{\phi}), \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega(\boldsymbol{\phi}), \\
\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}(\boldsymbol{\phi}) \quad & \text{on } \partial\Omega(\boldsymbol{\phi}), \\
\int_{\Omega(\boldsymbol{\phi})} p \, \mathrm{d}x = 0. &
\end{aligned}
$$

□

For later use, referring to the weak form of the Stokes problem (Problem 5.5.2) with a Dirichlet boundary condition, let the Lagrange function with respect to Problem 9.13.1 be

$$
\begin{aligned}
& \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}, q) \\
& \quad = \int_{\Omega(\boldsymbol{\phi})} \left\{ -\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top} \cdot \left(\boldsymbol{\nabla} \boldsymbol{v}^{\top} \right) + p \boldsymbol{\nabla} \cdot \boldsymbol{v} + q \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{b} \cdot \boldsymbol{v} \right\} \mathrm{d}x \\
& \qquad + \int_{\partial\Omega(\boldsymbol{\phi})} \{ (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}) \cdot (\mu \partial_{\nu} \boldsymbol{v} - q \boldsymbol{\nu}) + \boldsymbol{v} \cdot (\mu \partial_{\nu} \boldsymbol{u} - p \boldsymbol{\nu}) \} \, \mathrm{d}\gamma, \qquad (9.13.7)
\end{aligned}
$$

where $(\boldsymbol{u}, p)$ is not necessarily the solution of Problem 9.13.1 and $(\boldsymbol{v}, q)$ is taken to be an element of $U \times P$ introduced as a Lagrange multiplier. If $(\boldsymbol{u}, p)$ is the solution of Problem 9.13.1, the equation

$$
\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}, q) = 0
$$

holds with respect to an arbitrary $(\boldsymbol{v}, q) \in U \times P$. This equation is equivalent to the weak form of Problem 9.13.1.

9.13.2 Mean Flow Resistance Minimization Problem

Let us define a shape optimization problem with the associated cost functions defined as follows. With respect to the solution $(\boldsymbol{u}, p)$ of Problem 9.13.1,

$$
f_0(\boldsymbol{\phi}, \boldsymbol{u}, p) = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{b} \cdot \boldsymbol{u} \, \mathrm{d}x + \int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{u}_{\mathrm{D}} \cdot (\mu \partial_{\nu} \boldsymbol{u} - p \boldsymbol{\nu}) \, \mathrm{d}\gamma \qquad (9.13.8)
$$

is referred to as the mean flow resistance. The reason for this is as explained in Section 8.10.2. Moreover,

$$
f_1(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})} \mathrm{d}x - c_1 \qquad (9.13.9)
$$

is a cost function with respect to domain measure constraint. Here, c_1 is a positive constant such that $f_1(\boldsymbol{\phi}) \leq 0$ holds with respect to some $\boldsymbol{\phi} \in \mathcal{D}$.

We define a mean flow resistance minimization problem as follows.

Problem 9.13.2 (Mean flow resistance minimization problem) Let $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{Q}$ be defined as in Eq. (9.1.3), Eq. (9.13.2) and Eq. (9.13.4), respectively. Let f_0 and f_1 be Eq. (9.13.8) and Eq. (9.13.9), respectively. In this case, obtain $\Omega(\boldsymbol{\phi})$ which satisfies

$$\min_{(\boldsymbol{\phi},\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}},p)\in\mathcal{D}\times\mathcal{S}\times\mathcal{Q}} \left\{ f_0(\boldsymbol{\phi},\boldsymbol{u},p) \mid f_1(\boldsymbol{\phi}) \le 0,\ \text{Problem 9.13.1} \right\}.$$

□

9.13.3 Shape Derivatives of Cost Functions

The shape derivative of $f_1(\boldsymbol{\phi})$ has already been obtained using Eq. (9.12.24) or Eq. (9.12.58). Hence, only the shape derivative of $f_0(\boldsymbol{\phi},\boldsymbol{u},p)$ will be computed. Here too, let us consider the case of using the formulae based on the shape derivative of a function and the case using the formulae based on the partial shape derivative of a function separately. If the formulae based on the shape derivative of a function are used, the expression for the shape derivative up to the second order will be established. As preparation for this, let the Lagrange function of $f_0(\boldsymbol{\phi},\boldsymbol{u})$ be

$$\begin{aligned}
&\mathscr{L}_0(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0)\\
&\quad= f_0(\boldsymbol{\phi},\boldsymbol{u},p) - \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v},q)\\
&\quad= \int_{\Omega(\boldsymbol{\phi})} \left\{ \mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{v}_0^{\top} - p\boldsymbol{\nabla}\cdot\boldsymbol{v}_0 - \boldsymbol{b}\cdot(\boldsymbol{v}_0+\boldsymbol{u}) - q_0\boldsymbol{\nabla}\cdot\boldsymbol{u} \right\} \mathrm{d}x\\
&\qquad - \int_{\partial\Omega(\boldsymbol{\phi})} \left\{ (\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_{\nu}\boldsymbol{v}_0 - q_0\boldsymbol{\nu}) + (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_{\nu}\boldsymbol{u} - p\boldsymbol{\nu}) \right\} \mathrm{d}\gamma.
\end{aligned} \tag{9.13.10}$$

Here, $\mathscr{L}_{\mathrm{S}}$ is the Lagrange function of the state determination problem defined in Eq. (9.13.7). Moreover, it is assumed that $(\boldsymbol{v}_0, q_0)$ is a Lagrange multiplier with respect to the state determination problem prepared for f_0 and that $(\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}}, q_0)$ is an element of $U\times P$.

Shape Derivative of f_0 Using Formulae Based on Shape Derivative of a Function

If the formulae based on the shape derivative of a function are used, the following results are obtained. In this case, it is assumed that $\boldsymbol{b}$ and $\boldsymbol{u}_{\mathrm{D}}$ are fixed with the material.

In this case, the Fréchet derivative of $\mathscr{L}_0$ can be written as

$$\begin{aligned}
&\mathscr{L}_0'(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0)[\boldsymbol{\varphi},\hat{\boldsymbol{u}},\hat{p},\hat{\boldsymbol{v}}_0,\hat{q}_0]\\
&\quad= \mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0)[\boldsymbol{\varphi}] + \mathscr{L}_{0\boldsymbol{u}p}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0)[\hat{\boldsymbol{u}},\hat{p}]\\
&\qquad + \mathscr{L}_{0\boldsymbol{v}_0q_0}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0)[\hat{\boldsymbol{v}}_0,\hat{q}_0]
\end{aligned} \tag{9.13.11}$$

for any arbitrary variation $(\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{v}}_0, \hat{q}_0) \in X \times (U \times P)^2$. Here, the notations in Eq. (9.3.5) and Eq. (9.3.15) were used. Each term is considered below.

The third term on the right-hand side of Eq. (9.13.11) becomes

$$\begin{aligned}\mathscr{L}_{0\boldsymbol{v}_0 q_0}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{v}}_0, \hat{q}_0] &= -\mathscr{L}_{\mathrm{S}\boldsymbol{v}_0, q_0}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{v}}_0, \hat{q}_0] \\ &= -\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, p, \hat{\boldsymbol{v}}_0, \hat{q}_0). \end{aligned} \tag{9.13.12}$$

Eq. (9.13.12) is the Lagrange function of the state determination problem (Problem 9.13.1). Hence, if $(\boldsymbol{u}, p)$ is the weak solution of the state determination problem, the third term on the right-hand side of Eq. (9.13.11) is zero.

Moreover, the second term on the right-hand side of Eq. (9.13.11) becomes

$$\begin{aligned}&\mathscr{L}_{0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{u}}, \hat{p}] \\ &\quad= \int_{\Omega(\boldsymbol{\phi})} \left\{\mu\left(\boldsymbol{\nabla}\boldsymbol{u}'^{\top}\right) \cdot \boldsymbol{\nabla}\boldsymbol{v}_0^{\top} - \hat{p}\boldsymbol{\nabla}\cdot\boldsymbol{v}_0 - \boldsymbol{b}\cdot\hat{\boldsymbol{u}} - q_0\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}\right\} \mathrm{d}x \\ &\qquad - \int_{\partial\Omega(\boldsymbol{\phi})} \left\{\hat{\boldsymbol{u}}\cdot(\mu\partial_\nu\boldsymbol{v}_0 - q_0\boldsymbol{\nu}) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\hat{\boldsymbol{u}} - \hat{p}\boldsymbol{\nu})\right\} \mathrm{d}\gamma \\ &\quad= -\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{v}_0, q_0, \hat{\boldsymbol{u}}, \hat{p}) \end{aligned} \tag{9.13.13}$$

for any arbitrary variation $(\hat{\boldsymbol{u}}, \hat{p}) \in U \times P$ of $(\boldsymbol{u}, p)$. Hence, when the self-adjoint relationship

$$(\boldsymbol{u}, p) = (\boldsymbol{v}_0, q_0) \tag{9.13.14}$$

holds, the second term on the right-hand side of Eq. (9.13.11) also vanishes.

Furthermore, the first term on the right-hand side of Eq. (9.13.11) becomes

$$\begin{aligned}&\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}] \\ &\quad= \int_{\Omega(\boldsymbol{\phi})} \Big[-\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nabla}\boldsymbol{v}_0^{\top}\right) - \mu\boldsymbol{\nabla}\boldsymbol{v}_0^{\top}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right) \\ &\qquad\quad + p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nabla}\right)\cdot\boldsymbol{v}_0 + q_0\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nabla}\right)\cdot\boldsymbol{u} \\ &\qquad\quad + \left\{\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{v}_0^{\top} - p\boldsymbol{\nabla}\cdot\boldsymbol{v}_0 - q_0\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{b}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\Big]\mathrm{d}x \\ &\qquad - \int_{\partial\Omega(\boldsymbol{\phi})} \Big[\left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{w}(\boldsymbol{\varphi}, \boldsymbol{v}_0, q_0) + (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{w}(\boldsymbol{\varphi}, \boldsymbol{u}, p)\right\} \\ &\qquad\quad + \big\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{v}_0 - q_0\boldsymbol{\nu}) \\ &\qquad\qquad + (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{u} - p\boldsymbol{\nu})\big\}(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau\Big]\mathrm{d}\gamma, \end{aligned}$$

in view of Eq. (9.3.5) and Eq. (9.3.15) representing the results of Propositions 9.3.4 and 9.3.7. Here, we let

$$\begin{aligned}&\boldsymbol{w}(\boldsymbol{\varphi}, \boldsymbol{u}, p) \\ &\quad= \left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top} - p\boldsymbol{I}\right\}\left[\left\{\boldsymbol{\nu}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nu}\right)\right\}\boldsymbol{\nu} - \left\{\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\right)^{\top}\right\}\boldsymbol{\nu}\right], \end{aligned} \tag{9.13.15}$$

and $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau$ follows Eq. (9.2.6). $\boldsymbol{I}$ represents a d-order unit matrix. Furthermore, by applying the identity

$$\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\right)\cdot\boldsymbol{v}_0=\left(\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)^\top\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}^\top=\boldsymbol{I}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right), \tag{9.13.16}$$

we get

$$\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'}\left(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0\right)[\boldsymbol{\varphi}]\\
&\quad=\int_{\Omega(\boldsymbol{\phi})}\left[-\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right)-\left(\mu\boldsymbol{\nabla}\boldsymbol{v}_0^\top-q_0\boldsymbol{I}\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\right.\\
&\qquad\left.+\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\cdot\boldsymbol{\nabla}\boldsymbol{v}_0^\top-q_0\boldsymbol{\nabla}\cdot\boldsymbol{u}-\boldsymbol{b}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\right]\mathrm{d}x\\
&\qquad-\int_{\partial\Omega(\boldsymbol{\phi})}\left[\left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{w}(\boldsymbol{\varphi},\boldsymbol{v}_0,q_0)+(\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot\boldsymbol{w}(\boldsymbol{\varphi},\boldsymbol{u},p)\right\}\right.\\
&\qquad+\left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{v}_0-q_0\boldsymbol{\nu})\right.\\
&\qquad\left.\left.+(\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{u}-p\boldsymbol{\nu})\right\}(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau\right]\mathrm{d}\gamma.
\end{aligned} \tag{9.13.17}$$

With the above results in mind, it is assumed that $(\boldsymbol{u},p)$ is the weak solution of Problem 9.13.1 and that the self-adjoint relationship (Eq. (9.13.14)) holds true. Here, using the Dirichlet condition and the continuity equation of Problem 9.13.1, we get

$$\begin{aligned}
\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}]&=\mathscr{L}_{0\boldsymbol{\phi}'}\left(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0\right)[\boldsymbol{\varphi}]=\langle\boldsymbol{g}_0,\boldsymbol{\varphi}\rangle\\
&=\int_{\Omega(\boldsymbol{\phi})}\left(\boldsymbol{G}_{\Omega0}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}^\top+g_{\Omega0}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\right)\mathrm{d}x
\end{aligned} \tag{9.13.18}$$

following the notation of Eq. (7.5.15) for $\tilde{f}_0$, where

$$\boldsymbol{G}_{\Omega0}=-2\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top, \tag{9.13.19}$$

$$g_{\Omega0}=\mu\boldsymbol{\nabla}\boldsymbol{u}^\top\cdot\boldsymbol{\nabla}\boldsymbol{u}^\top-2\boldsymbol{b}\cdot\boldsymbol{u}. \tag{9.13.20}$$

From the above results, similar conclusions can be obtained for Theorem 9.8.2 with respect to $\boldsymbol{g}_0$ of Eq. (9.13.18).

Second-Order Shape Derivative of f_0 Using Formulae Based on Shape Derivative of a Function

Next, let us obtain the second-order shape derivative of mean flow resistance f_0. Here, the formulae based on the shape derivative of a function are used, following the procedures shown in Sect. 9.8.2.

In correspondence with Hypothesis 9.8.3, the first condition $\boldsymbol{b}=\boldsymbol{0}_{\mathbb{R}^d}$ is assumed and we also suppose that the second condition is again satisfied. However, Hypothesis 9.8.3 (3) is unnecessary.

The Lagrange function $\mathscr{L}_0$ of f_0 is defined by Eq. (9.13.10). Viewing $(\boldsymbol{\phi},\boldsymbol{u},p)$ as a design variable, we define its admissible set and admissible direction set respectively as

$$S=\left\{(\boldsymbol{\phi},\boldsymbol{u},p)\in\mathcal{D}\times\mathcal{S}\times\mathcal{Q}\mid\mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v},q)=0\text{ for all }(\boldsymbol{v},q)\in U\times P\right\},$$

$$
\begin{aligned}
T_S(\boldsymbol{\phi}, \boldsymbol{u}, p) = \{ & (\boldsymbol{\varphi}, \hat{\boldsymbol{v}}, \hat{\pi}) \in X \times U \times P \mid \\
& \mathscr{L}_{\mathrm{S}\boldsymbol{\phi}\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}, q)[\boldsymbol{\varphi}, \hat{\boldsymbol{v}}, \hat{\pi}] = 0 \text{ for all } (\boldsymbol{v}, q) \in U \times P \}.
\end{aligned}
$$

Considering Eq. (9.1.6), the second-order Fréchet partial derivative of $\mathscr{L}_0$ with respect to arbitrary variations $(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1, \hat{\pi}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2, \hat{\pi}_2) \in T_S(\boldsymbol{\phi}, \boldsymbol{u}, p)$ of the design variable $(\boldsymbol{\phi}, \boldsymbol{u}, p) \in S$ is given as follows:

$$
\begin{aligned}
& \mathscr{L}_{0(\boldsymbol{\phi}', \boldsymbol{u}, p)(\boldsymbol{\phi}', \boldsymbol{u}, p)}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1, \hat{\pi}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2, \hat{\pi}_2)] \\
&\quad = \left(\mathscr{L}_{0(\boldsymbol{\phi}', \boldsymbol{u}, p)}\right)_{(\boldsymbol{\phi}', \boldsymbol{u}, p)}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[(\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_1, \hat{\pi}_1), (\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_2, \hat{\pi}_2)] \\
&\qquad + \langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \left(\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1] + \mathscr{L}_{0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{v}}_1, \hat{\pi}_1]\right)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \left(\mathscr{L}_{0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1] + \mathscr{L}_{0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\hat{\boldsymbol{v}}_1, \hat{\pi}_1]\right)_{\boldsymbol{u}p}[\hat{\boldsymbol{v}}_2, \hat{\pi}_2] \\
&\qquad + \langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \left(\mathscr{L}_{0\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_2, \hat{\pi}_2] \\
&\qquad + \mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_2, \hat{\boldsymbol{v}}_1, \hat{\pi}_1] \\
&\qquad + \mathscr{L}_{0\boldsymbol{u}p\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{v}}_1, \hat{\pi}_1, \hat{\boldsymbol{v}}_2, \hat{\pi}_2] \\
&\qquad + \langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle,
\end{aligned} \tag{9.13.21}
$$

where $\langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle$ follows the definition given in Eq. (9.1.8).

Here, the first and fifth terms of the right-hand side in Eq. (9.13.21) become

$$
\begin{aligned}
& \left(\mathscr{L}_{0\boldsymbol{\phi}'}\right)_{\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \left[\left\{ -\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right) \right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \right. \\
&\qquad + \left\{ -\left(\mu \boldsymbol{\nabla} \boldsymbol{v}_0^\top - q_0 \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{u}^\top\right) \right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad + \mu \boldsymbol{\nabla} \boldsymbol{u}^\top \cdot \boldsymbol{\nabla} \boldsymbol{v}_0^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1)_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\qquad - 2\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right)\left(\boldsymbol{\nabla} \boldsymbol{u}^\top\right)^\top \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top - \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)\right) \\
&\qquad \left. + \mu \boldsymbol{\nabla} \boldsymbol{u}^\top \cdot \boldsymbol{\nabla} \boldsymbol{u}^\top \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top\right)^\top \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top - (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1) \right\} \right] \mathrm{d}x,
\end{aligned} \tag{9.13.22}
$$

which is obtained from Eq. (9.13.17) using the Dirichlet condition of Problem 9.13.1, the equation of continuity and the assumption $\boldsymbol{b} = \boldsymbol{0}_{\mathbb{R}^d}$. The first term of the integrand on the right-hand side of Eq. (9.13.22) can be expanded as follows:

$$
\begin{aligned}
& \left\{ -\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right) \right\}_{\boldsymbol{\phi}'}[\boldsymbol{\varphi}_2] \\
&\quad = \left\{ \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right) \right\}_{\boldsymbol{\nabla} \boldsymbol{u}^\top} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top\right) \\
&\qquad + \left\{ \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right) \right\}_{\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top\right) \\
&\qquad + \left\{ \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I}\right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right) \right\}_{\boldsymbol{\nabla} \boldsymbol{v}_0^\top} \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \left\{ \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \\
&= \mu \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \\
&\quad + \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I} \right) \cdot \left\{ \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) + \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \right\} \\
&\quad - \left\{ \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 .
\end{aligned} \tag{9.13.23}
$$

Similarly, the second term of the integrand on the right-hand side of Eq. (9.13.22) is the first term with $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}_0, q_0)$ switched over. The third term of the integrand on the right-hand side of Eq. (9.13.22) is

$$
\begin{aligned}
&\mu \boldsymbol{\nabla} \boldsymbol{u}^\top \cdot \boldsymbol{\nabla} \boldsymbol{v}_0^\top \left\{ \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right\}_{\boldsymbol{\phi}'} [\boldsymbol{\varphi}_2] \\
&\quad = \mu \boldsymbol{\nabla} \boldsymbol{u}^\top \cdot \boldsymbol{\nabla} \boldsymbol{v}_0^\top \left\{ - \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \right)^\top \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_2 \right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \right) \right\} .
\end{aligned} \tag{9.13.24}
$$

Hence, using the self-adjoint relationship, Eq. (9.13.22) becomes

$$
\begin{aligned}
&\left(\mathscr{L}_{0\boldsymbol{\phi}'} \right)_{\boldsymbol{\phi}'} (\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0) [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \langle \boldsymbol{g}(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \rangle \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[\mu \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \\
&\qquad + \left(\mu \boldsymbol{\nabla} \boldsymbol{u}^\top - p \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \\
&\qquad + \mu \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right) \\
&\qquad + \left(\mu \boldsymbol{\nabla} \boldsymbol{v}_0^\top - q_0 \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{\varphi}_2^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right) \Big] \mathrm{d}x .
\end{aligned} \tag{9.13.25}
$$

Next, we consider the second term on the right-hand side of Eq. (9.13.21). Using Eq. (9.13.17), the Dirichlet condition of Problem 9.13.1, the equation of continuity and the assumption $\boldsymbol{b} = \boldsymbol{0}_{\mathbb{R}^d}$, we get

$$
\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}p} (\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0) [\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_2, \hat{\pi}_2] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[- \left(\mu \boldsymbol{\nabla} \hat{\boldsymbol{v}}_2^\top - \hat{\pi}_2 \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \boldsymbol{v}_0^\top \right) \\
&\qquad - \left(\mu \boldsymbol{\nabla} \boldsymbol{v}_0^\top - q_0 \boldsymbol{I} \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_1^\top \boldsymbol{\nabla} \hat{\boldsymbol{v}}_2^\top \right) \\
&\qquad + \left\{ \mu \left(\boldsymbol{\nabla} \hat{\boldsymbol{v}}_2^\top - \hat{\pi}_2 \boldsymbol{I} \right) \cdot \boldsymbol{\nabla} \boldsymbol{v}_0^\top - q_0 \boldsymbol{\nabla} \cdot \hat{\boldsymbol{v}}_2 \right\} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_1 \Big] \mathrm{d}x .
\end{aligned} \tag{9.13.26}
$$

On the other hand, the variation of $(\boldsymbol{u}, p)$ satisfying the state determination problem with respect to an arbitrary domain variation $\boldsymbol{\varphi}_j \in Y$ for $j \in \{1, 2\}$ is written as $(\hat{\boldsymbol{v}}_j, \hat{\pi}_j) = \left(\boldsymbol{v}'(\boldsymbol{\phi}) [\boldsymbol{\varphi}_j], \pi'(\boldsymbol{\phi}) [\boldsymbol{\varphi}_j] \right)$. If the Fréchet partial derivative of the Lagrange function $\mathscr{L}_{\mathrm{S}}$ of the state determination problem defined in Eq. (9.13.7) is taken, then we obtain

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'\boldsymbol{u}p} (\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}, q) [\boldsymbol{\varphi}_j, \hat{\boldsymbol{v}}_j, \hat{\pi}_j] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \big[\mu \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \boldsymbol{\nabla} \boldsymbol{u}^\top \right) \cdot \left(\boldsymbol{\nabla} \boldsymbol{v}^\top \right) + \mu \boldsymbol{\nabla} \boldsymbol{u}^\top \cdot \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \boldsymbol{\nabla} \boldsymbol{v}^\top \right) \\
&\qquad - p \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \boldsymbol{\nabla} \right) \cdot \boldsymbol{v} - q \left(\boldsymbol{\nabla} \boldsymbol{\varphi}_j^\top \boldsymbol{\nabla} \right) \cdot \boldsymbol{u}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left\{ -\boldsymbol{\nabla}\boldsymbol{u}^\top \cdot \left(\mu \boldsymbol{\nabla}\boldsymbol{v}^\top - q\boldsymbol{I}\right) + p\boldsymbol{\nabla}\cdot\boldsymbol{v} \right\} \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j \\
&\quad - \left(\boldsymbol{\nabla}\hat{\boldsymbol{v}}_j^\top\right) \cdot \left(\mu\boldsymbol{\nabla}\boldsymbol{v}^\top - q\boldsymbol{I}\right) + \hat{\pi}_j \boldsymbol{\nabla}\cdot\boldsymbol{v} \Big] \mathrm{d}x \\
&= \int_{\Omega(\boldsymbol{\phi})} \Big[\Big\{ \mu \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\right)^\top - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right)\boldsymbol{I} \right) \boldsymbol{\nabla}\boldsymbol{u}^\top - \mu\boldsymbol{\nabla}\hat{\boldsymbol{v}}_j^\top \\
&\quad + \hat{\pi}_j\boldsymbol{I} + p\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right)\boldsymbol{I} - p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\right)^\top \Big\} \cdot \boldsymbol{\nabla}\boldsymbol{v}^\top \\
&\quad + q\left\{ -\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\boldsymbol{\nabla}\right)\cdot\boldsymbol{u} + \left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right) + \boldsymbol{\nabla}\cdot\hat{\boldsymbol{v}}_j \right\} \Big] \mathrm{d}x \\
&= 0 \qquad (9.13.27)
\end{aligned}
$$

for any arbitrary variation $(\boldsymbol{v}, q) \in U \times P$. From Eq. (9.13.27), the following identities:

$$
\boldsymbol{\nabla}\hat{\boldsymbol{v}}_j^\top = \left\{ \boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j \right\} \boldsymbol{\nabla}\boldsymbol{u}^\top, \qquad (9.13.28)
$$

$$
\boldsymbol{\nabla}\cdot\hat{\boldsymbol{v}}_j = \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\boldsymbol{\nabla}\right)\cdot\boldsymbol{u} - \left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right), \qquad (9.13.29)
$$

$$
\hat{\pi}_j\boldsymbol{I} = -p\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_j\right)\boldsymbol{I} - p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_j^\top\right)^\top \qquad (9.13.30)
$$

hold for any $(\boldsymbol{v}, q) \in U \times P$. Here, if $\hat{\boldsymbol{v}}_2$ and $\hat{\pi}_2$ satisfying Eq. (9.13.28) to Eq. (9.13.30) are substituted into $\hat{\boldsymbol{v}}_2$ and $\hat{\pi}_2$ of Eq. (9.13.26), we have

$$
\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{u}p}\left(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0\right)\left[\boldsymbol{\varphi}_1, \hat{\boldsymbol{v}}_2, \hat{\pi}_2\right] \\
&= \int_{\Omega(\boldsymbol{\phi})} \Big[-\Big\{ \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2 \right) \left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top\right) \\
&\quad + p\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right)\boldsymbol{I} - p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \Big\} \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{v}_0^\top\right) \\
&\quad - \left(\mu\boldsymbol{\nabla}\boldsymbol{v}_0^\top - q_0\boldsymbol{I}\right) \cdot \left\{ \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2 \right) \boldsymbol{\nabla}\boldsymbol{u}^\top \right\} \\
&\quad + \Big\{ \Big(\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2 \right) \left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top\right) \\
&\quad + p\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right)\boldsymbol{I} - p\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top \Big) \cdot \boldsymbol{\nabla}\boldsymbol{v}_0^\top - q_0\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \cdot \left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top \Big\} \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1 \Big] \mathrm{d}x. \\
&\qquad (9.13.31)
\end{aligned}
$$

Similarly, the third term on the right-hand side of Eq. (9.13.21) is Eq. (9.13.31) where $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are interchanged. The fourth term on the right-hand side of Eq. (9.13.21) becomes zero.

Summarizing the above results, the second-order shape derivative of $\tilde{f}_0$ becomes

$$
\begin{aligned}
&h_0\left(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{u}\right)\left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\right] \\
&= \int_{\Omega(\boldsymbol{\phi})} \Big[-2\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top \cdot \boldsymbol{\nabla}\boldsymbol{u}^\top\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right) \\
&\quad - \left\{ \left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top - p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top \right\} \cdot \Big\{ \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \\
&\qquad + \boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top + \boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)^\top
\end{aligned}
$$

$$- 4\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1 - 4\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\Big\}\Big]\,\mathrm{d}x. \tag{9.13.32}$$

Second-Order Shape Derivative of Cost Function Using Lagrange Multiplier Method

The application of the Lagrange multiplier method in obtaining the second-order shape derivative of the mean flow resistance f_0 is described as follows. Fixing $\boldsymbol{\varphi}_1$, we define the Lagrange function for $\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}_1] = \langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1\rangle$ in Eq. (9.13.18) by

$$\mathscr{L}_{\mathrm{I}0}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0) = \langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1\rangle - \mathscr{L}_{\mathrm{S}}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0), \tag{9.13.33}$$

where $\mathscr{L}_{\mathrm{S}}$ is given by Eq. (9.13.7), and $(\boldsymbol{w}_0, r_0) \in U \times P$ is the adjoint variable provided for $(\boldsymbol{u}, p)$ in $\boldsymbol{g}_0$.

Considering Eq. (9.1.6), with respect to arbitrary variations $(\boldsymbol{\varphi}_2, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}_0, \hat{r}_0) \in \mathcal{D} \times (U \times P)^2$ of $(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)$, the Fréchet derivative of $\mathscr{L}_{\mathrm{I}0}$ is written as

$$\begin{aligned}
&\mathscr{L}_{\mathrm{I}0}'(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)[\boldsymbol{\varphi}_2, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}_0, \hat{r}_0] \\
&\quad = \mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)[\boldsymbol{\varphi}_2] + \langle \boldsymbol{g}_0(\boldsymbol{\phi}), \boldsymbol{t}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)\rangle \\
&\qquad + \mathscr{L}_{\mathrm{I}0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)[\hat{\boldsymbol{u}}, \hat{p}] \\
&\qquad + \mathscr{L}_{\mathrm{I}0\boldsymbol{w}_0 r_0}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)[\hat{\boldsymbol{w}}_0, \hat{r}_0].
\end{aligned} \tag{9.13.34}$$

The fourth term on the right-hand side of Eq. (9.13.34) vanishes if $(\boldsymbol{u}, p)$ is the solution of the state determination problem.

The third term on the right-hand side of Eq. (9.13.34) is

$$\begin{aligned}
&\mathscr{L}_{\mathrm{I}0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{w}_0, r_0)[\hat{\boldsymbol{u}}, \hat{p}] \\
&\quad = \int_{\Omega(\boldsymbol{\phi})} \Big[-2\Big\{\Big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\big)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\Big)\mu\boldsymbol{\nabla}\boldsymbol{u}^\top + p\big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\big)^\top\Big\}\cdot\boldsymbol{\nabla}\hat{\boldsymbol{u}}^\top \\
&\qquad + 2(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1)\boldsymbol{b}\cdot\hat{\boldsymbol{u}} + 2\hat{p}\big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\big)^\top\cdot\big(\boldsymbol{\nabla}\boldsymbol{u}^\top\big) \\
&\qquad + \mu\boldsymbol{\nabla}\boldsymbol{w}_0^\top\cdot\Big(\boldsymbol{\nabla}\hat{\boldsymbol{u}}^\top\Big) - \hat{p}\boldsymbol{\nabla}\cdot\boldsymbol{w}_0^\top - r_0\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}\Big]\,\mathrm{d}x.
\end{aligned} \tag{9.13.35}$$

Here, the conditions that Eq. (9.13.35) is zero for arbitrary $(\hat{\boldsymbol{u}}, \hat{p}) \in U \times P$ and $\boldsymbol{u}$ satisfies the continuity equation are equivalent to setting $(\boldsymbol{w}_0, r_0)$ to be the solution of the following adjoint problem.

Problem 9.13.3 (Adjoint problem of $\boldsymbol{w}_0$ with respect to $\langle \boldsymbol{g}_0, \boldsymbol{\varphi}_1\rangle$) Under the assumption of Problem 9.13.2, let $\boldsymbol{\varphi}_1 \in Y$ be given. Find $(\boldsymbol{w}_0, r_0) = (\boldsymbol{w}_0(\vartheta_1), r_0(\vartheta_1)) \in U \times P$ satisfying

$$\begin{aligned}
-\boldsymbol{\nabla}^\top\big(\mu\boldsymbol{\nabla}\boldsymbol{w}_0^\top\big) + \boldsymbol{\nabla}^\top r_0 = {}& -2\boldsymbol{\nabla}^\top\Big\{\Big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\big)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\Big)\mu\boldsymbol{\nabla}\boldsymbol{u}^\top \\
&+ p\big(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\big)^\top\Big\} - 2(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1)\boldsymbol{b}^\top \quad \text{in } \Omega(\boldsymbol{\phi}),
\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{w}_0 &= 2\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top\cdot\boldsymbol{\nabla}\boldsymbol{u}^\top \quad \text{in } \Omega(\boldsymbol{\phi}),\\ \boldsymbol{w}_0 &= \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \partial\Omega(\boldsymbol{\phi}),\\ \int_{\Omega(\boldsymbol{\phi})} r_0\,\mathrm{d}x &= 0.\end{aligned}$$

□

Finally, the first and second terms on the right-hand side of Eq. (9.13.34) become

$$\begin{aligned}&\mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{w}_0,r_0)[\boldsymbol{\varphi}_2]+\langle\boldsymbol{g}_0(\boldsymbol{\phi}),\boldsymbol{t}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)\rangle\\ &\quad=\mathscr{L}_{0\boldsymbol{\phi}'\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{u},p)[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2]+\langle\boldsymbol{g}_0(\boldsymbol{\phi}),\boldsymbol{t}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)\rangle\\ &\qquad-\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{w}_0,r_0)[\boldsymbol{\varphi}_2]\end{aligned}\tag{9.13.36}$$

with respect to an arbitrary $\boldsymbol{\varphi}_1 \in Y$. The first and second terms on the right-hand side of Eq. (9.13.36) are given by Eq. (9.13.25) in which $(\boldsymbol{v}_0, q_0) = (\boldsymbol{u}, p)$ is substituted. The third term is

$$\begin{aligned}&-\mathscr{L}_{\mathrm{S}\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{w}_0,r_0)[\boldsymbol{\varphi}_2]\\ &\quad=\int_{\Omega(\boldsymbol{\phi})}\Big[\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\boldsymbol{\nabla}\boldsymbol{w}_0^\top\right)+\left(\mu\boldsymbol{\nabla}\boldsymbol{w}_0^\top-r_0\boldsymbol{I}\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\\ &\qquad+\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\cdot\boldsymbol{\nabla}\boldsymbol{w}_0^\top-r_0\boldsymbol{\nabla}\cdot\boldsymbol{u}-\boldsymbol{b}\cdot\boldsymbol{w}_0\right\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\Big]\mathrm{d}x.\end{aligned}\tag{9.13.37}$$

Here, $(\boldsymbol{u}, p)$ and $(\boldsymbol{w}_0, r_0)$ are the weak solutions of Problems 9.13.1 and 9.12.3, respectively. If we denote $f_0(\boldsymbol{\phi}, \boldsymbol{u}, p)$ by $\tilde{f}_0(\boldsymbol{\phi})$, then we obtain the relation

$$\begin{aligned}&\mathscr{L}_{\mathrm{I}0\boldsymbol{\phi}'}(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{w}_0(\boldsymbol{\varphi}_1),r_0(\boldsymbol{\varphi}_1))[\boldsymbol{\varphi}_2]+\langle\boldsymbol{g}_0(\boldsymbol{\phi}),\boldsymbol{t}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)\rangle\\ &\quad=\tilde{f}_0''(\boldsymbol{\phi})[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2]=\langle\boldsymbol{g}_{\mathrm{H}0}(\boldsymbol{\phi},\boldsymbol{\varphi}_1),\boldsymbol{\varphi}_2\rangle\\ &\quad=\int_{\Omega(\boldsymbol{\phi})}\Big[2\mu\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\\ &\qquad+2\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right\}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)\\ &\qquad-\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{w}_0^\top(\boldsymbol{\varphi}_1)\right)^\top+\left(\mu\boldsymbol{\nabla}\boldsymbol{w}_0^\top(\boldsymbol{\varphi}_1)-r_0\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right\}\\ &\qquad\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_2^\top\right)\\ &\qquad+\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top-p\boldsymbol{I}\right)\cdot\boldsymbol{\nabla}\boldsymbol{w}_0^\top(\boldsymbol{\varphi}_1)-\boldsymbol{b}\cdot\boldsymbol{w}_0(\boldsymbol{\varphi}_1)\right\}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_2\Big]\mathrm{d}x,\end{aligned}\tag{9.13.38}$$

where $\boldsymbol{g}_{\mathrm{H}0}$ is the Hesse gradient of the mean flow resistance.

If $\boldsymbol{b} = \boldsymbol{0}_{\mathbb{R}^d}$ is satisfied, with respect to the solution $(\boldsymbol{w}_0, r_0)$ of Problem 9.13.3,

$$\mu\boldsymbol{\nabla}\boldsymbol{w}_0^\top(\boldsymbol{\varphi}_1)-r_0\boldsymbol{I}=2\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top+\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top-\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)\mu\boldsymbol{\nabla}\boldsymbol{u}^\top$$

$$+ 2p \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top, \tag{9.13.39}$$

$$\boldsymbol{\nabla}\boldsymbol{w}_0^\top (\boldsymbol{\varphi}_1) = 2\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top + \left(\boldsymbol{\nabla}\boldsymbol{\varphi}_1^\top\right)^\top - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\right)\boldsymbol{\nabla}\boldsymbol{u}^\top \tag{9.13.40}$$

holds. Substituting Eq. (9.13.39) and Eq. (9.13.40) into Eq. (9.13.38), and using the relation $h_0(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = h_0(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1]$, it can be confirmed that Eq. (9.13.38) accords with Eq. (9.13.32).

Shape Derivative of f_0 Using Formulae Based on Partial Shape Derivative of a Function

If the formulae based on the partial shape derivative of a function are used, the corresponding results are as follows. Here, $\boldsymbol{b}$ and $\boldsymbol{u}_{\mathrm{D}}$ are assumed to be functions fixed in space. Moreover, it is assumed that $\boldsymbol{u}$ and $\boldsymbol{v}_0$ are elements of $W^{2,2q_{\mathrm{R}}}\left(D;\mathbb{R}^d\right)$, and p and q_0 are in $W^{1,2q_{\mathrm{R}}}(D;\mathbb{R})$, where $q_{\mathrm{R}} > d$.

Under these assumptions, the Fréchet derivative of $\mathscr{L}_0$ can be written as

$$\begin{aligned}\mathscr{L}_0'(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{v}}_0, \hat{q}_0] &= \mathscr{L}_{0\boldsymbol{\phi}}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\boldsymbol{\varphi}] \\ &+ \mathscr{L}_{0\boldsymbol{u}p}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{u}}, \hat{p}] + \mathscr{L}_{0\boldsymbol{v}_0 q_0}(\boldsymbol{\phi}, \boldsymbol{u}, p, \boldsymbol{v}_0, q_0)[\hat{\boldsymbol{v}}_0, \hat{q}_0]\end{aligned} \tag{9.13.41}$$

for any arbitrary variation $(\boldsymbol{\varphi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{v}}_0, \hat{q}_0) \in X \times (U \times P)^2$. Here, the notations of Eq. (9.3.21) and Eq. (9.3.27) were used. Each term is considered below.

The third term on the right-hand side of Eq. (9.13.41) accords with Eq. (9.13.12). Hence, if $(\boldsymbol{u}, p)$ is the weak solution of state determination problem (Problem 9.13.1), then this term is equal to zero.

Moreover, the second term on the right-hand side of Eq. (9.13.41) is the same as Eq. (9.13.13). Hence, if the self-adjoint relationship holds, then this term is also zero.

Furthermore, the first term on the right-hand side of Eq. (9.13.41) becomes

$$\begin{aligned}&\mathscr{L}_{0\boldsymbol{\phi}^*}(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_0)[\boldsymbol{\varphi}] \\ &\quad= \int_{\partial\Omega(\boldsymbol{\phi})} \left\{\mu\boldsymbol{\nabla}\boldsymbol{u}^\top \cdot \boldsymbol{\nabla}\boldsymbol{v}_0^\top - p\boldsymbol{\nabla}\cdot\boldsymbol{v}_0 - \boldsymbol{b}\cdot(\boldsymbol{u}+\boldsymbol{v}_0)\right\}\boldsymbol{\nu}\cdot\boldsymbol{\varphi}\,\mathrm{d}\gamma \\ &\qquad+ \int_{\partial\Omega(\boldsymbol{\phi})} \left[\left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot\bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{v}_0, q_0) + (\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}})\cdot\bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{u}, p)\right\}\right. \\ &\qquad\quad+ \left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{v}_0 - q_0\boldsymbol{\nu}) + (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{u} - p\boldsymbol{\nu})\right\}(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_\tau \\ &\qquad\quad+ (\partial_\nu + \kappa)\left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{v}_0 - q_0\boldsymbol{\nu})\right. \\ &\qquad\quad\left.\left.+ (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{u} - p\boldsymbol{\nu})\right\}\boldsymbol{\nu}\cdot\boldsymbol{\varphi}\right]\mathrm{d}\gamma \\ &\qquad- \int_{\partial\Omega(\boldsymbol{\phi})\cup\Theta(\boldsymbol{\phi})} \left\{(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{v}_0 - q_0\boldsymbol{\nu})\right. \\ &\qquad\quad\left.+ (\boldsymbol{v}_0-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_\nu\boldsymbol{u} - p\boldsymbol{\nu})\right\}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,\mathrm{d}\varsigma,\end{aligned}$$

from Eq. (9.3.21) and Eq. (9.3.27) expressing Propositions 9.3.10 and 9.3.13, where

$$\begin{aligned}\bar{\boldsymbol{w}}(\boldsymbol{\varphi}, \boldsymbol{u}, p) = &-\left\{\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top}\right)^{\top} - p\boldsymbol{I}\right\}\left[\sum_{i\in\{1,\dots,d-1\}} \left\{\boldsymbol{\tau}_i \cdot \left(\boldsymbol{\nabla}\boldsymbol{\varphi}^{\top}\boldsymbol{\nu}\right)\right\}\boldsymbol{\tau}_i\right] \\ &+ (\boldsymbol{\nu}\cdot\boldsymbol{\varphi})\left[\boldsymbol{\nabla}^{\top}\left\{\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top}\right)^{\top} - p\boldsymbol{I}\right\}\right]^{\top}, \end{aligned} \tag{9.13.42}$$

and $(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi})_{\tau}$ follows Eq. (9.2.6). Here, if the Dirichlet condition of Problem 9.13.1 is considered, the terms including $\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}$ and $\boldsymbol{v}_0 - \boldsymbol{u}_{\mathrm{D}}$ on $\mathscr{L}_{0\boldsymbol{\phi}^*}$ vanish.

With the above results in mind, if $\boldsymbol{u}$ and $\boldsymbol{v}_0$ fulfil the weak form of Problem 9.13.1 satisfying the self-adjoint relationship, we get

$$\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \mathscr{L}_{0\boldsymbol{\phi}^*}(\boldsymbol{\phi}, u, v_0)[\boldsymbol{\varphi}] = \langle \bar{\boldsymbol{g}}_0, \boldsymbol{\varphi}\rangle = \int_{\partial\Omega(\boldsymbol{\phi})} \bar{\boldsymbol{g}}_{\partial\Omega 0}\cdot\boldsymbol{\varphi}\,\mathrm{d}\gamma \tag{9.13.43}$$

using the notation of Eq. (7.5.15) for $\tilde{f}_0$, where

$$\bar{\boldsymbol{g}}_{\partial\Omega 0} = \left\{\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{\top} - 2\boldsymbol{b}\cdot\boldsymbol{u} - 2\partial_{\nu}(\boldsymbol{u}-\boldsymbol{u}_{\mathrm{D}})\cdot(\mu\partial_{\nu}\boldsymbol{u} - p\boldsymbol{\nu})\right\}\boldsymbol{\nu}. \tag{9.13.44}$$

Furthermore, on the homogeneous Dirichlet boundary, we have the equations

$$\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{\top} = \left\{\boldsymbol{\nu}(\partial_{\nu}\boldsymbol{u})^{\top}\right\}\cdot\left\{\boldsymbol{\nu}(\partial_{\nu}\boldsymbol{u})^{\top}\right\} = \partial_{\nu}\boldsymbol{u}\cdot\partial_{\nu}\boldsymbol{u} \tag{9.13.45}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = (\partial_{\nu}\boldsymbol{u})\cdot\boldsymbol{\nu} = 0. \tag{9.13.46}$$

In this case, we get

$$\bar{\boldsymbol{g}}_{\partial\Omega 0} = -\mu(\partial_{\nu}\boldsymbol{u}\cdot\partial_{\nu}\boldsymbol{u})\boldsymbol{\nu}. \tag{9.13.47}$$

From the above results, similar conclusions can be obtained for Theorem 9.8.6 with respect to function space containing $\bar{\boldsymbol{g}}_{\partial\Omega 0}$ of Eq. (9.13.44).

9.13.4 Relationship with Optimal Design Problem of One-Dimensional Branched Stokes Flow Field

Here, let us think about the relationship between the shape derivative of the cost function obtained with respect to the mean flow resistance minimization problem of a $d \in \{2, 3\}$-dimensional Stokes flow field and the cross-sectional derivative of the cost function obtained with respect to the mean flow resistance minimization problem (Problem 1.3.2) of the one-dimensional branched Stokes flow field looked at in Chap. 1. To correspond to the variables and linear spaces, relationships similar to those shown in Table 9.2 with respect to the linear elastic problem hold, with an addition of the pressure to the state variable and a change of the objective function.

In Problem 1.3.2, volume force was not assumed. If this assumption is applied to Problem 9.13.2, it corresponds to setting the objective function to

$$\begin{aligned} f_0(\boldsymbol{\phi}, \boldsymbol{u}, p) &= \int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{u}_{\mathrm{D}} \cdot (\mu \partial_\nu \boldsymbol{u} - p\boldsymbol{\nu})\, \mathrm{d}\gamma \\ &= -\sum_{i\in\{1,2\}} \int_0^{r_i} p_i u_{\mathrm{H}i}(r)\, 2\pi r\, \mathrm{d}r, \end{aligned} \tag{9.13.48}$$

where p_i and r_i for $i \in \{0,1,2\}$ follows the definition given in Problem 1.3.2, respectively, and $u_{\mathrm{H}i}$ is given by Eq. (1.3.1). In Eq. (9.13.48), $\partial_\nu \boldsymbol{u} = 0$ and $p_0 = 0$ were used. In Problem 1.3.2, u_1 and u_2 were defined as the volumes of the fluid flow per unit time on Γ_1 and Γ_2, respectively, and were fixed during the changes on the cross-sectional areas. In other words, $u_{\mathrm{H}1}$ and $u_{\mathrm{H}2}$ were varying with the boundary measure. This relation is written as

$$u'_{\mathrm{H}i}(r)[\boldsymbol{\varphi}] = -u_{\mathrm{H}i}(r)(\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau . \tag{9.13.49}$$

On the other hand, $\boldsymbol{u}_{\mathrm{D}}$ was taken to be fixed with material in Eq. (9.13.18) which gives the shape derivative of f_0. If this difference is considered, the shape derivative of f_0 defined by Eq. (9.13.48) becomes

$$\begin{aligned} \tilde{f}'_0(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \langle \boldsymbol{g}_0, \boldsymbol{\varphi} \rangle \\ &= \sum_{i\in\{0,1,2\}} l \int_{\Gamma_i} \left(\boldsymbol{G}_{\Omega 0 i} \cdot \boldsymbol{\nabla}\boldsymbol{\varphi}_i^\top + g_{\Omega 0 i} \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i \right) \mathrm{d}\gamma \\ &\quad + \sum_{i\in\{1,2\}} \int_0^{r_i} p_i u_{\mathrm{H}i}(r) (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau\, 2\pi r\, \mathrm{d}r. \end{aligned} \tag{9.13.50}$$

We express a point in the cylindrical domain in the one-dimensional Stokes flow field as $(x, r, \theta) \in (0, l) \times \Gamma_i$ and $\boldsymbol{u} = (u_{\mathrm{H}i}(r), 0, 0)^\top$. Here, because of the relationship

$$\int_{\Gamma_i} (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau\ \mathrm{d}\gamma = (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau\, a_i = b_i,$$

the following equations hold:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i = (\boldsymbol{\nabla} \cdot \boldsymbol{\varphi}_i)_\tau = \frac{b_i}{a_i}, \quad \boldsymbol{\nabla}\boldsymbol{\varphi}_i^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_i/a_i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9.13.51}$$

The flow velocity is given by

$$\boldsymbol{\nabla}\boldsymbol{u}^\top = \frac{p_i - \bar{p}}{4\mu l} \begin{pmatrix} 0 & 0 & 0 \\ 2r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned}\boldsymbol{G}_{\Omega 0i}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_i^{\top} &= -2\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}-p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\right\}\cdot\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_i^{\top}\right)\\ &= -8\mu\left(\frac{p_i-\bar{p}}{4\mu l}\right)^2\frac{b_i}{a_i}r^2,\\ g_{\Omega 0i}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_i &= \mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{\top}\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_i\right) = 4\mu\left(\frac{p_i-\bar{p}}{4\mu l}\right)^2\frac{b_i}{a_i}r^2.\end{aligned}$$

From these results and Eq. (1.3.2), we get

$$\begin{aligned}\tilde{f}_0'(\boldsymbol{\phi})[\boldsymbol{\varphi}] &= \sum_{i\in\{0,1,2\}} l\int_0^{r_i}\left(\boldsymbol{G}_{\Omega 0i}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_i^{\top}+g_{\Omega 0i}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_i\right)2\pi r\,\mathrm{d}r\\ &\quad-\sum_{i\in\{1,2\}} p_i\frac{u_i b_i}{a_i}\\ &= -\sum_{i\in\{0,1,2\}} 2\frac{u_i^2 b_i}{a_i^3} = \boldsymbol{g}_0\cdot\boldsymbol{b}.\end{aligned} \tag{9.13.52}$$

In Eq. (9.13.52), the equation of continuity with respect to domain variation

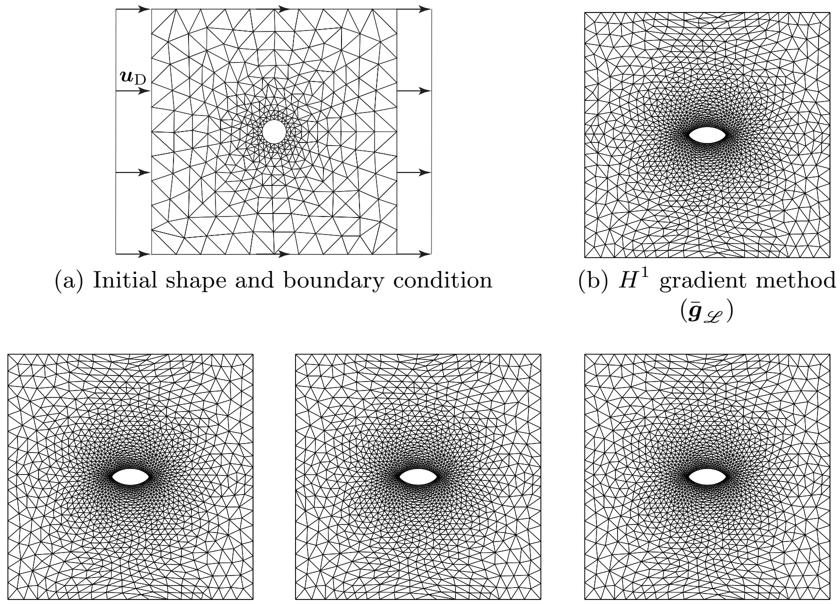
$$\sum_{i\in\{0,1,2\}}\int_{\Gamma_i} u_i\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_i\right)_{\tau}\,\mathrm{d}\gamma = \frac{u_0 b_0}{a_0}+\frac{u_1 b_1}{a_1}+\frac{u_2 b_2}{a_2} = 0 \tag{9.13.53}$$

was used. Here, $\boldsymbol{g}_0$ matches the cross-sectional-area gradient of the mean flow resistance f_0 with respect to the one-dimensional branched Stokes flow field obtained in Eq. (1.3.19).

Furthermore, the Hessian form of f_0 becomes

$$\begin{aligned}&h_0\left(\boldsymbol{\phi},\boldsymbol{u},p,\boldsymbol{v}_0,q_0\right)\left[\boldsymbol{\varphi}_{1i},\boldsymbol{\varphi}_2\right]\\ &= \sum_{i\in\{1,2\}} l\int_{\Gamma_i}\Big[-2\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_{2i}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_{1i}\right)\\ &\quad-\left\{\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^{\top}-p\boldsymbol{I}\right)\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\right\}\cdot\Big\{\boldsymbol{\nabla}\boldsymbol{\varphi}_{2i}^{\top}\boldsymbol{\nabla}\boldsymbol{\varphi}_{1i}^{\top}+\boldsymbol{\nabla}\boldsymbol{\varphi}_{1i}^{\top}\boldsymbol{\nabla}\boldsymbol{\varphi}_{2i}^{\top}\\ &\quad+\boldsymbol{\nabla}\boldsymbol{\varphi}_{2i}^{\top}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_{1i}^{\top}\right)^{\top}+\boldsymbol{\nabla}\boldsymbol{\varphi}_{1i}^{\top}\left(\boldsymbol{\nabla}\boldsymbol{\varphi}_{2i}^{\top}\right)^{\top}\\ &\quad-4\boldsymbol{\nabla}\boldsymbol{\varphi}_{2i}^{\top}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_{1i}-4\boldsymbol{\nabla}\boldsymbol{\varphi}_{1i}^{\top}\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_{2i}\Big\}\Big]\mathrm{d}\gamma\\ &\quad+\sum_{i\in\{0,1,2\}}\left\{\frac{\mathrm{d}}{\mathrm{d}a_i}\left(\frac{u_i^2 b_{1i}}{a_i^3}\right)b_{2i}+\frac{u_i^2 b_{1i}}{a_i^3}\left(\frac{b_{2i}}{a_i}\right)\right\}\\ &= \sum_{i\in\{0,1,2\}} 6\frac{u_i^2}{a_i^4}b_{1i}b_{2i} = \boldsymbol{b}_1\cdot\left(\boldsymbol{H}_0\boldsymbol{b}_2\right)\end{aligned} \tag{9.13.54}$$

using Eq. (9.13.32). Here, $\boldsymbol{H}_0$ matches the Hessian matrix of the mean flow resistance f_0 with respect to the cross-sectional areas of the one-dimensional branched Stokes flow field obtained in Eq. (1.3.26).



(a) Initial shape and boundary condition

(b) H^1 gradient method ($\bar{\boldsymbol{g}}_{\mathscr{L}}$)

(c) H^1 gradient method ($\boldsymbol{g}_{\mathscr{L}}$)

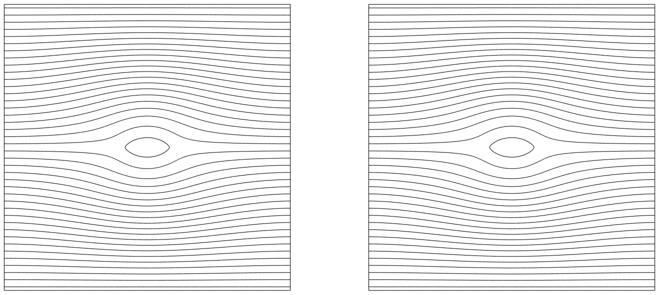
(d) H^1 Newton method ($h_0, \boldsymbol{g}_{\mathscr{L}}$)

(e) H^1 Newton method ($\boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{g}_{\mathscr{L}}$)

Fig. 9.24: Numerical example of mean flow resistance minimization problem: shape ($k = 40$).

9.13.5 Numerical Example

The results of mean flow resistance minimization for a two-dimensional Stokes flow field around an isolated object are shown in Figs. 9.24 to 9.27. The boundary condition of the state determination problem is assumed to be a uniform flow field in the horizontal direction on the outer boundary and zero on the boundary of the isolated object as shown in Fig. 9.24 (a). Moreover, with respect to the boundary condition for domain variation, the outer boundary was fixed (added in $\bar{\Omega}_{\mathrm{C}0}$ of Eq. (9.1.1)). The programs were written using the programming language FreeFEM (`https://freefem.org/`) [33] for the finite element method. In the finite element analyses of the Stokes problem, triangular elements of the second order with respect to the velocity and of the first order with respect to the pressure were used. Also, in the finite element analyses of the H^1 gradient method or the H^1 Newton method, the second-order triangular elements were used. On the other hand, in the case using the H^1 Newton method, the routine for the second-order method was started at $k_{\mathrm{N}} = 20$. The parameters (c_a in Eq. (9.10.1), c_Ω in Eq. (9.9.3), k_{N}, $c_{\Omega 1}$ and $c_{\Omega 0}$ in Eq. (9.9.17), c_h in Eq. (9.10.8) and the parameter (*errelas*) that controls the error level in the adaptive mesh) affect the result. For a complete understanding of the conditions,



(a) Streamline of initial shape (b) Streamline of optimal shape (H^1 Newton method, $k = 40$)

Fig. 9.25: Numerical example of mean flow resistance minimization problem: streamlines.

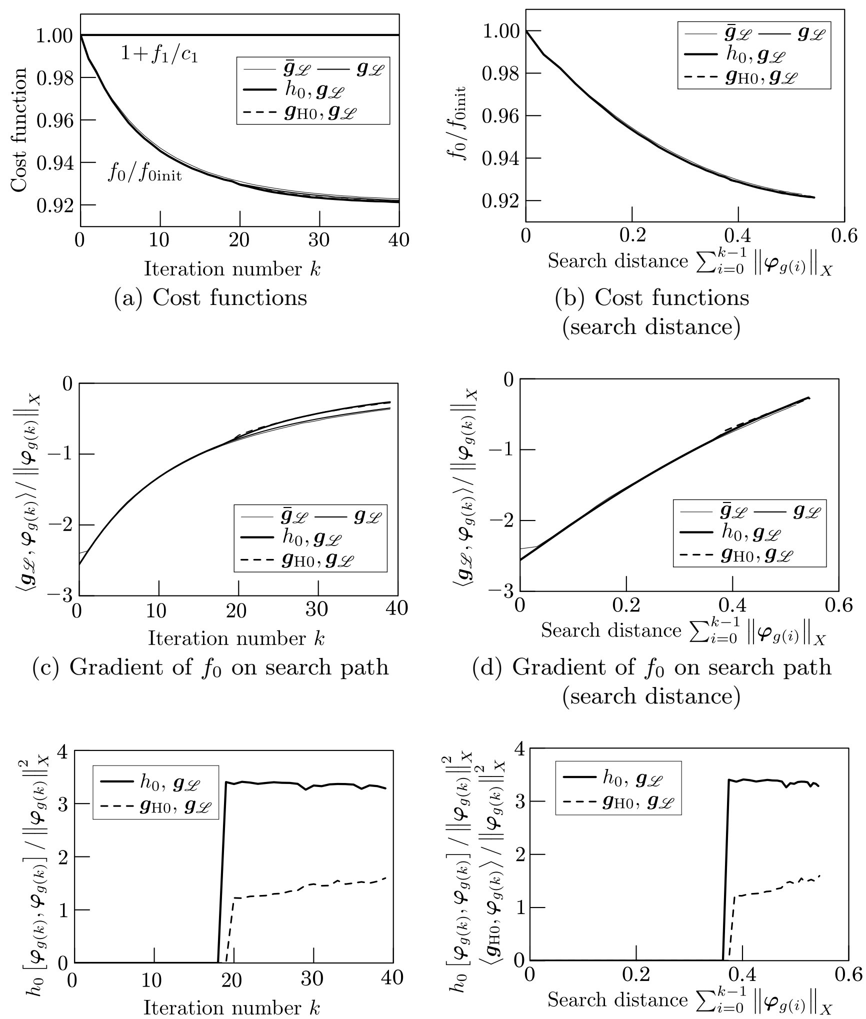
we suggest that the readers also examine the details of the programs.[5]

Figure 9.24 (b) to (e) show the shapes obtained by the four methods (H^1 gradient method using $\bar{\boldsymbol{g}}_{\mathscr{L}} = \bar{\boldsymbol{g}}_0 + \lambda_1 \bar{\boldsymbol{g}}_1$ of the boundary integral type, H^1 gradient method using $\boldsymbol{g}_{\mathscr{L}} = \boldsymbol{g}_0 + \lambda_1 \boldsymbol{g}_1$ of the domain integral type, H^1 Newton method using h_0 and $\boldsymbol{g}_{\mathscr{L}}$, and H^1 Newton method using $\boldsymbol{g}_{\mathrm{H}0}$ and $\boldsymbol{g}_{\mathscr{L}}$). Figures 9.25 (a) and (b) illustrate the streamlines in the initial shape and the optimal shape obtained by the H^1 Newton method, respectively. The streamlines are defined as the contour lines of the flow function $\psi : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ when the flow velocity $\boldsymbol{u}$ is given by $(\partial\psi/\partial x_2, -\partial\psi/\partial x_1)^{\top}$.

The graphs in Fig. 9.26 illustrate the histories the of cost functions and the gradients and Hessians of the object function f_0 on the search path with respect to the iteration number k and the search distance $\sum_{i=0}^{k-1} \left\| \boldsymbol{\varphi}_{g(i)} \right\|_X$. In this figure, $f_{0\mathrm{init}}$ denotes the value of f_0 at the initial density. Also, c_1 is set as the integral volume. The gradient of f_0 on the search path was calculated using the Lagrange function $\mathscr{L} = \mathscr{L}_0 + \lambda_1 f_1$ by $\left\langle \boldsymbol{g}_{\mathscr{L}}, \boldsymbol{\varphi}_{g(k)} \right\rangle / \left\| \boldsymbol{\varphi}_{g(k)} \right\|_X$. The Hessian of f_0 on the search path was computed by $h_0 \left[\boldsymbol{\varphi}_{g(k)}, \boldsymbol{\varphi}_{g(k)} \right] / \left\| \boldsymbol{\varphi}_{g(k)} \right\|_X^2$. In the case of the Newton method using the Hesse gradient, the formula $\left\langle g_{\mathrm{H}0}, \boldsymbol{\varphi}_{g(k)} \right\rangle / \left\| \boldsymbol{\varphi}_{g(k)} \right\|_X^2$ was used to calculate the Hessian. The norm $\left\| \boldsymbol{\varphi}_{g(i)} \right\|_X$ of the i-th search vector is defined by Eq. (9.12.67). The computational times until $k = 40$ by PC were 16.324, 43.628, 63.173, 81.039 sec by the H^1 gradient method of the boundary integral type, the H^1 gradient method of the domain integral type, the H^1 Newton method and the H^1 Newton method using the Hesse gradient, respectively.

Regarding the computational results obtained from the above-mentioned numerical illustrations, our findings were similar to those given in Sect. 9.12.5. The graphs in Fig. 9.26 (a) show that the convergence speed with respect to

[5] See Electronic supplementary material.



(a) Cost functions

(b) Cost functions (search distance)

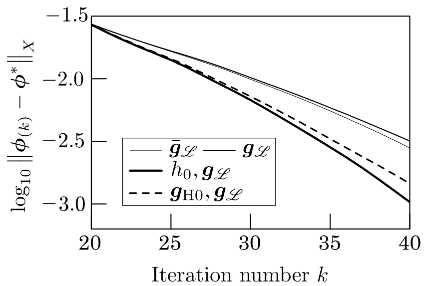
(c) Gradient of f_0 on search path

(d) Gradient of f_0 on search path (search distance)

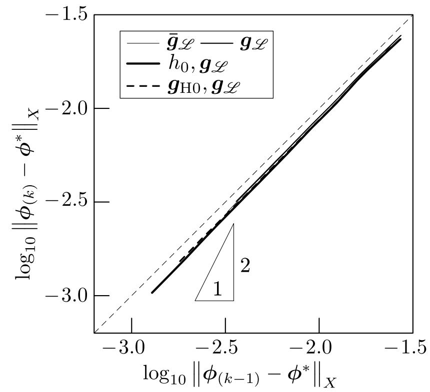
(e) Hessian of f_0 on search path

(f) Hessian of f_0 on search path (search distance)

Fig. 9.26: Numerical example of mean flow resistance minimization problem: cost functions, their gradients and Hessians on the search path ($\bar{\boldsymbol{g}}_{\mathscr{L}}$: H^1 gradient method using $\bar{\boldsymbol{g}}_{\mathscr{L}}$, $\boldsymbol{g}_{\mathscr{L}}$: H^1 gradient method using $\boldsymbol{g}_{\mathscr{L}}$, $h_0, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method, $\boldsymbol{g}_{\mathrm{H}0}, \boldsymbol{g}_{\mathscr{L}}$: H^1 Newton method using Hesse gradient).



(a) Iteration history



(b) $(k-1)$-th vs. k-th plot

Fig. 9.27: Numerical example of mean flow resistance minimization problem: distance $\left\|\phi_{(k)} - \phi^*\right\|_X$ from an approximate minimum point ϕ^* ($g_{\mathscr{L}}$: the gradient method, $h_{\mathscr{L}}, g_{\mathscr{L}}$: the Newton method, $g_{\mathrm{H}0}, h_1, g_{\mathscr{L}}$: the Newton method using the Hesse gradient).

the iteration number k is faster when using the H^1 Newton method than when employing the H^1 gradient method. However, we emphasize that $c_{\Omega 1}$ and $c_{\Omega 0}$ in Eq. (9.9.17) were actually replaced with smaller values (the step size was enlarged) within the area where numerical instability did not happen when the H^1 Newton method started, and it seems that the convergence speed was improved due to the increased in the step size. Moreover, the aspect around the minimum point can be observed in Fig. 9.26 (d) and (f). From these graphs, based on fact that the Hessian of f_0 on the search path is positive valued, we infer that the point of convergence is a local minimum point.

In addition, Fig. 9.27 (a) shows the graphs of the distance $\left\|\phi_{(k)} - \phi^*\right\|_X$ of the k-th approximate $\phi_{(k)}$ obtained by the four methods to the approximate minimum point ϕ^* with respect to the iteration number k. The approximate minimum point ϕ^* is substituted by the numerical solution of ϕ when the iteration time is taken larger than the given value in the H^1 Newton method. From this figure, it can easily be observed that the convergence orders for the results obtained via the H^1 Newton methods are higher than the first order. However, from Fig. 9.27 (b), which shows the plot of the k-th distance $\left\|\phi_{(k)} - \phi^*\right\|_X$ with respect to the $(k-1)$-th distance, it can be observed easily that the convergence order of the H^1 Newton method is less than the second order but is definitely more than the first order. The reason behind this result is considered as the same as that stated at the end of Sect. 9.12.5.

9.14 Summary

In Chap. 9, a shape optimization problem of domain variation type was constructed with respect to the domain on which a boundary value problem of a partial differential equation is defined and its solution looked at in detail. The key points are as below:

(1) In a shape optimization problem of domain variation type, the design variable is a function defined on an initial domain and represents the displacement of each of the points from the reference domain to the new domain after variation (Sect. 9.1). The linear space X and the admissible set $\mathcal{D}$ of the design variable are defined by Eq. (9.1.1) and Eq. (9.1.3), respectively. Furthermore, in Sect. 9.1.3, two notions of derivatives called the shape derivative and partial shape derivative were introduced with respect to a function and a functional defined on varying domains.

(2) In Sect. 9.2, the formulae for the shape derivatives relating to the Jacobi matrix of domain mapping were established. Using these formulae, it was shown in Sect. 9.3 that the shape derivatives of functions and functionals can be obtained and there corresponding forms were established. These formulae, in addition, were used to define a variety of variation rules for functions in Sect. 9.4.

(3) In Sect. 9.6, considering a Poisson problem as a state determination problem (Sect. 9.5), a shape optimization problem of domain variation type was defined on X.

(4) It was shown that the shape derivative of a cost function can be obtained via the Lagrange multiplier method. In this case, an evaluation method using the formulae based on the shape derivative of a function given in Theorem 9.8.2 and evaluation method using the formulae based on the shape derivative of a function stated in Theorem 9.8.6 can be considered. However, these shape derivatives are not necessarily in the linear space containing the admissible set for the design variables, as pointed out in Remark 9.8.7.

(5) In Sect. 9.9, an H^1 gradient method using the shape derivative of a cost function was defined on the space X. The solutions of the H^1 gradient method are contained in the admissible set (Theorem 9.9.6) excluding the neighborhoods of singular points. Furthermore, in Sect. 9.9.2, it was shown that if the second-order shape derivative of the cost function can be calculated, it is also possible to obtain a descent direction for the cost function using the H^1 Newton method.

(6) In Sect. 9.10, it was shown that a solution to a shape optimization problem of domain variation type with constraints can be constructed using the same framework as the gradient method with respect to constrained problems and Newton method with respect to constrained problems shown in Chap. 3.

(7) When the finite element method is used to solve a state determination problem, the adjoint problem with respect to f_i and the H^1 gradient method, the order evaluation of the finite element solution with respect to the search vector $\boldsymbol{\varphi}_g$ can be obtained (Theorem 9.11.5).

(8) In Sect. 9.12, the first and second order shape derivatives of some cost functions associated with a mean compliance minimization problem of a linear elastic body with domain measure constraint were established.

(9) In Sect. 9.13, the first and second-order shape derivatives of some cost functions associated with a mean flow resistance minimization problem of a Stokes flow field with domain measure constraint were established.

Formulations and solutions of certain topology optimization problems of density variation type and shape optimization problems of domain variation type were introduced in Chaps. 8 and 9, respectively. As concluding remarks, we give a comparison of these problems below, detailing their advantages and disadvantages.

In the case of the density variation type, the density defined on a fixed domain bears advantages and disadvantages. An advantage is that clear theoretical development could easily be carried out because it enters the conventional framework of a typical function optimization problem. Moreover, replacing the density of the design variable by other material parameters, various problems except the topology optimization problem can be formulated. For example, when we use a healthy rate of stiffness instead of the density, an identification problem of damage in a linear elastic body can be constructed [88]. On the other hand, a disadvantage can be mentioned due to the need of some additional scheme to determine the boundary of a continuum from the obtained density.

In contrast, in the case of the domain variation type, it is necessary to prepare various formulas to obtain the shape derivatives of cost functions. This is primarily due to the fact that the domains where the associated state determination problems were defined vary. Especially, when one calculates the second-order derivative, it is not easy to notice that a correction term (refer to Eq. (9.1.9) and Eq. (9.3.11)) proportional to the first-order shape derivative of the cost function that was obtained with respect to the product of the second variation vector and variation of the first perturbation vector. On the other hand, it is possible to perturb a boundary of a continuum directly in actual numerical analysis. Hence, it is superior in the sense that the shape can be found correctly.

In actual shape optimization problems, it is hoped that a suitable method could easily be chosen considering its desired features.

9.15 Practice Problems

9.1 Suppose condition (2) in Theorem 9.8.2 holds. Show that the second term on the right-hand side of Eq. (9.8.9) giving $\boldsymbol{g}_{pi}$ is in $L^\infty\left(\Gamma_p\left(\boldsymbol{\phi}\right);\mathbb{R}^d\right)$.

9.2 In Problems 9.5.4 and 9.12.1, which were considered as state determination problems in this chapter, a mixed Dirichlet–Neumann boundary condition was assumed. However, in order to obtain the results in Theorem 9.8.2, Hypothesis 9.5.3 (2) ($\beta < \pi/3$ when on mixed boundaries) has to be satisfied with respect to the opening angle β. If the mixed boundary condition is replaced with a Robin condition, Hypothesis 9.5.3 (1) ($\beta < 2\pi/3$ when on boundaries of the same type) then becomes applicable. Hence, if the extended Poisson problem taken up in Chap. 5 (Problem 5.1.3) is simplified by removing the terms unrelated to the boundary conditions and replacing with a domain variation type, then we obtain the following.

Problem 9.15.1 (Poisson problem of Robin type) Let $\boldsymbol{\phi} \in \mathcal{D}$ and $c_{\partial\Omega}(\boldsymbol{\phi}) : D \to \mathbb{R}$ and $p_{\mathrm{R}}(\boldsymbol{\phi}) : D \to \mathbb{R}$ be given functions fixed with the material. Find $u : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$
\begin{aligned}
-\Delta u = 0 \quad & \text{in } \Omega(\boldsymbol{\phi}), \\
\partial_\nu u + c_{\partial\Omega}(\boldsymbol{\phi}) u = p_{\mathrm{R}}(\boldsymbol{\phi}) \quad & \text{on } \partial\Omega(\boldsymbol{\phi}).
\end{aligned}
$$

□

Here, choose Problem 9.15.1 as the state determination problem and let the cost function be

$$
f_i(\boldsymbol{\phi}, u) = \int_{\partial\Omega(\boldsymbol{\phi})} \eta_{\mathrm{R}i}(\boldsymbol{\phi}, u)\, \mathrm{d}\gamma \tag{9.15.1}
$$

for $i \in \{0, 1, \ldots, m\}$, where $\eta_{\mathrm{R}i}(\boldsymbol{\phi}, u)$ is some function fixed with the material. In this case, compute the shape derivative $\boldsymbol{g}_i$ using the formulae based on the shape derivative of a function. Moreover, state the condition of the corner opening angle and the required regularities for $c_{\partial\Omega}$, p_{R}, $\eta_{\mathrm{R}i}$ and $\eta_{\mathrm{R}iu}$ in order to have a similar regularity result for $\boldsymbol{g}_i$ in Theorem 9.8.2.

9.3 In a shape optimization problem of domain variation type examined in this chapter, if there is a crack on $\partial\Omega(\boldsymbol{\phi})$ (opening angle $\beta = 2\pi$), or if there is a Dirichlet boundary and Neumann boundary on a smooth boundary (opening angle $\beta = \pi$), then Hypothesis 9.5.3 is not satisfied. This would then imply that the assumption $u \in \mathcal{S}$ in Theorem 9.8.2 is also not satisfied and therefore, it is not clear that the shape derivative is obtained as an element of X'. However, if the linear space of the design variable (domain variation) is replaced with

$$
X = \left\{ \boldsymbol{\phi} \in C^{0,1}\left(D; \mathbb{R}^d\right) \,\middle|\, \boldsymbol{\phi} = \mathbf{0}_{\mathbb{R}^d} \text{ on } \bar{\Omega}_{\mathrm{C}0} \right\},
$$

it is possible to show that the shape derivative of the corresponding cost function is a bounded linear functional with respect to this space. The shape derivative can then be computed using a generalized J integral [5].

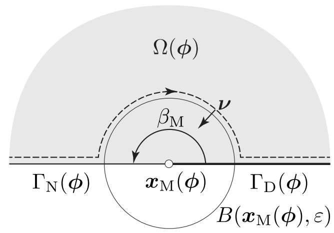


Fig. 9.28: Path of boundary integral $\mathscr{P}_u$.

Here, let us go through the calculation of an interim result used in obtaining the shape derivative. Suppose $\Omega(\boldsymbol{\phi})$ is a two-dimensional domain and $\boldsymbol{x}_{\mathrm{C}}$ is the tip of a crack (opening angle $\beta_{\mathrm{C}} = 2\pi$) and of an interior point on the homogeneous Dirichlet boundary or homogeneous Neumann boundary. Moreover, suppose $\boldsymbol{x}_{\mathrm{M}}$, as shown in Fig. 9.28, is a point on a smooth boundary and a boundary between the Dirichlet and Neumann boundaries (opening angle $\beta_{\mathrm{M}} = \pi$). In this case, think about obtaining the shape derivative of a cost function at $\boldsymbol{x}_{\mathrm{C}}$ and $\boldsymbol{x}_{\mathrm{M}}$ with the corresponding state determination problem defined as follows.

Problem 9.15.2 (Poisson problem of domain variation type)
Let $\boldsymbol{\phi} \in \mathcal{D}$ and $b(\boldsymbol{\phi})$ be a given function fixed with the material. Find $u : \Omega(\boldsymbol{\phi}) \to \mathbb{R}$ which satisfies

$$\begin{aligned} -\Delta u &= b(\boldsymbol{\phi}) \quad \text{in } \Omega(\boldsymbol{\phi}), \\ \partial_\nu u &= 0 \quad \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}), \\ u &= 0 \quad \text{on } \Gamma_{\mathrm{D}}(\boldsymbol{\phi}) \end{aligned}$$

□

Here, we replace the cost function of Eq. (9.6.1) with

$$f_i(\boldsymbol{\phi}, u) = \int_{\Omega(\boldsymbol{\phi})} \zeta_i(\boldsymbol{\phi}, u)\,\mathrm{d}x - c_i,$$

and assume, for simplicity, that ζ_i is not a function of $\boldsymbol{\nabla} u$. The shape derivative of f_i is computed as

$$\langle \boldsymbol{g}_i, \boldsymbol{\varphi} \rangle = -\mathscr{P}_u(\partial\Omega(\boldsymbol{\phi}), \boldsymbol{\varphi}, u)[v_i] + \langle \hat{\boldsymbol{g}}_{i\mathrm{C}}, \boldsymbol{\varphi} \rangle + \langle \hat{\boldsymbol{g}}_{i\mathrm{M}}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{g}_{i\mathrm{R}}, \boldsymbol{\varphi} \rangle$$

using the $\mathscr{P}$ integral defined in a generalized J integral [5], where

$$\begin{aligned} &-\mathscr{P}_u(\partial\Omega(\boldsymbol{\phi}), \boldsymbol{\varphi}, u)[v_i] \\ &\quad = \int_{\partial\Omega(\boldsymbol{\phi})} \big\{ (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \\ &\qquad - \partial_\nu u \boldsymbol{\nabla} v_i \cdot \boldsymbol{\varphi} - \partial_\nu v_i \boldsymbol{\nabla} u \cdot \boldsymbol{\varphi} \big\} \mathrm{d}\gamma, \end{aligned} \tag{9.15.2}$$

$$\begin{aligned}\left\langle \hat{\boldsymbol{g}}_{ij}, \boldsymbol{\varphi} \right\rangle &= \lim_{\epsilon \to 0} - \int_0^{\beta_j} \big\{ (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v_i)\, \boldsymbol{\nu} \cdot \boldsymbol{\varphi} - \partial_\nu u \boldsymbol{\nabla} v_i \cdot \boldsymbol{\varphi} \\ &\quad - \partial_\nu v_i \boldsymbol{\nabla} u \cdot \boldsymbol{\varphi} \big\} \epsilon \,\mathrm{d}\theta, \qquad (9.15.3)\\ \langle \boldsymbol{g}_{i\mathrm{R}}, \boldsymbol{\varphi} \rangle &= \int_{\partial\Omega(\boldsymbol{\phi})} b v_i \boldsymbol{\nu} \cdot \boldsymbol{\varphi} \,\mathrm{d}\gamma + \int_{\Omega(\boldsymbol{\phi})} (\zeta_{i\boldsymbol{\phi}} \cdot \boldsymbol{\varphi} + \zeta_i \boldsymbol{\nabla} \cdot \boldsymbol{\varphi}) \,\mathrm{d}x, \\ &\qquad (9.15.4)\end{aligned}$$

for $j \in \{\mathrm{C}, \mathrm{M}\}$, and $\beta_\mathrm{C} = 2\pi$ and $\beta_\mathrm{M} = \pi$ with respect to. Here, Eq. (9.15.3) is the shape derivative of f_i with respect to the variation of the singular point. u and v_i are given by

$$u\,(r, \theta) = k_j r^{\pi/\beta_j} \cos \frac{\pi}{\beta_j} \theta + u_\mathrm{R}, \qquad (9.15.5)$$

$$v_i\,(r, \theta) = l_{ij} r^{\pi/\beta_j} \cos \frac{\pi}{\beta_j} \theta + v_{i\mathrm{R}} \qquad (9.15.6)$$

using (r, θ) coordinate with $\boldsymbol{x}_j$ as the origin with respect to $j \in \{\mathrm{C}, \mathrm{M}\}$ as seen in Section 5.3. Here, k_j and l_{ij} are constants and u_R and $v_{i\mathrm{R}}$ are elements of $H^2\,(D; \mathbb{R})$. In this case, substitute Eq. (9.15.5) and Eq. (9.15.6) into Eq. (9.15.3) and obtain $\hat{\boldsymbol{g}}_{i\mathrm{C}}$ and $\hat{\boldsymbol{g}}_{i\mathrm{M}}$.

References

[1] Allaire, G., Jouve, F., and Toader, A. M. Structural optimization using sensitivity analysis and a level-set method. *J. Comput. Phys.*, 194:363–393, 2004.

[2] Azegami, H. A solution to domain optimization problems (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 60(574):1479–1486, 1994.

[3] Azegami, H. Regularized solution to shape optimization problem (in Japanese). *Transactions of the Japan Society for Industrial and Applied Mathematics*, 23(2):83–138, 2014.

[4] Azegami, H., Fukumoto, S., and Aoyama, T. Shape optimization of continua using NURBS as basis functions. *Structural and Multidisciplinary Optimization*, 47(2):247–258, 2013.

[5] Azegami, H., Ohtsuka, K., and Kimura, M. Shape derivative of cost function for singular point: Evaluation by the generalized J integral. *JSIAM Letters*, 6:29–32, 2014.

[6] Azegami, H., Sugai, Y., and Shimoda, M. Shape optimization with respect to buckling (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 66(647):1262–1267, 2000.

[7] Azegami, H. and Takeuchi, K. A smoothing method for shape optimization: Traction method using the Robin condition. *International Journal of Computational Methods*, 3(1):21–33, 2006.

[8] Azegami, H. and Wu, Z. C. Domain optimization analysis in linear elastic problems: Approach using traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 60(578):2312–2318, 1994.

[9] Azegami, H., Yokoyama, S., and Katamine, E. Solution to shape optimization problems of continua on thermal elastic deformation. In Tanaka, M. and Dulikravich, G. S., editors, *Inverse Problems in Engineering Mechanics III*, pages 61–66. Elsevier, Tokyo, 2002.

[10] Azegami, H., Zhou, L., Umemura, K., and Kondo, N. Shape optimization for a link mechanism. *Structural and Multidisciplinary Optimization*, 48(1):115–125, 2013.

[11] Banichuk, N. V. Optimality conditions and analytical methods of shape optimization. In Haug, E. J. and Cea, J., editors, *Optimization of Distributed Parameter Structures*, volume 2, pages 973–1004. Sijthoff & Noordhoff, Alphen aan den Rijn, 1981.

[12] Banichuk, N. V. *Problems and Methods of Optimal Structural Design.* Plenum Press, New York, 1983.

[13] Banichuk, N. V. *Introduction to Optimization of Structures.* Springer, New York, 1990.

[14] Belegundu, A. D. and Rajan, S. D. A shape optimization approach based on natural design varibles and shape functions. *Computer Methods in Applied Mechanics and Engineering*, 66:87–106, 1988.

[15] Boulkhemair, A., Chakib, A., and Nachaoui, A. Uniform trace theorem and application to shape optimization. *Applied and Computational Mathematics*, 7:192–205, 2008.

[16] Braibant, V. and Fleury, C. Shape optimal design using B-splines. *Computer Methods in Applied Mechanics and Engineering*, 44:247–267, 1984.

[17] Braibant, V. and Fleury, C. An approximation concepts approach to shape optimal design. *Computer Methods in Applied Mechanics and Engineering*, 53:119–148, 1985.

[18] Cea, J. Numerical methods of shape optimal design. In Haug, E. J. and Cea, J., editors, *Optimization of Distributed Parameter Structures*, volume 2, pages 1049–1088. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.

[19] Cea, J. Problems of shape optimization. In Haug, E. J. and Cea, J., editors, *Optimization of Distributed Parameter Structures*, volume 2, pages 1005–1048. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.

[20] Chenais, D. On the existence of a solution in a domain identification problem. *Journal of mathematical analysis and applications*, 52:189–219, 1975.

[21] Choi, K. K. Shape design sensitivity analysis of displacement and stress constraints. *Journal of Structural Mechanics*, 13:27–41, 1985.

[22] Choi, K. K. and Haug, E. J. Shape design sensitivity analysis of elastic structures. *Journal of Structural Mechanics*, 11:231–269, 1983.

[23] Choi, K. K. and Kim, N. H. *Structural Sensitivity Analysis and Optimization, 1 & 2.* Springer, New York, 2005.

[24] Ciarlet, P. G. *Three-Dimensional Elasticity.* North-Holland, Amsterdam; Tokyo, 1988.

[25] Delfour, M. C. and Zolésio, J. P. Tangent culculus and shape derivatives. In Cagnol, J., Polis, M. P., and Zolésio, J. P., editors, *Shape Optimization and Optimal Design: Proceedings of the IFIP Conference*, pages 37–60. Marcel Dekker, New York; Basel, 2001.

[26] Delfour, M. C. and Zolésio, J. P. *Shapes and Geometries: Metrics, Analysis, Differential Calculus, and Optimization, 2nd Ed.* Society for Industrial and Applied Mathematics, Philadelphia, 2011.

[27] Evans, L. C. and Gariepy, R. F. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992.

[28] Hadamard, J. *Mémoire des Savants Étragers. OEuvres de J. Hadamard*, chapter Mémoire sur le probléme d'analyse relatif á l'équilibre des plaques élastiques encastrées, Mémoire des savants étragers, pages 515–629. CNRS, Paris, 1968.

[29] Haslinger, J. and Mäkinen, R. A. E. *Introduction to Shape Optimization: Theory, Approximation, and Computation.* SIAM, Philadelphia, 2003.

[30] Haslinger, J. and Neittaanmäki, P. *Finite Element Approximation for Optimal Shape Design: Theory and Application.* John Wiley & Sons, Chichester, 1988.

[31] Haslinger, J. and Neittaanmäki, P. *Finite Element Approximation for Optimal Shape, Material and Topology Design, 2nd Ed.* John Wiley & Sons, Chichester, 1996.

[32] Haug, E. J., Choi, K. K., and Komkov, V. *Design Sensitivity Analysis of Structural Systems.* Academic Press, Orlando, 1986.

[33] Hecht, F. New development in freefem++. *Journal of Numerical Mathematics*, 20(3-4):251–265, 2012.

[34] Henrot, A. and Pierre, M. *Shape Variation and Optimization: A Geometrical Analysis.* Tracts in Mathematics 28. European Mathematical Society, Zürich, 2018.

[35] Horák, V. *Inverse Variational Principles of Continuum Mechanics.* Academia, nakladatelství Československé akademie véd, Praha, 1969.

[36] Ihara, H., Azegami, H., and Shimoda, M. Shape optimization for displacement path control problem considering geometrical non-linearity (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 67(656):611–617, 2001.

[37] Ihara, H., Azegami, H., Shimoda, M., and Watanabe, K. Solution to shape optimization problem considering material non-linearity (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 66(646):1111–1118, 2000.

[38] Ihara, H., Shimoda, M., Azegami, H., and Sakurai, T. Shape design by integrating shape optimization with topology optimization for multiobjective structures: an approach using homogenization method and traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series A*, 62(596):1091–1097, 1996.

[39] Ihara, H., Shimoda, M., Azegami, H., and Sakurai, T. Numerical analysis method of controlling deformation mode using topology-shape optimization technique (in Japanese). *Transactions of Society of Automotive Engineers of Japan*, 29(1):117–122, 1998.

[40] Imam, M. H. Three-dimensional shape optimization. *International Journal for Numerical Methods in Engineering*, 18:661–673, 1982.

[41] Inzarulfaisham, A. R. and Azegami, H. Shape optimization of linear elastic continua for moving nodes of natural vibration modes to assigned positions and its application to chassis-like frame structures. *Transactions of the Japan Society for Computational Engineering and Science*, 7:43–50, 2004.

[42] Inzarulfaisham, A. R. and Azegami, H. Solution to boundary shape optimization problem of linear elastic continua with prescribed natural vibration mode shapes. *Structural and Multidisciplinary Optimization*, 27(3):210–217, 2004.

[43] Iwai, T., Sugimoto, A., Aoyama, T., and Azegami, H. Shape optimization problem of elastic bodies for controlling contact pressure. *JSIAM Letters*, 2:1–4, 2010.

[44] Iwata, Y., Azegami, H., Aoyama, T., and Katamine, E. Numerical solution to shape optimization problems for non-stationary Navier-Stokes problems. *JSIAM Letters*, 2:37–40, 2010.

[45] Kaizu, S. and Azegami, H. Optimal shape problems and traction method (in Japanese). *Transactions of the Japan Society for Industrial and Applied Mathematics*, 16(3):277–290, 2006.

[46] Katamine, E. and Azegami, H. Solution to viscous flow field domain optimization problems: Approach by the traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 60(579):3859–3866, 1994.

[47] Katamine, E. and Azegami, H. Domain optimization analysis of potential flow field (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 61(581):103–108, 1995.

[48] Katamine, E. and Azegami, H. Domain optimization analysis of viscous flow field: In the case of considering convective term (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 61(585):1646–1653, 1995.

[49] Katamine, E., Azegami, H., and Hirai, M. Solution of shape identification problems on thermoelastic solids. *International Journal of Computational Methods*, 3(3):279–293, 2006.

[50] Katamine, E., Azegami, H., and Kojima, M. Boundary shape determination on steady-state heat conduction fields (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 65(629):275–281, 1999.

[51] Katamine, E., Azegami, H., and Matuura, Y. Solution to shape identification problem on unsteady heat-conduction fields (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 66(641):227–234, 2000.

[52] Katamine, E., Azegami, H., Tsubata, T., and Itoh, S. Solution to shape optimization problems of viscous flow fields. *International Journal of Computational Fluid Dynamics*, 19(1):45–51, 2005.

[53] Katamine, E., Azegami, H., and Yamaguchi, S. Shape identification analyses of potential flow field: Prescribed problems of pressure distribution and solution by the traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 64(620):1063–1070, 1998.

[54] Katamine, E., Iwata, Y., and Azegami, H. Shape optimization of un-steady heat-conduction fields for thermal dissipation maximization (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 74(743):1609–1616, 2008.

[55] Katamine, E., Kawase, Y., and Azegami, H. Shape optimization of forced heat-convection fields (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 73(733):1884–1891, 2007.

[56] Katamine, E., Nishimura, N., and Azegami, H. Shape optimization of steady-state viscous flow fields for drag minimization and lift maximization (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series B*, 74(748):2426–2434, 2008.

[57] Katamine, E., Tsubata, T., and Azegami, H. Solution to shape optimization problem of viscous flow fields considering convection term. In Tanaka, M., editor, *Inverse Problems in Engineering Mechanics IV*, pages 401–408. Elsevier, Tokyo, 2003.

[58] Katamine, E., Yoshioka, H., Matsuura, K., and Azegami, H. Shape optimization of thermoelastic fields for mean compliance minimization (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series C*, 77(783):4015–4023, 2011.

[59] Kimura, M. Shape derivative of minimum potential energy: abstract theory and applications. *Jindřich Nečas Center for Mathematical Modeling Lecture notes Volume IV, Topics in Mathematical Modeling*, pages 1–38, 2008.

[60] Kimura, M. and Wakano, I. New mathematical approach to the energy release rate in crack extension (in Japanese). *Transactions of the Japan Society for Industrial and Applied Mathematics*, 16(3):345–358, 2006.

[61] Lions, J. L. *Optimal Control of Systems Governed by Partial Differential Equations.* translated by S. K. Mitter. Springer, Berlin, 1971.

[62] McLean, W. *Strongly Elliptic Systems and Boundary Iintegral Equations.* Cambridge University Press, Cambridge, 2000.

[63] Meisters, G. H. and Olech, C. Locally one-to-one mappings and a classical theorem on schlicht functions. *Duke Mathematical Journal*, 30:63–80, 1963.

[64] Mohammadi, B. and Pironneau, O. *Applied Shape Optimization for Fluids.* Oxford University Press, Oxford, New York, 2001.

[65] Mohammadi, B. and Pironneau, O. *Applied Shape Optimization for Fluids, 2nd Edition.* Oxford University Press, Oxford, New York, 2010.

[66] Murai, D. *Error analyses in numerical solutions for initial and boundary value problems of partial differential equations and shape optimization problems (in Japanese).* PhD thesis, Nagoya University, 2011.

[67] Murai, D. and Azegami, H. Error analysis of H1 gradient method for shape-optimization problems of continua. *JSIAM Letters*, 5:29–32, 2013.

[68] Murat, F. and Simon, S. Etudes de problémes d'optimal design. In *Lecture Notes in Computer Science 41*, pages 54–62. Springer, Berlin, 1976.

[69] Ohtsuka, K. and Khludnev, A. Generalized J-integral method for sensitivity analysis of static shape design. *Control and Cybernetics*, 29:513–533, 2000.

[70] Ohtsuka, K. and Takaishi, T. *Finite Element Analysis using Mathematical Programming Language FreeFEM++ (in Japanese).* Kyoritsu Shuppan, Tokyo, 2014.

[71] Pironneau, O. On optimum profiles in Stokes flow. *Journal of Fluid Mechanics*, 59(1):117–128, 1973.

[72] Pironneau, O. On optimum design in fluid mechanics. *Journal of Fluid Mechanics*, 64(1):97–110, 1974.

[73] Pironneau, O. *Optimal Shape Design for Elliptic Systems*. Springer, New York, 1984.

[74] Polya, G. Torsion rigidity, principal frequency, electrostatic capacity and symmetrization. *Quarterly of Applied Mathematics*, 6:267–277, 1948.

[75] Raasch, I., Chargin, M. S., and Bruns, R. Optimierung von pkw-bauteilen in bezug auf form und dimensionierung. *VDI Berichte*, (699):713–748, 1988.

[76] Shimoda, M., Azegami, H., Ihara, H., and Sakurai, T. Shape optimization of linear elastic structures subject to multiple loading conditions: A traction method approach to minimum volume design (in Japanese). *Transactions of the JSME, Series A*, 61(587):1545–1552, 1995.

[77] Shimoda, M., Azegami, H., and Sakurai, T. Multiobjective shape optimization of linear elastic structures considering multiple loading conditions: Dealing with mean compliance minimization problems (in Japanese). *Transactions of the JSME Series A*, 61(582):359–366, 1995.

[78] Shimoda, M., Azegami, H., and Sakurai, T. Boundary shape determination of continua with desired stress distribution (in Japanese). *Transactions of the JSME, Series A*, 62(602):2393–2400, 1996.

[79] Shimoda, M., Azegami, H., and Sakurai, T. Shape determination of continua for homologous deformation (in Japanese). *Transactions of the JSME, Series A*, 62(604):2831–2837, 1996.

[80] Shimoda, M., Azegami, H., and Sakurai, T. Numerical solution for min-max problems in shape optimization: Minimum design of maximum stress and displacement (in Japanese). *Transactions of the JSME, Series A*, 63(607):610–617, 1997.

[81] Shimoda, M., Tsuji, J., and Azegami, H. Minimum weight shape design for the natural vibration problem of plate and shell structures. In Hernandez, S. and Brebbia, C. A., editors, *Computer Aided Optimum Design in Engineering IX*, pages 147–156. WIT Press, Southampton, UK, 2005.

[82] Shimoda, M., Tsuji, J., and Azegami, H. Non-parametric shape optimization method for thin-walled structures under strength criterion. In Hernandez, S. and Brebbia, C. A., editors, *Computer Aided Optimum Design in Engineering X*, pages 179–188. WIT Press, Southampton, UK, 2007.

[83] Shimoda, M., Wu, Z. C., Azegami, H., and Sakurai, T. Numerical method for domain optimization problems using a general purpose FEM code: Traction method approach (in Japanese). *Transactions of the JSME, Series A*, 60(578):2418–2425, 1994.

[84] Shintani, K., Nagatani, T., Ito, S., and Azegami, H. Examination of shape optimization for the nonlinear buckling phenomenon of suspension parts (in Japanese). *Transactions of the JSME, Series A*, 74(748):1187–1198, 2011.

[85] Simon, J. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis and Optimization*, 2(7-8):649–687, 1980.

[86] Simon, J. Second variations for domain optimization problems. *Control Theory of Distributed Parameter Systems and Applications*, 91:361–378, 1989.

[87] Sokolowski, J. and Zolésio, J. P. *Introduction to Shape Optimization: Shape Sensitivity Analysis.* Springer, New York, 1992.

[88] Tago, T., Aoki, T., and Azegami, H. Identification of building damage using vibrational eigenvalue and eigenmode pairs. *Japan Journal of Industrial and Applied Mathematics*, 32(2):297–313, 2015.

[89] Tautenhahn, U. On the asymptotical regularization of nonlinear ill-posed problems. *Inverse Problems*, 10:1405–1418, 1994.

[90] Vanderplaats, G. N. *Numerical Optimization Techniques for Engineering Design: With Applications.* McGraw-Hill, NewYork, 1984.

[91] Vanderplaats, G. N. and Miura, H. GENESIS-structural synthesis software using advanced approximation techniques. *AIAA Report (92-4839-CP)*, pages 180–190, 1992.

[92] Wang, M. Y., Wang, X., and Guo, D. A level set method for structural topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 192:227–246, 2003.

[93] Wu, Z. and Azegami, H. Domain optimization analysis in frequency response problems: Approach using traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series C*, 61(590):3968–3975, 1995.

[94] Wu, Z. and Azegami, H. Domain optimization analysis in natural vibration problems: mass minimization problems (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series C*, 61(587):2653–2696, 1995.

[95] Wu, Z., Sogabe, Y., and Azegami, H. Domain optimization analysis in frequency response problems considering proportional viscous damping (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series C*, 64(623):2618–2624, 1998.

[96] Wu, Z. C. and Azegami, H. Domain optimization analysis in natural vibration problems: approach using traction method (in Japanese). *Transactions of the Japan Society of Mechanical Engineers, Series C*, 61(583):930–937, 1995.

[97] Wu, Z. Q., Sogabe, Y., Arimitsu, Y., and Azegami, H. Shape optimization of transient response problems. In Tanaka, M. and Dulikravich, G. S., editors, *Inverse Problems in Engineering Mechanics II*, pages 285–294. Elsevier, Tokyo, 2000.

[98] Wu, Z. Q., Sogabe, Y., and Azegami, H. Shape optimization analysis for frequency response problems of solids with proportional viscous damping. *Key Engineering Materials*, 145-149:227–232, 1997.

[99] Yamada, T., Izui, K., Nishiwaki, S., and Takezawa, A. A topology optimization method based on the level set method incorporating a fictitious interface energy. *Computer Methods in Applied Mechanics and Engineering*, 199:2876–2891, 2010.

[100] Zolésio, J. P. Domain variational formulation for free boundary problems. In Haug, E. J. and Cea, J., editors, *Optimization of Distributed Parameter Structures*, volume 2, pages 1152–1194. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.

[101] Zolésio, J. P. The material derivative (or speed) method for shape optimization. In Haug, E. J. and Cea, J., editors, *Optimization of Distributed Parameter Structures*, volume 2, pages 1089–1151. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.