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Chapter 5

Boundary Value Problems of Partial Differential Equations

As seen in Chap. 1, optimal design problems are optimization problems whose state equations are considered as equality constraints. In Chap. 1, we have considered design variables and state variables as elements of a finite-dimensional vector space. However, in this book, our main interest focuses on the shape optimization problem of continuum. In this case, boundary value problems of partial differential equations, such as linear elastic bodies and Stokes flow field, are included in the equality constraints as state equations.

In this chapter, the definitions and the results of functional analyses discussed in Chap. 4 are used to study the methods of expressing boundary value problems of elliptic partial differential equation in their corresponding variational form (here on referred to as the weak form) as well as theorems relating to the existence of unique solutions. This weak form is not only used when considering methods in numerical analysis with respect to boundary value problems of elliptic partial differential equations shown in Chap. 6, but also as Lagrange functions with respect to shape and topology optimization problems where boundary value problems are included in the equality constraint (see Chaps. 8 and 9).

5.1 Poisson Problem

Let us consider a Poisson problem as a simple example of a boundary value problem of an elliptic partial differential equation (Definition A.7.1) and look at its definition and the process of transforming the system in its weak form. A Poisson problem can be thought of, for instance, as a situation when thermal conductivity is 1 in a stationary heat conduction problem (Section A.6).

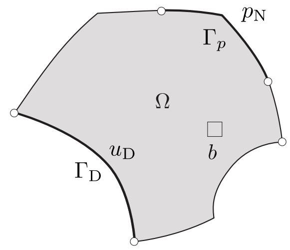


Fig. 5.1: Domain Ω and its boundary $\partial\Omega = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$.

Let the domain Ω be a Lipschitz domain (Section A.5) of $d \in \{2, 3\}$ dimensions such as in Fig. 5.1. Let Γ_{D} be a partial open set of $\partial\Omega$, the boundary of Ω at which temperature is given in the heat conductivity problem. The remaining boundary $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \bar{\Gamma}_{\mathrm{D}}$ is taken to be the boundary at which heat flux is given. Furthermore, let $\Gamma_p \subset \Gamma_{\mathrm{N}}$ be the boundary at which the heat flow is non-zero. In this chapter Γ_p and $\Gamma_{\mathrm{N}} \setminus \bar{\Gamma}_p$ are not distinguished from one another but they will be considered separately in Chap. 9. Moreover, $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ expresses the Laplace operator. Meanwhile, $\boldsymbol{\nu}$ expresses the outward unit normal vector defined on the boundary (Definition A.5.4) and $\partial_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\nabla}$. In this case, a Poisson problem with mixed boundary conditions is defined as follows.

Problem 5.1.1 (Poisson problem) Let the functions $b : \Omega \to \mathbb{R}$, $p_{\mathrm{N}} : \Gamma_{\mathrm{N}} \to \mathbb{R}$, and $u_{\mathrm{D}} : \Omega \to \mathbb{R}$ be given. Find the function $u : \Omega \to \mathbb{R}$ such that the system

$$-\Delta u = b \quad \text{in } \Omega, \tag{5.1.1}$$
$$\partial_\nu u = p_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{5.1.2}$$
$$u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{5.1.3}$$

is satisfied. □

In Problem 5.1.1, Eq. (5.1.1) is called a Poisson equation. Moreover, when $b = 0$, it is called a Laplace equation or homogeneous Poisson equation. The problem in that case (Problem 5.1.1) is called a Laplace problem.

Moreover, the boundary condition of Eq. (5.1.3) expresses the relationship established by the traces on Γ_{D} of the function u and u_{D} defined on Ω. In this situation, there is a need to define an appropriate function space of u and u_{D} such that a trace can be taken. On the other hand, $\partial_\nu u$ of Eq. (5.1.2) shows the relationship of a trace on Γ_{N}. In order for this relationship to have a meaning, assumptions such that trace on the boundary of $\boldsymbol{\nabla} u$ can be taken have to be specified. However, as shown below, if Problem 5.1.1 is changed to an integral equation (weak form), it should be noted that such an assumption becomes unnecessary.

From the above considerations, u_{D} is assumed to be an element of $H^1(\Omega; \mathbb{R})$

and the set of functions satisfying Eq. (5.1.3) is taken to be

$$U(u_{\mathrm{D}}) = \left\{ v \in H^1(\Omega;\mathbb{R}) \;\middle|\; v = u_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}} \right\}.$$

As seen in Section 4.6, $U(u_{\mathrm{D}})$ is an affine subspace of the Hilbert space

$$U = \left\{ v \in H^1(\Omega;\mathbb{R}) \middle| \; v = 0 \text{ on } \Gamma_{\mathrm{D}} \right\}. \tag{5.1.4}$$

The fact that U is a Hilbert space will be needed when fitting the Poisson problem into the framework of abstract variational problem later on.

If both sides of Eq. (5.1.1) are multiplied by an arbitrary $v \in U$ and integrated over Ω, the Gauss–Green theorem (Theorem A.8.2) can be employed to establish

$$-\int_{\Omega} \Delta u v \,\mathrm{d}x = \int_{\Omega} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} \partial_{\nu} u v \,\mathrm{d}\gamma = \int_{\Omega} b v \,\mathrm{d}x, \tag{5.1.5}$$

where the fact that $v = 0$ on Γ_{D} was used. On the other hand, if an arbitrary $v \in U$ is multiplied to both sides of Eq. (5.1.2) and integrated over Γ_{N}, then the equation

$$\int_{\Gamma_{\mathrm{N}}} \partial_{\nu} u v \,\mathrm{d}\gamma = \int_{\Gamma_{\mathrm{N}}} p_{\mathrm{N}} v \,\mathrm{d}\gamma \tag{5.1.6}$$

is established. Here, substituting Eq. (5.1.6) into Eq. (5.1.5) gives

$$\int_{\Omega} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x = \int_{\Omega} b v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} p_{\mathrm{N}} v \,\mathrm{d}\gamma. \tag{5.1.7}$$

Equation (5.1.7) is called a weak form of a Poisson problem.

Let us mention in advance the fact that the arbitrary function $v \in U$ used when obtaining the weak-form equation will be used as a Lagrange multiplier with respect to a boundary value problem when considering a shape or topology optimization problem in which a boundary value problem is included in the equality constraints.

Furthermore, the left-hand side of Eq. (5.1.7) has the property of being bilinear with respect to u and v. Moreover, the right-hand side of Eq. (5.1.7) is linear with respect to v. Here, as was seen in Section 4.6 we define

$$a(u, v) = \int_{\Omega} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x, \tag{5.1.8}$$

$$l(v) = \int_{\Omega} b v \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} p_{\mathrm{N}} v \,\mathrm{d}\gamma. \tag{5.1.9}$$

Using these definitions, the weak form of Problem 5.1.1 is given as follows.

Problem 5.1.2 (Weak form of Poisson problem) Let U be defined as in Eq. (5.1.4) and the consider the functions $b \in L^2(\Omega;\mathbb{R})$, $p_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}};\mathbb{R})$ and

$u_{\mathrm{D}} \in H^1(\Omega; \mathbb{R})$. Moreover, let $a(\cdot, \cdot)$ and $l(\cdot)$ be given by Eq. (5.1.8) and Eq. (5.1.9), respectively. Find u such that $\tilde{u} = u - u_{\mathrm{D}} \in U$ satisfying

$$a(u, v) = l(v), \tag{5.1.10}$$

for all $v \in U$. □

Here, let us compare Problem 5.1.1 with Problem 5.1.2. In Problem 5.1.1, in order for Eq. (5.1.1) to have meaning, u needs to be second-order differentiable. Moreover, $\partial_\nu u$ needs to be defined on Γ_{N}. On the other hand, in Problem 5.1.2, there is no need for u to be second-order differentiable, instead in order for the integral of Eq. (5.1.8) to be defined, there is a need for the first-order derivative of both u and v to be square integrable. In this way, depending on the conditions that the solutions should satisfy, Problem 5.1.1 is referred to as the strong form of Poisson problem and Problem 5.1.2 as the weak form of Poisson problem. Moreover, the solution u of Problem 5.1.2 is called the weak solution. Furthermore, as shown in Sect. 5.2, the fact that a unique solution exists is guaranteed by the weak solution.

The following terminology is used in a boundary value problem of a differential equation:

- Equation (5.1.3) is called a Dirichlet condition or fundamental boundary condition or first-type boundary condition. The boundary for which the Dirichlet condition is given is called a Dirichlet boundary. Problem 5.1.1 or Problem 5.1.2 for which this condition is given over the entire boundary is called a Dirichlet problem.
- Equation (5.1.2) can also be called a Neumann condition or natural boundary condition or second-type boundary condition. The boundary with Neumann condition is called a Neumann boundary. Problem 5.1.1 or Problem 5.1.2 with this condition given over the entire boundary is called a Neumann problem. However, there is a need to note that a Neumann problem does not have a unique solution (Exercise 5.2.6).
- When both the Dirichlet condition and Neumann condition exist, it is referred to as mixed boundary value problem.
- For the Dirichlet condition or Neumann condition, if $u_{\mathrm{D}} = 0$ or $p_{\mathrm{N}} = 0$ respectively, it is called a homogeneous type. When $u_{\mathrm{D}} \neq 0$ or $p_{\mathrm{N}} \neq 0$, it is called an inhomogeneous type.

5.1.1 Extended Poisson Problem

Let us consider an extended Poisson problem. This problem is used when specifying an abstract gradient method in Chap. 8. Moreover, a linear elastic problem extended in a similar manner to that shown here will be used when specifying an abstract gradient method in Chap. 9 too.

We use the symbols used in Problem 5.1.1 to extend the Poisson problem in the following way.

Problem 5.1.3 (Extended Poisson problem) Let the functions $b : \Omega \to \mathbb{R}$, $c_\Omega : \Omega \to \mathbb{R}$, $p_\mathrm{R} : \partial\Omega \to \mathbb{R}$ and $c_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ be given. Find the function $u : \Omega \to \mathbb{R}$ satisfying

$$-\Delta u + c_\Omega u = b \quad \text{in } \Omega, \tag{5.1.11}$$
$$\partial_\nu u + c_{\partial\Omega} u = p_\mathrm{R} \quad \text{on } \partial\Omega. \tag{5.1.12}$$

□

In Problem 5.1.3, Eq. (5.1.12) is called a Robin condition or third-type boundary condition. In reference to Problem 5.1.3, the problem when this condition is given across the entire boundary is called a Robin problem.

The weak form of Problem 5.1.3 can be obtained in the following way. Here

$$U = H^1(\Omega; \mathbb{R}) \tag{5.1.13}$$

is set. Multiplying both sides of Eq. (5.1.11) by an arbitrary $v \in U$ and integrating over Ω, then using the Gauss–Green theorem (Theorem A.8.2) gives

$$\begin{aligned}\int_\Omega (-\Delta u + c_\Omega u)\, v \;\mathrm{d}x &= \int_\Omega (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + c_\Omega u v)\,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu u v \;\mathrm{d}\gamma \\ &= \int_\Omega b v \;\mathrm{d}x.\end{aligned} \tag{5.1.14}$$

On the other hand, if both sides of Eq. (5.1.12) are multiplied by an arbitrary $v \in U$ and integrated over $\partial\Omega$, the equality

$$\int_{\partial\Omega} \partial_\nu u v \;\mathrm{d}\gamma = \int_{\partial\Omega} (p_\mathrm{R} - c_{\partial\Omega} u)\, v \;\mathrm{d}\gamma \tag{5.1.15}$$

holds. Here, if Eq. (5.1.15) is substituted into Eq. (5.1.14), the equation

$$\int_\Omega (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + c_\Omega u v)\,\mathrm{d}x + \int_{\partial\Omega} c_{\partial\Omega} u v \;\mathrm{d}\gamma = \int_\Omega b v \;\mathrm{d}x + \int_{\partial\Omega} p_\mathrm{R} v \;\mathrm{d}\gamma \tag{5.1.16}$$

is obtained. Eq. (5.1.16) is the weak form of Problem 5.1.3.

Here, note that the left-hand side of Eq. (5.1.16) is bilinear with respect to u and v and the right-hand side is linear with respect to v. Let $a : U \times U \to \mathbb{R}$ and $l : U \to \mathbb{R}$ be

$$a(u, v) = \int_\Omega (\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + c_\Omega u v)\,\mathrm{d}x + \int_{\partial\Omega} c_{\partial\Omega} u v \;\mathrm{d}\gamma, \tag{5.1.17}$$
$$l(v) = \int_\Omega b v \;\mathrm{d}x + \int_{\partial\Omega} p_\mathrm{R} v \;\mathrm{d}\gamma. \tag{5.1.18}$$

Here, the weak form of Problem 5.1.3 becomes as follows.

Problem 5.1.4 (Weak form of extended Poisson problem) Let U be Eq. (5.1.13) and the functions $b \in L^2(\Omega;\mathbb{R})$, $c_\Omega \in L^\infty(\Omega;\mathbb{R})$, $p_\mathrm{R} \in L^2(\partial\Omega;\mathbb{R})$, $c_{\partial\Omega} \in L^\infty(\partial\Omega;\mathbb{R})$. Moreover, let $a(\cdot,\cdot)$ and $l(\cdot)$ be given by Eq. (5.1.17) and Eq. (5.1.18), respectively. In this case, obtain a $u \in U$ which satisfies

$$a(u,v) = l(v) \tag{5.1.19}$$

with respect to an arbitrary $v \in U$. □

5.2 Abstract Variational Problem

In Sections 5.1 and 5.1.1, the weak forms of the Poisson problem and extended Poisson problem were shown as Eq. (5.1.10) and Eq. (5.1.19), respectively. These are classified as boundary value problems of elliptic partial differential equations based on classification of linear second-order partial differential equations (Definition A.7.1). If it is a boundary value problem of an elliptic partial differential equation, it can be expected that either of the weak forms can be expressed using a bilinear form a and linear form l. Hence, let us define an abstract variational problem which abstracts the weak form of elliptic partial differential equation and investigate the existence of a unique solution to such a problem.

In this section, U is taken to be a real Hilbert space. Let us define two characteristics with respect to a bilinear form on U (Section 4.4).

Definition 5.2.1 (Coercive bilinear form on real Hilbert space) Let $a: U \times U \to \mathbb{R}$ be a bilinear form on U. If some constant $\alpha > 0$ exists with respect to an arbitrary $v \in U$ and

$$a(v,v) \geq \alpha \|v\|_U^2$$

holds, a is referred to as coercive or elliptic. □

If U is $\mathbb{R}^d$, the bilinear form equation with respect to $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ can be written as $a(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{y})$. Here, $\boldsymbol{A}$ is a matrix of $\mathbb{R}^{d\times d}$. When $\boldsymbol{A} = \boldsymbol{A}^\top$, coerciveness of a is equivalent to $\boldsymbol{A}$ being positive definite.

Definition 5.2.2 (Boundedness of bilinear form on real Hilbert space) Let $a: U \times U \to \mathbb{R}$ be a bilinear form on U. If there exists some $\beta > 0$ with respect to an arbitrary $u, v \in U$ and

$$|a(u,v)| \leq \beta \|u\|_U \|v\|_U$$

holds, a is said to be bounded. □

If $U = \mathbb{R}^d$, the boundedness of the bilinear form $a(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{y})$ becomes equivalent to the norm of matrix $\boldsymbol{A}$ being bounded (see Eq. (4.4.3)).

Let us consider the next problem using the definitions above.

Problem 5.2.3 (Abstract variational problem) Let $a: U \times U \to \mathbb{R}$ be a bilinear form on U and $l = l(\,\cdot\,) = \langle l, \cdot \rangle \in U'$. In this case, obtain a $u \in U$ which satisfies

$$a(u, v) = l(v)$$

with respect to an arbitrary $v \in U$. □

Let $U = \mathbb{R}^d$. If the matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ in the bilinear form $a(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{y})$ and $\boldsymbol{b} \in \mathbb{R}^d$ are given, an abstract variational problem becomes a problem seeking $\boldsymbol{x} \in \mathbb{R}^d$ which satisfies

$$(\boldsymbol{A}\boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{b} \cdot \boldsymbol{y} \tag{5.2.1}$$

with respect to an arbitrary $\boldsymbol{y} \in \mathbb{R}^d$.

5.2.1 Lax–Milgram Theorem

The fact that there exists a unique solution to Problem 5.2.3 is guaranteed by the Lax–Milgram theorem. In this theorem, it is assumed that a bilinear form a is coercive and bounded. Since these characteristics are the same as the definition of an inner product in Hilbert spaces, this theorem is proven using Riesz's representation theorem (Theorem 4.4.17) (cf. [2, Theorem 1.3, p. 29], [3, Theorem 1, p. 297], [9, Theorem 2.6, p. 48]).

Theorem 5.2.4 (Lax–Milgram theorem) In Problem 5.2.3, let a be coercive and bounded. Moreover, let $l \in U'$. In this case, there is a unique solution $u \in U$ for Problem 5.2.3 and

$$\|u\|_U \leq \frac{1}{\alpha} \|l\|_{U'}$$

holds with respect to α used in Definition 5.2.1. □

If $U = \mathbb{R}^d$, assuming $\boldsymbol{A}$ is symmetric bounded and positive definite, there exists an inverse matrix to $\boldsymbol{A}$ and $\boldsymbol{x}$ satisfying Eq. (5.2.1) becomes

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}. \tag{5.2.2}$$

Here, the inequality

$$\|\boldsymbol{x}\|_{\mathbb{R}^d} \leq \frac{1}{\alpha} \|\boldsymbol{b}\|_{\mathbb{R}^d}$$

holds, where α is the minimum eigenvalue of $\boldsymbol{A}$. Moreover, when $\boldsymbol{A}$ is asymmetric, the positive definiteness of $\boldsymbol{A}$ is replaced with that of $\left(\boldsymbol{A}^\top + \boldsymbol{A}\right)/2$, because $\boldsymbol{x} \cdot \left\{\left(\boldsymbol{A}^\top + \boldsymbol{A}\right)\boldsymbol{x}\right\} \geq 2\alpha \|\boldsymbol{x}\|_{\mathbb{R}^d}^2$ holds if $\boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{x}) \geq \alpha \|\boldsymbol{x}\|_{\mathbb{R}^d}^2$ with respect to an arbitrary $\boldsymbol{x} \in \mathbb{R}^d$.

Next, let us show the existence of a unique solution to the Poisson problem using the Lax–Milgram theorem.

Exercise 5.2.5 (Existence of unique solution to Poisson problem) In Problem 5.1.2, when $|\Gamma_{\mathrm{D}}|$ $(= \int_{\Gamma_{\mathrm{D}}} \mathrm{d}\gamma)$ is positive, show that there exists a unique solution $\tilde{u} = u - u_{\mathrm{D}} \in U$. □

Answer The assumptions of the Lax–Milgram theorem with respect to Problem 5.1.2 need to be shown. Consider the Hilbert space $U = \left\{ u \in H^1(\Omega;\mathbb{R}) \middle|\ u = 0 \text{ on } \Gamma_{\mathrm{D}} \right\}$. Moreover, if we let

$$\hat{l}(v) = l(v) - a(u_{\mathrm{D}}, v), \tag{5.2.3}$$

Problem 5.1.2 can be written as a problem seeking $\tilde{u} = u - u_{\mathrm{D}} \in U$ which satisfies

$$a(\tilde{u}, v) = \hat{l}(v),$$

with respect to an arbitrary $v \in U$. In view of these relationships, the assumptions of the Lax–Milgram theorem hold in the following ways:

(1) a is coercive. In fact,

$$a(v,v) = \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} v \,\mathrm{d}x = \|\boldsymbol{\nabla} v\|^2_{L^2(\Omega;\mathbb{R}^d)} \geq \frac{1}{c^2} \|v\|^2_{H^1(\Omega;\mathbb{R})}$$

holds because of Poincaré's inequality (Corollary A.9.4). If we let $1/c^2$ be α, from Definition 5.2.1, a is coercive.

(2) a is bounded. In fact, using Hölder's inequality (Theorem A.9.1), the inequality

$$\begin{aligned} |a(u,v)| &= \left| \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x \right| \leq \|\boldsymbol{\nabla} u\|_{L^2(\Omega;\mathbb{R}^d)} \|\boldsymbol{\nabla} v\|_{L^2(\Omega;\mathbb{R}^d)} \\ &\leq \|u\|_{H^1(\Omega;\mathbb{R})} \|v\|_{H^1(\Omega;\mathbb{R})} \end{aligned}$$

is established. This relationship shows that it holds when $\beta = 1$ in Definition 5.2.2.

(3) $\hat{l} \in U'$. In fact, from the fact that $\partial\Omega$ assumes a Lipschitz boundary, the norm of the trace operator (Theorem 4.4.2)

$$\|\gamma\|_{\mathcal{L}\left(H^1(\Omega;\mathbb{R}); H^{1/2}(\partial\Omega;\mathbb{R})\right)} = \sup_{v \in H^1(\Omega;\mathbb{R}) \setminus \{0_{H^1(\Omega;\mathbb{R})}\}} \frac{\|v\|_{H^{1/2}(\partial\Omega;\mathbb{R})}}{\|v\|_{H^1(\Omega;\mathbb{R})}}. \tag{5.2.4}$$

is bounded. This is set as $c_1 > 0$. Moreover, the inequalities

$$\begin{aligned} \left|\hat{l}(v)\right| &\leq \int_\Omega |bv| \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} |p_{\mathrm{N}} v| \,\mathrm{d}\gamma + \int_\Omega |\boldsymbol{\nabla} u_{\mathrm{D}} \cdot \boldsymbol{\nabla} v| \,\mathrm{d}x \\ &\leq \|b\|_{L^2(\Omega;\mathbb{R})} \|v\|_{L^2(\Omega;\mathbb{R})} + \|p_{\mathrm{N}}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R})} \|v\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R})} \\ &\quad + \|\boldsymbol{\nabla} u_{\mathrm{D}}\|_{L^2(\Omega;\mathbb{R}^d)} \|\boldsymbol{\nabla} v\|_{L^2(\Omega;\mathbb{R}^d)} \\ &\leq \left(\|b\|_{L^2(\Omega;\mathbb{R})} + c_1 \|p_{\mathrm{N}}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R})} + \|u_{\mathrm{D}}\|_{H^1(\Omega;\mathbb{R})} \right) \|v\|_{H^1(\Omega;\mathbb{R})} \end{aligned}$$

are established if Hölder's inequality is used. In Problem 5.1.2, $b \in L^2(\Omega;\mathbb{R})$, $p_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}};\mathbb{R})$ and $u_{\mathrm{D}} \in H^1(\Omega;\mathbb{R})$ were assumed. Thus, the right-hand side of $(\,\cdot\,)$ is bounded and l is a bounded linear functional on U.

Therefore, $\tilde{u} = u - u_{\mathrm{D}} \in U$ exists uniquely. □

Moreover, if the Lax–Milgram theorem is applied with respect to the Neumann problem, we get the following.

Exercise 5.2.6 (Indeterminateness of solution to Neumann problem) If $|\Gamma_{\mathrm{D}}| = 0$ in Problem 5.1.2, show that there does not exist a unique $u \in U$ which satisfies Problem 5.1.2. Moreover, show how the problem needs to be amended in order to guarantee the existence of a unique solution. □

Answer In the solution to Exercise 5.2.5, to show the coercivity of a, a corollary of Poincaré's inequality (Corollary A.9.4) was used from the fact that $|\Gamma_{\mathrm{D}}| > 0$ was assumed. However, in Neumann problems $|\Gamma_{\mathrm{D}}| = 0$, hence the corollary of Poincaré's inequality cannot be used and so a cannot be said to be coercive. Hence, the Lax–Milgram theorem cannot be used. However, if we let

$$u_{\mathrm{D}} = \frac{1}{|\Omega|} \int_{\Omega} u \, \mathrm{d}x \tag{5.2.5}$$

and use Poincaré's inequality (Theorem A.9.3), the inequality

$$a(v, v) = \int_{\Omega} \nabla v \cdot \nabla v \, \mathrm{d}x = \|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^d)} \geq \frac{1}{c^2} \|v - u_{\mathrm{D}}\|^2_{L^2(\Omega;\mathbb{R}^d)}$$

holds and a becomes coercive. Therefore, there is the existence of a unique solution if the Neumann problem is rewritten as a problem seeking $\tilde{u} = u - u_{\mathrm{D}} \in U$ with respect to u_{D} satisfying Eq. (5.2.5). □

From the result of Exercise 5.2.6, the solution to the Neumann problem is said to have uncertainty of constant.

Furthermore, the following assumptions are needed in order to guarantee the existence of a unique solution with respect to an extended Poisson problem (Problem 5.1.3).

Exercise 5.2.7 (Existence of solution to extended Poisson problem) In Problem 5.1.4, one of the following is assumed to hold:

(1) $c_{\Omega} \in L^{\infty}(\Omega; \mathbb{R})$ takes a positive value on most of Ω.

(2) $c_{\partial\Omega} \in L^{\infty}(\partial\Omega; \mathbb{R})$ takes a positive value over most of $\partial\Omega$.

In this case, show that there exists a unique solution $u \in U$ of Problem 5.1.4. □

Answer The assumptions of the Lax–Milgram theorem with respect to Problem 5.1.4 need to be verified. Let $U = H^1(\Omega; \mathbb{R})$ be a Hilbert space. Furthermore, the following holds:

(1) a is coercive. In fact, from the assumption, $\operatorname{ess\,inf}_{\boldsymbol{x}\in\Omega} c_{\Omega}(\boldsymbol{x})$ and $\operatorname{ess\,inf}_{\boldsymbol{x}\in\partial\Omega} c_{\partial\Omega}(\boldsymbol{x})$ are set as $c_1 > 0$ and $c_2 > 0$, respectively and the norm of the inverse operator of the trace operator $\gamma : H^1(\Omega; \mathbb{R}) \to L^2(\partial\Omega; \mathbb{R})$:

$$\left\|\gamma^{-1}\right\|_{\mathcal{L}\left(L^2(\partial\Omega;\mathbb{R});H^1(\Omega;\mathbb{R})\right)} = \sup_{v\in L^2(\partial\Omega;\mathbb{R})\setminus\{0_{L^2(\partial\Omega;\mathbb{R})}\}} \frac{\|v\|_{H^1(\Omega;\mathbb{R})}}{\|v\|_{L^2(\partial\Omega;\mathbb{R})}}. \tag{5.2.6}$$

is set to be $c_3 > 0$,

$$\begin{aligned} a(v,v) &\geq \|\boldsymbol{\nabla} v\|^2_{L^2\left(\Omega;\mathbb{R}^d\right)} + c_1 \|v\|^2_{L^2(\Omega;\mathbb{R})} + c_2 \|v\|^2_{L^2(\partial\Omega;\mathbb{R})} \\ &\geq \left(\min\{1, c_1\} + \frac{c_2}{c_3^2}\right) \|v\|^2_{H^1(\Omega;\mathbb{R})} \end{aligned}$$

holds. If $(\,\cdot\,)$ of the right-hand side is set to be α, a becomes coercive from Definition 5.2.1.

(2) a is bounded. In fact, when the norm $\|\gamma\|_{\mathcal{L}\left(H^1(\Omega;\mathbb{R}); H^{1/2}(\partial\Omega;\mathbb{R})\right)}$ of the trace operator of Eq. (5.2.4) is set to be c_4,

$$\begin{aligned} |a(u,v)| \leq{}& \|\boldsymbol{\nabla} u\|_{L^2\left(\Omega;\mathbb{R}^d\right)} \|\boldsymbol{\nabla} v\|_{L^2\left(\Omega;\mathbb{R}^d\right)} \\ &+ \|c_\Omega\|_{L^\infty(\Omega;\mathbb{R})} \|u\|_{L^2(\Omega;\mathbb{R})} \|v\|_{L^2(\Omega;\mathbb{R})} \\ &+ \|c_{\partial\Omega}\|_{L^\infty(\partial\Omega;\mathbb{R})} \|u\|_{L^2(\partial\Omega;\mathbb{R})} \|v\|_{L^2(\partial\Omega;\mathbb{R})} \\ \leq{}& \left(1 + \|c_\Omega\|_{L^\infty(\Omega;\mathbb{R})} + c_4^2 \|c_{\partial\Omega}\|_{L^\infty(\partial\Omega;\mathbb{R})}\right) \|u\|_{H^1\left(\Omega;\mathbb{R}^d\right)} \|v\|_{H^1\left(\Omega;\mathbb{R}^d\right)} \end{aligned}$$

is established from $c_\Omega \in L^\infty(\Omega;\mathbb{R})$ and $c_{\partial\Omega} \in L^\infty(\partial\Omega;\mathbb{R})$. If $(\,\cdot\,)$ of the right-hand side is set to be β, a becomes bounded from Definition 5.2.2.

(3) $l \in U'$. In fact, when the norm $\|\gamma\|_{\mathcal{L}\left(H^1(\Omega;\mathbb{R}); H^{1/2}(\partial\Omega;\mathbb{R})\right)}$ of the trace operator of Eq. (5.2.4) is set to be c_4, the inequality

$$\begin{aligned} |l(v)| &\leq \int_\Omega |bv| \;\mathrm{d}x + \int_{\partial\Omega} |p_\mathrm{R} v| \;\mathrm{d}\gamma \\ &\leq \|b\|_{L^2(\Omega;\mathbb{R})} \|v\|_{L^2(\Omega;\mathbb{R})} + \|p_\mathrm{R}\|_{L^2(\partial\Omega;\mathbb{R})} \|v\|_{L^2(\partial\Omega;\mathbb{R})} \\ &\leq \left(\|b\|_{L^2(\Omega;\mathbb{R})} + c_4 \|p_\mathrm{R}\|_{L^2(\partial\Omega;\mathbb{R})}\right) \|v\|_{H^1(\Omega;\mathbb{R})} \end{aligned}$$

is established. In Problem 5.1.4, since $b \in L^2(\Omega;\mathbb{R})$ and $p_\mathrm{R} \in L^2(\partial\Omega;\mathbb{R})$ in $(\,\cdot\,)$ of the right-hand side become bounded, then l becomes a bound linear functional on U.

Therefore, from the Lax–Milgram theorem, $u \in U$ exists uniquely. □

5.2.2 Abstract Minimization Problem

In an abstract variational problem (Problem 5.2.3), if $a: U \times U \to \mathbb{R}$ is symmetric, the abstract variational problem is shown to be equivalent to the abstract minimization problem. Let us confirm that in this section.

Let U be a real Hilbert space and $a: U \times U \to \mathbb{R}$ be a bilinear form on U. If for arbitrary $u, v \in U$,

$$a(u,v) = a(v,u)$$

holds, a is called symmetric.

If U is $\mathbb{R}^d$ and with respect to $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $a(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{y})$, a being symmetric is equivalent to the matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ being symmetric $\boldsymbol{A} = \boldsymbol{A}^\top$.

The following problem is called an abstract minimization problem.

Problem 5.2.8 (Abstract minimization problem) Let $a: U \times U \to \mathbb{R}$ be a bilinear form on U, $l = l(\,\cdot\,) = \langle l, \cdot\rangle \in U'$ and $f: U \to \mathbb{R}$. In this case, obtain $u \in U$ such that

$$\min_{u \in U} \left\{ f(u) = \frac{1}{2} a(u, u) - l(u) \right\}.$$

□

If $U = \mathbb{R}^d$, it becomes a problem seeking $\boldsymbol{x} \in \mathbb{R}^d$ satisfying

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \left\{ f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x} \cdot (\boldsymbol{A}\boldsymbol{x}) - \boldsymbol{b} \cdot \boldsymbol{x} \right\}. \tag{5.2.7}$$

The following results can be obtained with respect to Problem 5.2.8 (cf. [2, Theorem 1.1, p. 24], [6, Theorem 2.1, p. 33], [9, Theorem 2.7, p. 50]).

Theorem 5.2.9 (Solution to abstract minimization problem) Consider Problem 5.2.8 and let a be coercive, bounded and symmetric. In this case, with respect to an arbitrary $l \in U'$, $u \in U$ satisfying Problem 5.2.8 exists uniquely and agrees with the solution to Problem 5.2.3. □

If $U = \mathbb{R}^d$ and $\boldsymbol{A}$ is bounded, positive definite and symmetric, $\boldsymbol{x} \in \mathbb{R}^d$ satisfying Eq. (5.2.7) is the same as Eq. (5.2.2).

If a is symmetric in the weak form of the Poisson problem (Problem 5.1.2), then Problem 5.1.2 is equivalent to the following problem in view of the solution of Exercise 5.2.5 and Theorem 5.2.9.

Problem 5.2.10 (Minimization problem of Poisson problem) Let a and $\hat{l}$ be Eq. (5.1.8) and Eq. (5.2.3), respectively. In this case, obtain $\tilde{u} = u - u_{\mathrm{D}} \in U$ which satisfies

$$\min_{\tilde{u} \in U} \left\{ f(\tilde{u}) = \frac{1}{2} a(\tilde{u}, \tilde{u}) - \hat{l}(\tilde{u}) \right\}.$$

□

5.3 Regularity of Solutions

The Poisson problem is an abstract variational problem and we have seen how the existence of a unique solution can be guaranteed by the Lax–Milgram theorem. In this case, if $\hat{l}$ of Eq. (5.2.3) constructed form the given functions b, p, u_{D} in the Poisson problem (Problem 5.1.1) is in U'. This shows that the solution $u - u_{\mathrm{D}}$ of the Poisson problem exists in $U = \left\{ u \in H^1(\Omega; \mathbb{R}) \middle|\ u = 0 \text{ on } \Gamma_{\mathrm{D}} \right\}$. However, this condition is necessary for the existence of a solution. So, even if smoother given functions are assumed, it is expected that the solution to the Poisson problem will be correspondingly smooth. In Chaps. 8 and 9 smoothness greater than H^1 class is needed with respect to the solution to a boundary value problem. Here let us examine this notion of smoothness.

In this book, the smoothness of a function represents the order of differentiability and the exponent of integrability for the function. They are referred to as the regularity of function. In contrast, if there are not enough regularities or there are only a few, it is referred to as irregularity. The regularity (or irregularity) of a function can be expressed as "C^1 class" by adding "class" to the symbol representing the function space.

There are two factors determining the irregularity of the solution to a boundary value problem. Let us look at these in the following subsections.

5.3.1 Regularity of Given Functions

Firstly, let us think about the relationship between the regularity of solution and regularity of given functions b, p, u_{D} of the Poisson problem (Problem 5.1.1). Suppose the boundary $\partial\Omega$ is sufficiently smooth. In this case, from the fact that

$$-\Delta u = b \quad \text{in } \Omega, \quad \partial_\nu u = p_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}, \quad u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}},$$

holds, if we assume

$$b \in L^2(\Omega; \mathbb{R}), \quad p_{\mathrm{N}} \in H^1(\Omega; \mathbb{R}), \quad u_{\mathrm{D}} \in H^2(\Omega; \mathbb{R}),$$

we get $u \in H^2\left(\Omega \setminus \bar{B}; \mathbb{R}\right)$, where the neighborhood of the boundary between the Dirichlet boundary and the Neumann boundary is denoted by B. In more detail, if we set $b \in L^2(\Omega; \mathbb{R})$, we obtain $u \in H^2\left(\Omega \setminus \bar{B}; \mathbb{R}\right)$ from the fact that the Poisson equation is satisfied. Moreover, since the boundary $\partial\Omega$ is sufficiently smooth, then $\boldsymbol{\nu} \in C(\Gamma_{\mathrm{N}}; \mathbb{R})$. So if $p_{\mathrm{N}} \in H^1(\Omega; \mathbb{R})$, $\partial_\nu u = \boldsymbol{\nu} \cdot \boldsymbol{\nabla} u \in H^{1/2}\left(\Gamma_{\mathrm{N}} \setminus \bar{B}; \mathbb{R}\right)$ is obtained from $p_{\mathrm{N}} \in H^{1/2}(\Gamma_{\mathrm{N}}; \mathbb{R})$. From this, $u \in H^2\left(\Omega \setminus \bar{B}; \mathbb{R}\right)$ can be obtained. Moreover, according to the Sobolev embedding theorem (Theorem 4.3.14), $H^2(\Omega; \mathbb{R}) \subset C^{0,\sigma}\left(\bar{\Omega}; \mathbb{R}\right)$ holds with respect to $\sigma \in (0, 1/2)$ and $d \in \{2, 3\}$. Hence, u becomes a continuous function. It is said that there are no irregularities in the solution u in this case. If a given function is changed to an even smoother function then a correspondingly smoother u can be obtained.

5.3.2 Regularity of Boundary

On the other hand, even if given functions are assumed to be sufficiently smooth, if the boundary is not smooth, there can be irregularities in the solution. Let us look at such a situation in detail. In this section, Ω is assumed to be a two-dimensional domain and focus is given to the neighborhood around a corner such as $\boldsymbol{x}_0$ in Fig. 5.2. Such a corner point corresponds to looking at a point in the perpendicular cross-section with respect to a smooth cut-out line in a three-dimensional domain with a V-shaped cut-out.

A discontinuous point on boundary $\partial\Omega$ with respect to C^1 class such as $\boldsymbol{x}_0$ on Fig. 5.2 is called a corner point. A set of corner points will be denoted by

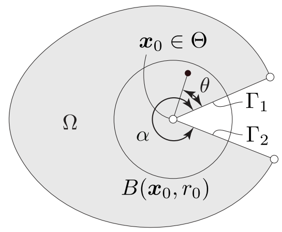


Fig. 5.2: Two-dimensional domain with a corner.

Θ. Let r_0 be a positive constant and $B(\boldsymbol{x}_0, r_0)$ the neighborhood (open set) around $\boldsymbol{x}_0$ with radius r_0. Set the opening angle at $\boldsymbol{x}_0$ in the internal domain to be $\alpha \in (0, 2\pi)$. The boundaries (open set) on both sides of $\boldsymbol{x}_0$ in $B(\boldsymbol{x}_0, r_0)$ are set to be Γ_1 and Γ_2, respectively. The boundaries Γ_1 and Γ_2 are assumed to be smooth (C^1 class). Moreover, the polar coordinates with $\boldsymbol{x}_0$ as the origin are denoted as (r, θ).

From the fact that u is smooth (analytic) at points further away from $\boldsymbol{x}_0$, if some $r \in (0, r_0]$ is fixed, u can be expanded as

$$u(r, \theta) = \sum_{i \in \{1,2,\ldots\}} k_i u_i(r) \tau_i(\theta) + u_{\mathrm{R}} \tag{5.3.1}$$

(cf. [7], [8, Chap. 8, p. 257] , [10], [5, Preface, p. ix, and Chap. 4, p. 182]). Here, u_{R} expresses the remainder term determined by the regularity of the given function. In contrast, the first term on the right-hand side of Eq. (5.3.1) arising due to the corner point is called the main term. For each $i \in \{1, 2, \ldots\}$, the main terms k_i are real constants and $u_i(r)$ represent real-valued functions determined dependent on r. Moreover, $\tau_i(\theta)$ are determined in the following way by real-valued functions of $\theta \in (0, \alpha)$ dependent on boundary conditions. When Γ_1 and Γ_2 are both homogeneous Dirichlet boundaries ($u = 0$) and both homogeneous Neumann boundaries ($\partial_\nu u = 0$), these respectively become

$$\tau_i(\theta) = \sin \frac{i\pi}{\alpha}\theta, \tag{5.3.2}$$

$$\tau_i(\theta) = \cos \frac{i\pi}{\alpha}\theta. \tag{5.3.3}$$

In reality, Eq. (5.3.2) satisfies $\tau_i(0) = \tau_i(\alpha) = 0$. Equation (5.3.3) satisfies the condition that $(\mathrm{d}\tau_i/\mathrm{d}\theta)(0) = (\mathrm{d}\tau_i/\mathrm{d}\theta)(\alpha) = 0$. Moreover, if Γ_1 and Γ_2 are mixed boundaries with a homogeneous Dirichlet and Neumann boundary, it becomes

$$\tau_i(\theta) = \sin \frac{i\pi}{2\alpha}\theta. \tag{5.3.4}$$

On the other hand, with respect to the Laplace operator Δ,

$$\Delta(r^\omega \sin \omega\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(r^\omega \sin \omega\theta) = 0 \tag{5.3.5}$$

holds, where ω is a real number satisfying $\omega > 1/4$ which is not 1. The condition $\omega > 1/4$ corresponds to the fact that, in the condition shown later, as Γ_1 and Γ_2 have mixed boundary conditions and get closer to a crack ($\alpha \to 2\pi$), they become $\omega \to 1/4$. Moreover, $\omega = 1$ corresponds to the condition that the boundary is smooth. In addition, Eq. (5.3.5) is also obtained by Cauchy–Riemann equations, which forms a necessary and sufficient condition for a complex function to be complex differentiable (holomorphic), with respect to the imaginary part $u_i = \mathrm{Im}\,[z^\omega] = r^m \sin \omega\theta$ of a complex function $f(z) = z^\omega$ using the correspondence between a complex number $z = x_1 + \mathrm{i}x_2 = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$ (i is the imaginary unit) and $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Equation (5.3.5) shows that if a function has the format $r^\omega \sin \omega\theta$, the Laplace equation (same with the homogeneous Poisson equation) is satisfied.

Focusing on this relationship, when $\tau_i(\theta)$ is given in the format $\sin \omega\theta$, if

$$u_i(r) = r^\omega,$$

the Laplace equation is satisfied. From this result, the following results are obtained in the neighborhood $B(\boldsymbol{x}_0, r_0) \cap \Omega$ around $\boldsymbol{x}_0$ with radius r_0:

(1) When Γ_1 and Γ_2 are both homogeneous Dirichlet boundaries ($u = 0$),

$$u(r, \theta) = k r^{\pi/\alpha} \sin \frac{\pi}{\alpha}\theta + u_{\mathrm{R}}. \tag{5.3.6}$$

(2) When Γ_1 and Γ_2 are both homogeneous Neumann boundaries ($\partial_\nu u = 0$),

$$u(r, \theta) = k r^{\pi/\alpha} \cos \frac{\pi}{\alpha}\theta + u_{\mathrm{R}}. \tag{5.3.7}$$

(3) If it is a mixed boundary where Γ_1 is a homogeneous Dirichlet boundary and Γ_2 is a homogeneous Neumann boundary,

$$u(r, \theta) = k r^{\pi/(2\alpha)} \sin \frac{\pi}{2\alpha}\theta + u_{\mathrm{R}}. \tag{5.3.8}$$

Here, k is a constant dependent on α.

Moreover, the following result can be obtained for a Sobolev space containing functions of the format r^ω.

Proposition 5.3.1 (Regularity of singularity term) Let Ω be a two-dimensional bounded domain and $\boldsymbol{x}_0$ be a corner point of opening angle $\alpha \in (0, 2\pi)$ on $\partial\Omega$. The function u is given by

$$u = r^\omega \tau(\theta)$$

in the neighborhood $B(\boldsymbol{x}_0, r_0) \cap \Omega$ around $\boldsymbol{x}_0$, where $\tau(\theta)$ is taken to be an element of $C^\infty((0, \alpha), \mathbb{R})$. In this case, if

$$\omega > k - \frac{2}{p} \tag{5.3.9}$$

holds for $k \in \mathbb{N} \cup \{0\}$ and $p \in (1, \infty)$, u is in $W^{k,p}(B(\boldsymbol{x}_0, r_0) \cap \Omega; \mathbb{R})$. □

Proof The k-th order derivative of $u = r^{\omega}\tau(\theta)$ is constructed as a sum of the terms including $r^{\omega-k}\tilde{\tau}(\theta)$. Here, $\tilde{\tau}(\theta)$ is an element of $C^{\infty}((0,\alpha),\mathbb{R})$. Hence, in order for the p-th-order Lebesgue integral on $B(\boldsymbol{x}_0, r_0) \cap \Omega$ of k-th order derivative of u to be finite, the condition

$$\int_0^{r_0}\int_0^{\alpha} r^{p(\omega-k)} r\tilde{\tau}(\theta)\ \mathrm{d}\theta\mathrm{d}r < \infty$$

needs to hold. For this,

$$p(\omega - k) + 1 > -1$$

is obtained. This relationship gives Eq. (5.3.9). □

From the fact that the main term of solution u to the Poisson problem around the corner point is a function of the form r^{ω} and Proposition 5.3.1, the following results can be obtained with respect to a corner point such as that in Fig. 5.3.

Theorem 5.3.2 (Regularity of a solution around a corner) Let Ω be a two-dimensional bounded domain and $\boldsymbol{x}_0 \in \Theta$ be a corner point of opening angle $\alpha \in (0, 2\pi)$. In this case the solution u of the Poisson problem (Problem 5.1.1) is in $H^s(B(\boldsymbol{x}_0, r_0) \cap \Omega; \mathbb{R})$ in the neighborhood of $\boldsymbol{x}_0$. Here:

(1) if the boundaries Γ_1 and Γ_2 of both sides of $\boldsymbol{x}_0$ share the same type of boundary condition, then $\alpha \in [\pi, 2\pi)$ implies that $s \in (3/2, 2]$.

(2) if Γ_1 and Γ_2 are mixed boundaries, then $\alpha \in [\pi/2, \pi)$ implies $s \in (3/2, 2]$ and $\alpha \in [\pi, 2\pi)$ means that $s \in (5/4, 3/2]$.

□

Proof If Γ_1 and Γ_2 are the same type of boundary, Eq. (5.3.6) and Eq. (5.3.7) give $\omega = \pi/\alpha$. Here, when the opening angle is $\alpha \in [\pi, 2\pi)$, $\omega \in (1/2, 1]$. In this case, if the inequality condition

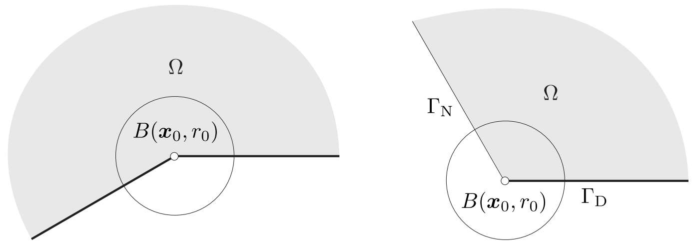
$$s_1 - \frac{2}{p} = \frac{3}{2} - \frac{2}{2} = \frac{1}{2} < \omega \le s_2 - \frac{2}{p} = 2 - \frac{2}{2} = 1$$

holds with respect to Eq. (5.3.9), then $\omega \in (1/2, 1]$ implies that s is (1).

On the other hand, if Γ_1 and Γ_2 are mixed boundaries, Eq. (5.3.8) gives $\omega = \pi/(2\alpha)$. Hence, if the opening angle is $\alpha \in [\pi/2, \pi)$, then $\omega \in (1/2, 1]$ and s satisfying Eq. (5.3.9) becomes a result such as that in the first half of (2). Moreover, if the opening angle is $\alpha \in [\pi, 2\pi)$, then $\omega \in (1/4, 1/2]$. In this case, if we have that

$$s_1 - \frac{2}{p} = \frac{5}{4} - \frac{2}{2} = \frac{1}{4} < \omega \le s_2 - \frac{2}{p} = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}$$

holds with respect to Eq. (5.3.9), then s becomes a result such as the latter half of (2) with respect to $\omega \in (1/4, 1/2]$. □



(a) Opening angle is $\alpha > \pi$ between boundaries of the same type

(b) Opening angle is $\alpha > \pi/2$. between the mixed boundaries

Fig. 5.3: Two-dimensional domain having a corner with irregularity.

Assumptions in Theorem 5.3.2 did not include a crack ($\alpha = 2\pi$). If $\boldsymbol{x}_0$ is a crack tip,

$$u \in H^{3/2-\epsilon}\left(B\left(\boldsymbol{x}_0, r_0\right) \cap \Omega; \mathbb{R}\right) \tag{5.3.10}$$

can be written with respect to $\epsilon > 0$. Moreover, even when $\boldsymbol{x}_0$ is a boundary of mixed boundaries and the boundary is smooth around $\boldsymbol{x}_0$ ($\alpha = \pi$), it can be written as Eq. (5.3.10).

In order to guarantee that u is a function of $W^{1,\infty}$ class, the following result can be used.

Theorem 5.3.3 (Regularity of a solution around a corner) Let Ω be a two-dimensional bounded domain and $\boldsymbol{x}_0 \in \Theta$ be a corner point of opening angle $\alpha \in (0, 2\pi)$. The solution u of the Poisson problem (Problem 5.1.1) is in $W^{1,\infty}\left(B\left(\boldsymbol{x}_0, r_0\right) \cap \Omega; \mathbb{R}\right)$,

(1) if $\alpha < \pi$ in the case that the boundaries Γ_1 and Γ_2 of both sides of $\boldsymbol{x}_0$ share the same type of boundary condition,

(2) if $\alpha < \pi/2$ in the case that Γ_1 and Γ_2 are mixed boundaries.

□

Proof If Γ_1 and Γ_2 are the same type of boundary, Eq. (5.3.6) and Eq. (5.3.7) give $\omega = \pi/\alpha$. Here, when the opening angle is $\alpha < \pi$, $\omega > 1$. In this case, (1) holds with respect to Eq. (5.3.9). On the other hand, if Γ_1 and Γ_2 are mixed boundaries, Eq. (5.3.8) gives $\omega = \pi/(2\alpha)$. Hence, if the opening angle is $\alpha < \pi/2$, then $\omega > 1$. From Eq. (5.3.9), we have that (2) holds with respect to Eq. (5.3.9). □

Moreover, from Theorem 5.3.2 (2), if Γ_1 and Γ_2 are mixed boundaries, it becomes apparent that even when the boundary is smooth, the same irregularity is observed as that at a crack tip. One method for preventing the occurence of such irregularity is to rewrite the mixed boundary value problem as an extended Poisson problem, such as Problem 5.1.3. In this case, by assuming a smooth function in $c_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ such that it changes from a Dirichlet boundary to a Neumann boundary, a mixed boundary value problem with no singularities can be constructed.

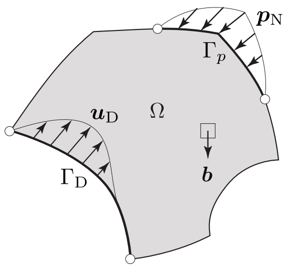


Fig. 5.4: Two-dimensional linear elastic problem.

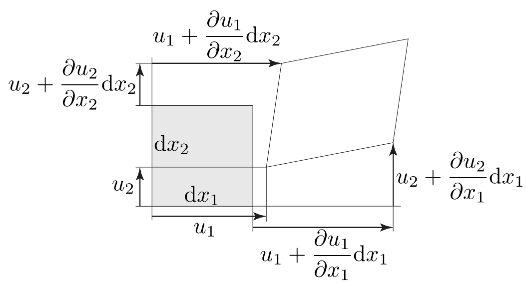


Fig. 5.5: Displacement $\boldsymbol{u}$ and its gradient $\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top$ in 2D linear elastic body.

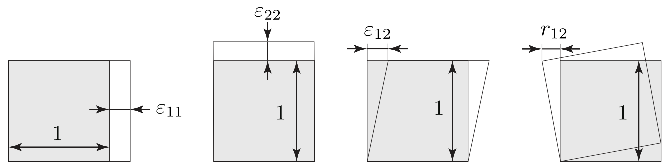
5.4 Linear Elastic Problem

In this book, attempts are made to represent specific examples of shape optimization problems using a linear elastic body and Stokes flow field. As preparation for that, we shall now define a linear elastic problem and look at the existence of unique solutions and their weak forms.

Let $\Omega \subset \mathbb{R}^d$ be a $d \in \{2,3\}$-dimensional Lipschitz domain. Let $\Gamma_{\mathrm{D}} \subset \partial\Omega$ be a boundary when displacement is given (Dirichlet boundary) and the remaining boundary $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \bar{\Gamma}_{\mathrm{D}}$ be a boundary where traction is given (Neumann boundary). Moreover, $\Gamma_p \subset \Gamma_{\mathrm{N}}$ is taken to represent a boundary where the traction is non-zero. Here, Γ_p and $\Gamma_{\mathrm{N}} \setminus \bar{\Gamma}_p$ are not distinguished but they will be in Chap. 9. Figure 5.4 shows a linear elastic body in the two-dimensional case. However, as seen in Exercise 5.2.6, in order to get rid of the uncertainty of constant, $|\Gamma_{\mathrm{D}}| > 0$ is assumed. Moreover, $\boldsymbol{b} : \Omega \to \mathbb{R}^d$ is taken to be a volume force, $\boldsymbol{p}_{\mathrm{N}} : \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ is the traction and $\boldsymbol{u}_{\mathrm{D}} : \Omega \to \mathbb{R}^d$ is the given displacement. A linear elastic problem is defined as a problem seeking displacements $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ when these are given.

5.4.1 Linear Strain

A linear elastic problem of a one-dimensional continuous body was defined in Chap. 1. Here, let us extend this to $d \in \{2,3\}$ dimensions. Firstly, let us define



(a) ε_{11} (b) ε_{22} (c) $\varepsilon_{12} = \varepsilon_{21}$ (d) r_{12}

Fig. 5.6: Linear strain $\boldsymbol{E}(\boldsymbol{u})$ and rotation tensor $\boldsymbol{R}(\boldsymbol{u})$ in 2D linear elastic body.

the term strain. In a one-dimensional linear elastic body, the displacement u was a real-valued function defined on $(0, l)$. Strain was defined using its gradient $\mathrm{d}u/\mathrm{d}x$. If the linear elastic body is $d \in \{2, 3\}$-dimensional, the displacement $\boldsymbol{u}$ becomes a d-dimensional vector and its gradient $\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top = (\partial u_i/\partial x_j)_{ij}$ becomes a second-order tensor (matrix) with the value $\mathbb{R}^{d\times d}$. Figure 5.5 shows the relationship between $\boldsymbol{u}$ and $\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top$. This tensor is split into the symmetric and non-symmetric components as

$$\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top = \boldsymbol{E}(\boldsymbol{u}) + \boldsymbol{R}(\boldsymbol{u}) \tag{5.4.1}$$

In this case,

$$\boldsymbol{E}(\boldsymbol{u}) = \boldsymbol{E}^\top(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{ij} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^\top + \left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top\right), \tag{5.4.2}$$

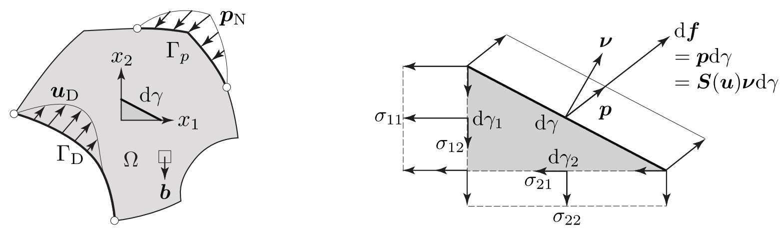
$$\boldsymbol{R}(\boldsymbol{u}) = -\boldsymbol{R}^\top(\boldsymbol{u}) = (r_{ij}(\boldsymbol{u}))_{ij} = \frac{1}{2}\left(\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)^\top - \boldsymbol{\nabla}\boldsymbol{u}^\top\right). \tag{5.4.3}$$

Here, the symmetric component $\boldsymbol{E}(\boldsymbol{u})$ represents the deformations such as those from (a) to (c) in Fig. 5.6 when Ω is a two-dimensional domain, and is called the linear strain, or simply strain of a d-dimensional linear elastic body if there is no confusion. Moreover, the non-symmetric component $\boldsymbol{R}(\boldsymbol{u})$ represents the rotational motion such as (d) in Fig. 5.6 with respect to a two-dimensional domain Ω, and is called the rotation tensor of a d-dimensional linear elastic body.

The linear strain and rotation tensor defined in Eq. (5.4.2) and Eq. (5.4.3) were defined using the gradient tensor of $\boldsymbol{u}$ when $\boldsymbol{u}$ is $\boldsymbol{0}_{\mathbb{R}^d}$ (before deformation). Hence, there is a need to focus on the fact that $\boldsymbol{u}$ cannot take a large value. When it is assumed that $\boldsymbol{u}$ is finite, finite deformation theory using Green strain or Almansi strain which is defined with the second-order terms of elements of the gradient tensor of displacement is employed. In this case the differential equation becomes non-linear. The non-linearity in this case is called a geometric non-linearity. This book is limited to linear problems.

5.4.2 Cauchy Tensor

In contrast, with respect to a linear strain defined from displacement, stress can be defined from the distribution of force. Consider a small domain inside



(a) Small line component $\mathrm{d}\gamma$ in domain. (b) Cauchy stress and stress.

Fig. 5.7: Cauchy stress $\boldsymbol{S}$ and stress $\boldsymbol{p}$ of 2D linear elastic body.

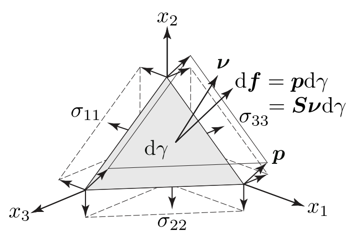


Fig. 5.8: Cauchy stress $\boldsymbol{S}$ and stress $\boldsymbol{p}$ of 3D elastic body.

the domain Ω. When $d = 2$, a triangle such as the one in Fig. 5.7 (b) is imagined, while when $d = 3$, a triangular pyramid such as that in Fig. 5.8 is considered. The normal of their tilt boundary is $\boldsymbol{\nu}$. Force per unit boundary measure (length when $d = 2$, area when $d = 3$) working on the tilt boundary is taken to be $\boldsymbol{p} \in \mathbb{R}^d$. The function $\boldsymbol{p}$ represents the stress. Moreover, with respect to $i, j \in \{1, \dots, d\}$, when σ_{ij} is the force in the x_j-direction per unit boundary measure working on a boundary with normal in the x_i-direction, $\boldsymbol{S} = (\sigma_{ij}) \in \mathbb{R}^{d \times d}$ is called Cauchy stress, or simply stress if there is no confusion.

Cauchy stress $\boldsymbol{S}$ and stress $\boldsymbol{p}$ can be related in the following way.

Proposition 5.4.1 (Cauchy tensor) When $\boldsymbol{p}$ is a stress and $\boldsymbol{S}$ is its Cauchy stress,

$$\boldsymbol{S}^{\top}\boldsymbol{\nu} = \boldsymbol{S}\boldsymbol{\nu} = \boldsymbol{p} \tag{5.4.4}$$

holds. □

Proof We will show the case when $d = 2$. From the balance of force in the direction x_i with respect to $i \in \{1, 2\}$, the relation

$$\sigma_{1i}\mathrm{d}\gamma_1 + \sigma_{2i}\mathrm{d}\gamma_2 = p_i \mathrm{d}\gamma$$

holds (Fig. 5.7 (b)). Here, using $\nu_1 = \mathrm{d}\gamma_1/\mathrm{d}\gamma$ and $\nu_2 = \mathrm{d}\gamma_2/\mathrm{d}\gamma$ gives

$$\sigma_{1i}\nu_1 + \sigma_{2i}\nu_2 = p_i. \tag{5.4.5}$$

Equation (5.4.5) represents Eq. (5.4.4). On the other hand, the identity

$$\sigma_{21} = \sigma_{12}$$

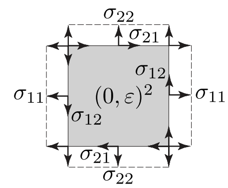


Fig. 5.9: Balance of moments at a small area in a two-dimensional linear elastic body ($\epsilon \ll 1$).

holds from the balance of moments (Fig. 5.9). A similar relationship holds when $d = 3$ too. □

5.4.3 Constitutive Equation

In a similar way to that used in Chap. 1 when one-dimensional linear elastic problems were defined, constitutive equation or constitutive law, which relates the strain defined using displacement and the stress defined using force, is now needed. In a linear elastic body of d dimensions, it is given by

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{u}) &= \boldsymbol{S}^\top(\boldsymbol{u}) = (\sigma_{ij}(\boldsymbol{u}))_{ij} \\ &= \boldsymbol{C}\boldsymbol{E}(\boldsymbol{u}) = \left(\sum_{(k,l)\in\{1,\dots,d\}^2} c_{ijkl}\varepsilon_{kl}(\boldsymbol{u})\right)_{ij}.\end{aligned} \tag{5.4.6}$$

Here, $\boldsymbol{C} = (c_{ijkl})_{ijkl} : \Omega \to \mathbb{R}^{d\times d\times d\times d}$ is a function of fourth-order tensor value representing the rigidity and assumes the following characteristics. Firstly, from the symmetry of $\boldsymbol{S}(\boldsymbol{u})$ and $\boldsymbol{E}(\boldsymbol{u})$, the relationships

$$c_{ijkl} = c_{jikl}, \quad c_{ijkl} = c_{ijlk}. \tag{5.4.7}$$

hold. Moreover, assuming $\boldsymbol{C}$ is L^∞ class, there exist positive constants α and β such that

$$\boldsymbol{A}\cdot(\boldsymbol{C}\boldsymbol{A}) \geq \alpha \|\boldsymbol{A}\|^2, \tag{5.4.8}$$

$$|\boldsymbol{A}\cdot(\boldsymbol{C}\boldsymbol{B})| \leq \beta \|\boldsymbol{A}\| \|\boldsymbol{B}\| \tag{5.4.9}$$

hold almost everywhere in Ω with respect to arbitrary symmetric tensor $\boldsymbol{A} = (a_{ij})_{ij} \in \mathbb{R}^{d\times d}$ and $\boldsymbol{B} = (b_{ij})_{ij} \in \mathbb{R}^{d\times d}$. In this book, the scalar product of matrices is represented as $\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{i,j\{1,\dots,d\}} a_{ij}b_{ij}$. The fact that Eq. (5.4.8) holds is referred to as $\boldsymbol{C}$ being elliptic. Moreover, the fact that Eq. (5.4.9) holds is referred to as $\boldsymbol{C}$ being bounded. When $\boldsymbol{C}$ is not a function of $\boldsymbol{u}$ (stress is a linear function of strain), Eq. (5.4.6) is referred to as the generalized Hooke's law. The non-linearity such that $\boldsymbol{C}$ becomes a function of $\boldsymbol{u}$ is called material non-linearity. This sort of non-linearity will also not be treated in this book.

Moreover, the following can be said about the number of real numbers which can be chosen independently in the rigidity $\boldsymbol{C}$, when $d = 3$:

(1) $\boldsymbol{C}$ is constructed of $3^4 = 81$ real numbers.

(2) It reduces to 36 from Eq. (5.4.7).

(3) If strain energy density w exists and

$$w = \frac{1}{2}\boldsymbol{E}(\boldsymbol{u}) \cdot (\boldsymbol{C}\boldsymbol{E}(\boldsymbol{u})), \quad \boldsymbol{S}(\boldsymbol{u}) = \frac{\partial w}{\partial \boldsymbol{E}(\boldsymbol{u})}$$

holds, the the relation

$$c_{ijkl} = c_{klij} \tag{5.4.10}$$

holds by the symmetry of second-order form. In this case it reduces down to 21.

(4) It reduces to nine in the case of orthotropic materials.

(5) It reduces to two in the case of isotropic materials.

Suppose the two constants in the case of isotropic material are written as λ_{L} and μ_{L} and

$$\boldsymbol{S}(\boldsymbol{u}) = 2\mu_{\mathrm{L}}\boldsymbol{E}(\boldsymbol{u}) + \lambda_{\mathrm{L}}\mathrm{tr}(\boldsymbol{E}(\boldsymbol{u}))\boldsymbol{I},$$

where $\mathrm{tr}(\boldsymbol{E}(\boldsymbol{u})) = \sum_{i\in\{1,\dots,d\}} e_{ii}(\boldsymbol{u})$. In this case, the quantities λ_{L} and μ_{L} are called Lamé's parameters. Moreover, μ_{L} is also referred to as shear modulus. In addition, when the two constants are expressed as e_{Y} and ν_{P} and

$$\boldsymbol{E}(\boldsymbol{u}) = \frac{1+\nu_{\mathrm{P}}}{e_{\mathrm{Y}}}\boldsymbol{S}(\boldsymbol{u}) - \frac{\nu_{\mathrm{P}}}{e_{\mathrm{Y}}}\mathrm{tr}(\boldsymbol{S}(\boldsymbol{u}))\boldsymbol{I}$$

is assumed, e_{Y} and ν_{P} are called longitudinal elastic modulus (Young's modulus) and Poisson's ratio, respectively. Other than this, bulk modulus k_{b} is also used. A relationship such as

$$k_{\mathrm{b}} = \lambda_{\mathrm{L}} + \frac{2\mu_{\mathrm{L}}}{3}, \quad e_{\mathrm{Y}} = 2\mu_{\mathrm{L}}(1+\nu_{\mathrm{P}}), \quad \lambda_{\mathrm{L}} = \frac{2\mu_{\mathrm{L}}\nu_{\mathrm{P}}}{1-2\nu_{\mathrm{P}}}$$

holds with respect to these constants.

5.4.4 Equilibrium Equations of Force

A linear elastic problem is constructed using the balance condition of forces based on a linear strain and Cauchy stress being linked via generalized Hooke's law Eq. (5.4.6).

When an arbitrary small square element is chosen within a two-dimensional linear elastic body, the force working on that element is as shown by the arrows in Fig. 5.10. Here, the equilibrium equation of force in the x_1-direction and x_2-direction becomes

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + b_1 = 0,$$

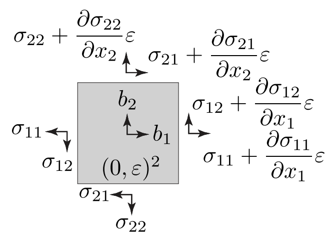


Fig. 5.10: Balance of forces in a small area ($\epsilon \ll 1$).

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + b_2 = 0.$$

In the case of a $d \in \{2,3\}$-dimensional linear elastic body, it can be written as

$$-\boldsymbol{\nabla}^{\top} \boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{b}^{\top}. \tag{5.4.11}$$

Eq. (5.4.11) is a second-order differential equation with respect to $\boldsymbol{u}$ if we look at the fact that $\boldsymbol{\nabla}^{\top}\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{\nabla}\cdot\left\{\boldsymbol{C}\left(\frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\top} + \left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top}\right)\right)\right\}$. Furthermore, from the fact that $\boldsymbol{C}$ satisfies ellipticity, Eq. (5.4.11) is classed as an elliptic partial differential equation.

Adding boundary conditions to the equilibrium equation (Eq. (5.4.11)) of force gives a linear elastic problem such as the one below.

Problem 5.4.2 (Linear elastic problem) Let the functions $\boldsymbol{b} : \Omega \to \mathbb{R}^d$, $\boldsymbol{p}_{\mathrm{N}} : \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ and $\boldsymbol{u}_{\mathrm{D}} : \Omega \to \mathbb{R}^d$ be given. Obtain $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ which satisfies

$$-\boldsymbol{\nabla}^{\top}\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{b}^{\top} \quad \text{in } \Omega, \tag{5.4.12}$$

$$\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{p}_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{5.4.13}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}. \tag{5.4.14}$$

□

5.4.5 Weak Form

In order to show the existence of a unique solution to the linear elastic problem, let us rewrite Problem 5.4.2 in the weak form. Let the function space with respect to $\boldsymbol{u}$ be

$$U = \left\{ \boldsymbol{v} \in H^1\left(\Omega;\mathbb{R}^d\right) \;\middle|\; \boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}} \right\}. \tag{5.4.15}$$

By multiplying both sides of Eq. (5.4.12) by an arbitrary $\boldsymbol{v} \in U$ and integrating over Ω, then using the Gauss–Green theorem (Theorem A.8.2), the equation

$$-\int_{\Omega}\left(\boldsymbol{\nabla}^{\top}\boldsymbol{S}(\boldsymbol{u})\right)\boldsymbol{v}\,\mathrm{d}x = -\int_{\Gamma_{\mathrm{N}}}\left(\boldsymbol{S}(\boldsymbol{u})\boldsymbol{\nu}\right)\cdot\boldsymbol{v}\,\mathrm{d}\gamma + \int_{\Omega}\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{v})\,\mathrm{d}x$$

$$= \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}x \tag{5.4.16}$$

can be obtained. Moreover, if both sides of Eq. (5.4.13) are multiplied by an arbitrary $\boldsymbol{v} \in U$ and integrated over Γ_{N}, the equation

$$\int_{\Gamma_{\mathrm{N}}} \left(\boldsymbol{S}\left(\boldsymbol{u}\right) \boldsymbol{\nu} \right) \cdot \boldsymbol{v} \, \mathrm{d}\gamma = \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v} \, \mathrm{d}\gamma \tag{5.4.17}$$

is obtained. Substituting Eq. (5.4.17) in the first term of the second equation in Eq. (5.4.16) gives

$$\int_{\Omega} \boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{v}\right) \, \mathrm{d}x = \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v} \, \mathrm{d}\gamma.$$

This equation, which holds for arbitrary $\boldsymbol{v} \in U$, is referred to as the weak form of the linear elastic problem.

Also, if we set

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{S}\left(\boldsymbol{u}\right) \cdot \boldsymbol{E}\left(\boldsymbol{v}\right) \, \mathrm{d}x, \tag{5.4.18}$$

$$l\left(\boldsymbol{v}\right) = \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v} \, \mathrm{d}\gamma, \tag{5.4.19}$$

the weak-form linear elastic problem becomes as follows.

Problem 5.4.3 (Weak form of linear elastic problem) Let U be given by Eq. (5.4.15) and the functions $\boldsymbol{b} \in L^2\left(\Omega; \mathbb{R}^d\right)$, $\boldsymbol{p}_{\mathrm{N}} \in L^2\left(\Gamma_{\mathrm{N}}; \mathbb{R}^d\right)$, $\boldsymbol{u}_{\mathrm{D}} \in H^1\left(\Omega; \mathbb{R}^d\right)$ and $\boldsymbol{C} \in L^\infty\left(\Omega; \mathbb{R}^{d \times d \times d \times d}\right)$. Set $a\left(\,\cdot\,,\,\cdot\,\right)$ and $l\left(\,\cdot\,\right)$ as Eq. (5.4.18) and Eq. (5.4.19), respectively. In this case, seek $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U$ which satisfies

$$a\left(\boldsymbol{u}, \boldsymbol{v}\right) = l\left(\boldsymbol{v}\right)$$

with respect to an arbitrary $\boldsymbol{v} \in U$. □

5.4.6 Existence of Solution

If v is viewed as a virtual displacement, from the fact that $l\left(\boldsymbol{v}\right)$ is the virtual work done by external forces and $a\left(\boldsymbol{u}, \boldsymbol{v}\right)$ is the virtual work done by internal forces, the weak form of a linear elastic problem represents the principle of virtual work. The existence of unique solutions with respect to this weak form is shown as follows.

Exercise 5.4.4 (Existence of unique solution to linear elastic problem) In Problem 5.4.3, show that the solution $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U$ exists uniquely for $|\Gamma_{\mathrm{D}}| > 0$. □

Answer Let us confirm that the assumptions of the Lax–Milgram theorem hold. Let U be a Hilbert space and let

$$\hat{l}(\boldsymbol{v}) = l(\boldsymbol{v}) - a(\boldsymbol{u}_{\mathrm{D}}, \boldsymbol{v})$$

with respect to an arbitrary $\boldsymbol{v} \in U$. Problem 5.4.3 can be rewritten as the problem seeking $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U$ which satisfies

$$a(\tilde{\boldsymbol{u}}, \boldsymbol{v}) = \hat{l}(\boldsymbol{v}).$$

Under these assumptions, the fact that the Lax–Milgram theorem holds can be confirmed in the following ways:

(1) a is coercive. In fact, the rigid motion is not generated because of $|\Gamma_{\mathrm{D}}| > 0$. Therefore, from Korn's second inequality (Theorem A.9.6), the estimate

$$\|\boldsymbol{v}\|^2_{H^1(\Omega;\mathbb{R}^d)} \le c \|\boldsymbol{E}(\boldsymbol{v})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}$$

holds with respect to a positive constant c. From ellipticity of $\boldsymbol{C}$ due to Eq. (5.4.8), the inequality

$$\begin{aligned} a(\boldsymbol{v}, \boldsymbol{v}) &= \int_\Omega \boldsymbol{E}(\boldsymbol{v}) \cdot (\boldsymbol{C}\boldsymbol{E}(\boldsymbol{v}))\, \mathrm{d}x \\ &\ge c_1 \|\boldsymbol{E}(\boldsymbol{v})\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \ge \frac{c_1}{c} \|\boldsymbol{v}\|^2_{H^1(\Omega;\mathbb{R}^d)} \end{aligned}$$

holds with respect to $\boldsymbol{v} \in U$. Here, c_1 is a positive constant multiplying together α of Eq. (5.4.8) and $|\Omega|$. If c_1/c is reset to be α, from Definition 5.2.1, a is coercive.

(2) a is bounded. In fact, if the positive constant multiplying β of Eq. (5.4.9) and $|\Omega|$ is replaced by β, Definition 5.2.2 confirms the boundedness of a.

(3) $\hat{l} \in U'$. In fact, since $\partial\Omega$ assumes a Lipschitz boundary, the norm $\|\gamma\|_{\mathcal{L}(H^1(\Omega;\mathbb{R}^d);H^{1/2}(\partial\Omega;\mathbb{R}^d))}$ of the trace operator (Theorem 4.4.2) is bounded. Let this be $c_2 > 0$. Moreover, using Hölder's inequality, the following result holds:

$$\begin{aligned} \left|\hat{l}(\boldsymbol{v})\right| &\le \int_\Omega |\boldsymbol{b} \cdot \boldsymbol{v}|\, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} |\boldsymbol{p}_{\mathrm{N}} \cdot \boldsymbol{v}|\, \mathrm{d}\gamma + \int_\Omega \beta |\boldsymbol{E}(\boldsymbol{u}_{\mathrm{D}}) \cdot \boldsymbol{E}(\boldsymbol{v})|\, \mathrm{d}x \\ &\le \|\boldsymbol{b}\|_{L^2(\Omega;\mathbb{R}^d)} \|\boldsymbol{v}\|_{L^2(\Omega;\mathbb{R}^d)} + \|\boldsymbol{p}_{\mathrm{N}}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^d)} \|\boldsymbol{v}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^d)} \\ &\quad + \beta \|\boldsymbol{E}(\boldsymbol{u}_{\mathrm{D}})\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \|\boldsymbol{E}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \\ &\le \Big(\|\boldsymbol{b}\|_{L^2(\Omega;\mathbb{R}^d)} + c_2 \|\boldsymbol{p}_{\mathrm{N}}\|_{L^2(\Gamma_{\mathrm{N}};\mathbb{R}^d)} \\ &\qquad + \beta \|\boldsymbol{E}(\boldsymbol{u}_{\mathrm{D}})\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\Big) \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}. \end{aligned}$$

Therefore, from the Lax–Milgram theorem there is a unique $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U$ which satisfies Problem 5.4.3. □

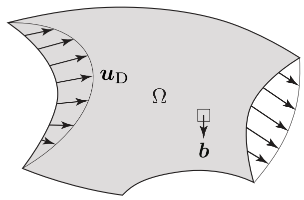


Fig. 5.11: Two-dimensional Stokes problem.

5.5 Stokes Problem

Next, let us define a Stokes problem as an example of a flow field and look at its weak form and the existence of a unique solution. A Stokes problem is used as a mathematical model of a flow field which is slow so that inertia can be ignored relative to viscosity in the flow field of a viscous fluid.

In this case too, $\Omega \subset \mathbb{R}^d$ is taken to be a Lipschitz domain of $d \in \{2, 3\}$ dimensions. Again, let $\boldsymbol{b} : \Omega \to \mathbb{R}^d$ be the volume force. Let the entire boundary $\partial\Omega$ of Ω be a Dirichlet boundary with respect to flow velocity given by $\boldsymbol{u}_{\mathrm{D}} : \Omega \to \mathbb{R}^d$ such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{D}} = 0 \quad \text{in } \Omega. \tag{5.5.1}$$

Let μ be a positive constant representing coefficient of viscosity. Figure 5.11 shows a Stokes problem in two dimensions.

When these assumptions are given, a Stokes problem can be defined as a problem seeking flow velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and pressure $p : \Omega \to \mathbb{R}$ in the following way. Here, $(\boldsymbol{\nu} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} = \left(\boldsymbol{\nabla}\boldsymbol{u}^{\top}\right)^{\top} \boldsymbol{\nu}$ is written as $\partial_\nu \boldsymbol{u}$.

Problem 5.5.1 (Stokes problem) Let $\boldsymbol{b} : \Omega \to \mathbb{R}^d$, $\boldsymbol{u}_{\mathrm{D}} : \Omega \to \mathbb{R}^d$ and $\mu \in \mathbb{R}$ be given. Find $(\boldsymbol{u}, p) : \Omega \to \mathbb{R}^{d+1}$ such that the following equations,

$$-\boldsymbol{\nabla}^{\top}\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top}\right) + \boldsymbol{\nabla}^{\top} p = \boldsymbol{b}^{\top} \quad \text{in } \Omega, \tag{5.5.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{5.5.3}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad \text{on } \partial\Omega, \tag{5.5.4}$$

$$\int_{\Omega} p \,\mathrm{d}x = 0, \tag{5.5.5}$$

are satisfied. □

In Problem 5.5.1, Eq. (5.5.2) is called a Stokes equation and Eq. (5.5.3) is called a continuity equation. These are used to model the flow field of an incompressible fluid with the Newton viscosity.

Moreover, Eq. (5.5.2) can be written as

$$-\boldsymbol{\nabla}^{\top}\left(\mu \boldsymbol{\nabla} \boldsymbol{u}^{\top} - p\boldsymbol{I}\right) = \boldsymbol{b}^{\top} \quad \text{in } \Omega, \tag{5.5.6}$$

where $\boldsymbol{I}$ represents a unit matrix of d-th order. Moreover, defining Cauchy stress as

$$\boldsymbol{S}(\boldsymbol{u},p) = -p\boldsymbol{I} + 2\mu\boldsymbol{E}(\boldsymbol{u}) \tag{5.5.7}$$

using $\boldsymbol{E}(\boldsymbol{u})$ defined in Eq. (5.4.2), Eq. (5.5.2) can be written as

$$-\boldsymbol{\nabla}^\top \boldsymbol{S}(\boldsymbol{u},p) = \boldsymbol{b}^\top \quad \text{in } \Omega. \tag{5.5.8}$$

If Eq. (5.5.3) holds, these are equivalent to one another. In this chapter, Eq. (5.5.2) is used in order to look at the relationship with the abstract saddle point variational problem in Sect. 5.6.

The weak form with respect to Problem 5.5.1 can be obtained in the following way. Let the function space with respect to $\boldsymbol{u}$ be

$$U = H_0^1\left(\Omega;\mathbb{R}^d\right) = \left\{ \boldsymbol{u}\in H^1\left(\Omega;\mathbb{R}^d\right) \mid \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \partial\Omega \right\}. \tag{5.5.9}$$

Multiplying both sides of Eq. (5.5.2) by an arbitrary $\boldsymbol{v}\in U$ and integrating over Ω, then using the Gauss–Green theorem (Theorem A.8.2) gives

$$\begin{aligned}
&\int_\Omega \left\{ \boldsymbol{\nabla}^\top\left(\mu\boldsymbol{\nabla}\boldsymbol{u}^\top\right) - \boldsymbol{\nabla}^\top p + \boldsymbol{b}^\top \right\} \boldsymbol{v}\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \left(\mu\partial_\nu \boldsymbol{u} - p\boldsymbol{\nu}\right)\cdot\boldsymbol{v}\,\mathrm{d}\gamma \\
&\qquad + \int_\Omega \left(-\mu\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}^\top\right) + p\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{b}\cdot\boldsymbol{v}\right)\mathrm{d}x \\
&\quad = \int_\Omega \left(-\mu\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}^\top\right) + p\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{b}\cdot\boldsymbol{v}\right)\mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

The fact that this equation holds with respect to an arbitrary $\boldsymbol{v}\in U$ is referred to as the weak form of the Stokes equation.

On the other hand, let the function space with respect to p be

$$P = \left\{ q\in L^2(\Omega;\mathbb{R}) \;\middle|\; \int_\Omega q\,\mathrm{d}x = 0 \right\}. \tag{5.5.10}$$

Multiplying Eq. (5.5.3) by an arbitrary $q\in P$ and integrating over Ω gives

$$\int_\Omega q\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}x = 0.$$

The fact that this equation holds with respect to an arbitrary $q\in P$ is called the weak form of the continuity equation.

With respect to the Stokes problem, let

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mu\left(\boldsymbol{\nabla}\boldsymbol{u}^\top\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}^\top\right)\,\mathrm{d}x, \tag{5.5.11}$$

$$b(\boldsymbol{v}, q) = -\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{v} \, \mathrm{d}x, \tag{5.5.12}$$

$$l(\boldsymbol{v}) = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}x. \tag{5.5.13}$$

In this case, the weak form of the Stokes problem can be written as follows.

Problem 5.5.2 (Weak form of Stokes problem) Let U and P be given by Eq. (5.5.9) and Eq. (5.5.10), respectively. Suppose $\boldsymbol{u}_\mathrm{D} \in H^1(\Omega; \mathbb{R}^d)$ satisfies Eq. (5.5.1). Let μ be a positive constant and $a(\,\cdot\,,\,\cdot\,)$, $b(\,\cdot\,,\,\cdot\,)$ and $l(\,\cdot\,)$ are taken to be Eq. (5.5.11), Eq. (5.5.12) and Eq. (5.5.13), respectively. In this case, find $(\tilde{\boldsymbol{u}}, p) = (\boldsymbol{u} - \boldsymbol{u}_\mathrm{D}, p) \in U \times P$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = l(\boldsymbol{v}), \tag{5.5.14}$$

$$b(\boldsymbol{u}, q) = 0, \tag{5.5.15}$$

for an arbitrary $(\boldsymbol{v}, q) \in U \times P$. $\square$

5.6 Abstract Saddle Point Variational Problem

Now we have the weak form of the Stokes problem, let us look at what assumptions guarantee the existence of a unique solution of the given weak form.

A linear elastic problem is an elliptic partial differential equation with respect to displacement $\boldsymbol{u}$. Hence, the existence of a unique solution could be shown using the results with respect to an abstract variational problem or an abstract minimization problem. In contrast, a Stokes problem demands that the pressure p is added as an unknown variable in addition to the flow velocity $\boldsymbol{u}$, and that the continuity equation is satisfied simultaneously. This structure can seen to be a problem called the abstract saddle point variational problem or abstract saddle point problem corresponding to the abstract variational problem with constraints or abstract minimization problem with constraints. Here, let us show the existence of a unique solution with respect to the Stokes problem using certain definitions and results.

Let U and P be real Hilbert spaces, and the functions $a : U \times U \to \mathbb{R}$ and $b : U \times P \to \mathbb{R}$ be bounded bilinear operators defined on $U \times U$ and $U \times P$, respectively (Section 4.4.4). Also, let their norms be given by

$$\|a\| = \|a\|_{\mathcal{L}(U,U;\mathbb{R})} = \sup_{\boldsymbol{u},\boldsymbol{v} \in U \setminus \{\boldsymbol{0}_U\}} \frac{|a(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{u}\|_U \|\boldsymbol{v}\|_U},$$

$$\|b\| = \|b\|_{\mathcal{L}(U,P;\mathbb{R})} = \sup_{\boldsymbol{u} \in U \setminus \{\boldsymbol{0}_U\},\ q \in P \setminus \{0_P\}} \frac{|b(\boldsymbol{u}, q)|}{\|\boldsymbol{u}\|_U \|q\|_P},$$

respectively.

A set of functions which satisfy the continuity equation that become Hilbert spaces is defined as follows.

Definition 5.6.1 (Divergence free Hilbert space U_{div}) Let $b : U \times P \to \mathbb{R}$ be a bilinear form. In this case,

$$U_{\mathrm{div}} = \{ \boldsymbol{v} \in U | \ b(\boldsymbol{v}, q) = 0 \text{ for all } q \in P \}$$

is called a divergence free Hilbert space of U. □

Using these definitions, we consider the following problem.

Problem 5.6.2 (Abstract saddle point variational problem) Let $a : U \times U \to \mathbb{R}$ and $b : U \times P \to \mathbb{R}$ be bounded bilinear operators and $l \in U'$ and $r \in P'$ be given. Find $(\boldsymbol{u}, p) \in U \times P$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \langle l, \boldsymbol{v} \rangle, \\ b(\boldsymbol{u}, q) &= \langle r, q \rangle, \end{aligned}$$

with respect to an arbitrary $(\boldsymbol{v}, q) \in U \times P$. □

5.6.1 Existence Theorem of Solution

In reference to the existence of a unique solution to the abstract saddle point variational problem (Problem 5.6.2), the following result is known (cf. [4, Corollary 4.1, p. 61], [1, Theorem 1.1, p. 42], [6, Theorem 7.3, p. 135], [9, Theorem 4.3, p. 116]).

Theorem 5.6.3 (Solution to abstract saddle point variational problem) Suppose $a : U \times U \to \mathbb{R}$ is a coercive and bounded bilinear operator on U_{div} (i.e., there exists some $\alpha > 0$ and

$$|a(\boldsymbol{v}, \boldsymbol{v})| \geq \alpha \|\boldsymbol{v}\|_U^2$$

is satisfied with respect to an arbitrary $\boldsymbol{v} \in U_{\mathrm{div}}$). Also, let $b : U \times P \to \mathbb{R}$ be a bounded bilinear operator and that some $\beta > 0$ exists satisfying the inequality

$$\inf_{q \in P \setminus \{0_P\}} \sup_{\boldsymbol{v} \in U \setminus \{\boldsymbol{0}_U\}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_U \|q\|_P} \geq \beta. \tag{5.6.1}$$

In this case, the solution $(\boldsymbol{u}, p) \in U \times P$ to Problem 5.6.2 exists uniquely and with respect to $c > 0$ depending on α, β, $\|a\|$ and $\|b\|$,

$$\|u\|_U + \|p\|_P \leq c\left(\|l\|_{U'} + \|r\|_{P'}\right)$$

holds. □

Equation (5.6.1) is called the inf-sup condition or Ladysenskaja–Babuška–Brezzi condition, Babuška–Brezzi–Kikuchi condition, etc.

If Theorem 5.6.3 is used, the existence of a unique solution to the Stokes problem can be shown in the following way.

Exercise 5.6.4 (Existence of unique solution to the Stokes problem) In Problem 5.5.2, it is supposed that some function $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}} \in U_{\mathrm{div}}$ exists and satisfies $\tilde{\boldsymbol{u}} = \mathbf{0}_{\mathbb{R}^d}$ on $\partial\Omega$. In this case, show that $(\tilde{\boldsymbol{u}}, p) \in U \times P$ satisfying Eq. (5.5.14) and Eq. (5.5.15) exists uniquely. □

Answer Let us confirm that the assumptions of Theorem 5.6.3 hold in the following way. Let U and P be Hilbert spaces. Moreover, Problem 5.5.2 is equivalent to a problem seeking $(\tilde{\boldsymbol{u}}, p) \in U \times P$ which satisfies

$$a(\tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \hat{l}(\boldsymbol{v}), \quad b(\tilde{\boldsymbol{u}}, q) = \hat{r}(q),$$

with respect to an arbitrary $(\boldsymbol{v}, q) \in U \times P$, where

$$\hat{l}(\boldsymbol{v}) = l(\boldsymbol{v}) - a(\boldsymbol{u}_{\mathrm{D}}, \boldsymbol{v}), \quad \hat{r}(q) = -b(\boldsymbol{u}_{\mathrm{D}}, q).$$

We show that a is bounded and coercive on U_{div}. Clearly, a is bounded and coercive on U in view of Exercise 5.4.4. Next, we note that b is bounded and satisfies the inf-sup condition. In fact, when $U_{\mathrm{div}}^{\perp}$ is taken to be the orthogonal complement of U_{div}, an operator such that the domain of operator div is limited to $U_{\mathrm{div}}^{\perp}$ is taken to be τ. τ is bounded ($|\mathrm{div}\boldsymbol{v}| / \|\boldsymbol{v}\|_U < \infty$), linear and injective (with respect to $\boldsymbol{v}_1, \boldsymbol{v}_2 \in U_{\mathrm{div}}^{\perp}$, if $\tau\boldsymbol{v}_1 = \tau\boldsymbol{v}_2$, then $\boldsymbol{v}_1 = \boldsymbol{v}_2$). This is because with respect to $\boldsymbol{v} \in U_{\mathrm{div}}^{\perp}$, if $\tau\boldsymbol{v} = \mathrm{div}\boldsymbol{v} = 0$, then $\boldsymbol{v} \in U_{\mathrm{div}}$, and we get $\boldsymbol{v} \in U_{\mathrm{div}}^{\perp} \cap U_{\mathrm{div}} = \{\mathbf{0}_U\}$. Furthermore, it can be shown that τ is a surjection from $U_{\mathrm{div}}^{\perp}$ to P (see, e.g., [4] for the proof). Hence, the following inequalities,

$$\begin{aligned}
&\inf_{q \in P \setminus \{0_P\}} \sup_{\boldsymbol{v} \in U \setminus \{\mathbf{0}_U\}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_U \|q\|_P} \\
&= \inf_{q \in P \setminus \{0_P\}} \sup_{\boldsymbol{v} \in U \setminus \{\mathbf{0}_U\}} \frac{(-\mathrm{div}\boldsymbol{v}, q)_{L^2(\Omega;\mathbb{R})}}{\|\boldsymbol{v}\|_U \|q\|_P} \\
&\geq \inf_{q \in P \setminus \{0_P\}} \sup_{\boldsymbol{v} \in U \setminus \{\mathbf{0}_U\}} \frac{(-\tau\boldsymbol{v}, q)_P}{\|\tau^{-1}(-q)\|_U \|q\|_P} \geq \inf_{q \in P \setminus \{0_P\}} \frac{(q, q)_P}{\|\tau^{-1}(-q)\|_U \|q\|_P} \\
&\geq \frac{1}{\|\tau^{-1}\|_{\mathcal{L}\left(P; U_{\mathrm{div}}^{\perp}\right)}} > 0,
\end{aligned}$$

are established. On the other hand, $\hat{l} \in U'$ is already verified in Exercise 5.4.4. Moreover, from assumptions of Eq. (5.5.1), we see that $\hat{r}(q) = 0 \in P'$. As seen above, Theorem 5.6.3 can be applied and $(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}}, p) \in U \times P$ satisfying Problem 5.5.2 exists uniquely. □

5.6.2 Abstract Saddle Point Problem

In the abstract saddle point variational problem (Problem 5.6.2), if $a : U \times U \to \mathbb{R}$ is symmetric, its abstract saddle point variational problem is equivalent to an abstract saddle point problem such as the one below.

Problem 5.6.5 (Abstract saddle point problem) Let $a : U \times U \to \mathbb{R}$ and $b : U \times P \to \mathbb{R}$ be bounded bilinear operators. Given $(l, r) \in U' \times P'$, define

$$\mathscr{L}(\boldsymbol{v}, q) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, q) - \langle l, \boldsymbol{v} \rangle - \langle r, q \rangle.$$

In this case, find $(\boldsymbol{u}, p) \in U \times P$ such that

$$\mathscr{L}(\boldsymbol{u}, q) \leq \mathscr{L}(\boldsymbol{u}, p) \leq \mathscr{L}(\boldsymbol{v}, p)$$

for any $(\boldsymbol{v}, q) \in U \times P$. □

The following results can be obtained with respect to Problem 5.6.5 (cf. [4, Theorem 4.2, p. 62], [9, Theorem 4.4, p. 118]).

Theorem 5.6.6 (Agreement of abstract saddle point problems) If a is symmetric ($a(\boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{v}, \boldsymbol{u})$) and semi-positive definite (with respect to an arbitrary $\boldsymbol{v} \in U$, $a(\boldsymbol{v}, \boldsymbol{v}) \geq 0$ holds), then the solution to Problem 5.6.2 and the solution to Problem 5.6.5 agree. □

Let us check that in an abstract saddle point problem (Problem 5.6.5), $q \in P$ is a Lagrange multiplier with respect to equality constraint such as the continuity equation. Problem 5.6.5 is the problem, when setting

$$f(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - \langle l, \boldsymbol{v} \rangle,$$

seeking $(\boldsymbol{v}, q)$ which satisfies

$$\min_{(\boldsymbol{v}, q) \in U \times P} \{ f(\boldsymbol{v}) \mid b(\boldsymbol{v}, q) - \langle r, q \rangle = 0 \}. \tag{5.6.2}$$

Here, $\mathscr{L}(\boldsymbol{v}, q)$ of Problem 5.6.5 is the Lagrange function of this problem and $q \in P$ is a Lagrange multiplier with respect to equality constraints. Theorem 5.6.6 which shows that the solution to Eq. (5.6.2) matches the saddle points of Problem 5.6.5 is a result corresponding to the duality theorem (Theorem 2.9.2).

5.7 Summary

In Chap. 5, we defined boundary value problems of elliptic partial differential equations, sought their weak form and studied the existence of a solution and its regularity. The key points from this chapter are as follows.

(1) The existence of a unique solution for a boundary value problem of an elliptic partial differential equation (Poisson problem) is guaranteed when the assumptions of the Lax–Milgram theorem are satisfied with respect to the weak form (Sections 5.1 and 5.2).

(2) The regularity of a solution with respect to a boundary value problem of an elliptic partial differential equation depends on the regularity of given functions and the regularity of the boundary (Sect. 5.3).

(3) A linear elastic problem is a boundary value problem of an elliptic partial differential equation. The existence of a unique solution with respect to this problem is guaranteed when the assumptions of the Lax–Milgram theorem are satisfied with respect to the weak form (Sect. 5.4).

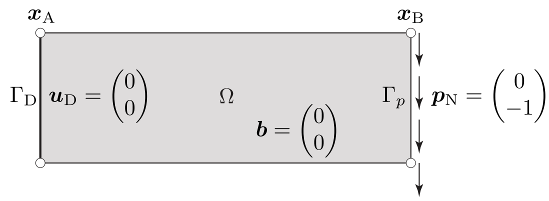


Fig. 5.12: Cantilever problem of linear elastic body.

(4) A Stokes problem is a boundary value problem of an elliptic partial differential equation with a continuity equation as equality constraint. The existence of a unique solution of a Stokes problem is guaranteed when the assumptions for the existence of solutions with respect to an abstract saddle point variational problem using an inf-sup condition with respect to the weak form (Sect. 5.5) are satisifed.

5.8 Practice Problems

5.1 With respect to $b : \Omega \to \mathbb{R}$, $u_{\mathrm{D}} : \Omega \to \mathbb{R}$, obtain the weak form of the boundary value problem seeking $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{aligned} -\Delta u + u &= b \quad && \text{in } \Omega, \\ u &= u_{\mathrm{D}} \quad && \text{on } \partial\Omega. \end{aligned}$$

Moreover, determine the appropriate function spaces for b and u_{D} with respect to the unique solution u of the corresponding weak form of the above system.

5.2 Consider a cantilever problem of a linear elastic body such as that in Fig. 5.12. In this case, show that, even though the point $\boldsymbol{x}_{\mathrm{A}}$ is not a singular point, $\boldsymbol{x}_{\mathrm{B}}$ is.

5.3 A dynamic linear elastic problem can be expressed as follows: "With respect to $\boldsymbol{b} : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$, $\boldsymbol{p}_{\mathrm{N}} : \Gamma_{\mathrm{N}} \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$, $\boldsymbol{u}_{\mathrm{D}} : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$, $\boldsymbol{u}_{\mathrm{D}0} : \Omega \to \mathbb{R}^d$, $\boldsymbol{u}_{\mathrm{D}\top} : \Omega \to \mathbb{R}^d$ and $\rho > 0$, obtain $\boldsymbol{u} : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$ which satisfies

$$\begin{aligned} \rho \ddot{\boldsymbol{u}}^{\top} - \boldsymbol{\nabla} \cdot \boldsymbol{S}(\boldsymbol{u}) &= \boldsymbol{b}^{\top} \quad && \text{in } \Omega \times (0, t_{\mathrm{T}}), \\ \boldsymbol{S}(\boldsymbol{u}) \boldsymbol{\nu} &= \boldsymbol{p}_{\mathrm{N}} \quad && \text{on } \Gamma_{\mathrm{N}} \times (0, t_{\mathrm{T}}), \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}} \quad && \text{on } \Gamma_{\mathrm{D}} \times (0, t_{\mathrm{T}}), \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}0} \quad && \text{in } \Omega \times \{0\}, \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}\top} \quad && \text{in } \Omega \times \{t_{\mathrm{T}}\}, \end{aligned}$$

where $\dot{\boldsymbol{u}} = \partial \boldsymbol{u} / \partial t$ with respect to time $t \in (0, t_{\mathrm{T}})$." Obtain the weak form of this problem.

5.4 In Exercise **5.3**, when $\boldsymbol{b} = \mathbf{0}_{\mathbb{R}^d}$, $\boldsymbol{p}_{\mathrm{N}} = \mathbf{0}_{\mathbb{R}^d}$ and $\boldsymbol{u}_{\mathrm{D}} = \mathbf{0}_{\mathbb{R}^d}$, when with respect to $(\boldsymbol{x}, t) \in \Omega \times (0, t_{\mathrm{T}})$, a solution of variable separation (standing wave) is assumed to be

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\phi}(\boldsymbol{x}) \mathrm{e}^{\lambda t}.$$

The problem of seeking $\boldsymbol{\phi} : \Omega \to \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ is called an eigenfrequency problem. Obtain the weak form of this problem.

5.5 With respect to $\boldsymbol{b} : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$, $\boldsymbol{u}_{\mathrm{D}} : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d$, $\mu > 0$ and $\rho > 0$, the problem seeking $(\boldsymbol{u}, p) : \Omega \times (0, t_{\mathrm{T}}) \to \mathbb{R}^d \times \mathbb{R}$ which satisfies

$$\begin{aligned} \rho \dot{\boldsymbol{u}} + \rho (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} - \mu \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{b} \quad & \text{in } \Omega \times (0, t_{\mathrm{T}}), \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega \times (0, t_{\mathrm{T}}), \\ \boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad & \text{on } \{\partial\Omega \times (0, t_{\mathrm{T}})\} \cup \{\Omega \times \{0\}\} \end{aligned}$$

is called a Navier–Stokes problem. The first equation is called a Navier–Stokes equation and the second equation is called a continuity equation. Obtain the weak form of this problem.

5.6 With respect to an isotropic linear elastic body, show that the relationship $e_{\mathrm{Y}} = 2\mu_{\mathrm{L}} (1 + \nu_{\mathrm{P}})$ holds between Young's modulus e_{Y}, elastic shear modulus μ_{L} and Poisson's ratio ν_{P}.

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