Contents

Chapter 4

Basics of Variational Principles and Functional Analysis

In last chapters, we looked at the theory of solutions relating to optimization problems in finite-dimension. From this chapter onward, we shall consider optimization problems where the design variables are of function of time and space.

We shall first examine variational principles in the field of mechanics and see if they have the structure of a function optimization problem, and if the equation of motion, etc. corresponds to the optimality condition. We can then start to prepare the tools which are needed to deal with function optimization problems. In optimization problems until Chap. 3, the linear space containing the design variables was a finite-dimensional vector space. In contrast, this chapter prepares the function space as the linear space containing design variables. However, in order to be able to explain function spaces, we must start with the definition of a linear space and provide explanations regarding several abstract spaces such as a continuous (complete) distance space in which a limit operation can be used and a linear space in which inner product can be studied. The various function spaces will be explained with consideration to their relationship with the abstract spaces.

Once a function space is defined, we want to consider a mapping between function spaces. Here such a mapping is defined as an operator and its boundedness and linearity will be explained first. The trace operator is an example of an operator and it will be introduced in Chap. 5 and used subsequently when showing the existence of solutions for boundary value problems of partial differential equations and in error evaluation of numerical analysis. After that, operators with a range limited to real numbers are defined and referred to as functionals. Even among such functionals, the set of bounded and linear functionals is referred to as a dual space with respect to the function

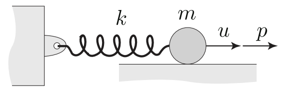


Fig. 4.1: Spring mass system with one degree of freedom.

space which is a domain in the functional. A dual space is a function space used as a gradient in Fréchet derivatives among the generalized derivatives of an operator that will be shown later.

After gathering the tools required in function optimization problems as mentioned above, we return to the variational principles again to completely describe function spaces used in the variational principles. In this regard, variational principles are viewed as optimization problems in function spaces. Hence, optimality conditions can be confirmed to be given by the conditions under which the Fréchet differential is zero (KKT conditions in constrained problems). The understanding of variational principles as a function optimization problem will be of immediate use in Chap. 5.

4.1 Variational Principles

Hamilton's principle and minimum principle of potential energy which are well-known in mechanics show that a motion equation or a equilibrium equation of forces can be obtained as a stationary point of some energy. Furthermore, the optimum control law of a control system is known as being obtainable via Pontryagin's minimum principle with respect to an optimum control problem. Here we will look at how these problems have the structure of a function optimization problem.

4.1.1 Hamilton's Principle

Let us consider a spring mass system such as one shown in Fig. 4.1. Let k and m be positive constants representing the spring constant and mass. Let t_{T} be a positive constant representing final time, and $p : (0, t_{\mathrm{T}}) \to \mathbb{R}$ and $u : (0, t_{\mathrm{T}}) \to \mathbb{R}$ be functions of time $(0, t_{\mathrm{T}})$ expressing external force and displacement, respectively. Here, given p, we seek the motion equation for determining u using Hamilton's principle.

With respect to time $t \in (0, t_{\mathrm{T}})$, let $\dot{u} = \partial u / \partial t$ denote the velocity and functions $\kappa(u, \dot{u})$ and $\pi(u, \dot{u})$ of u and $\dot{u}$ be the kinetic energy and potential energy, respectively. In this case, the Lagrange function in mechanics with respect to the spring mass system of Fig. 4.1 is given by

$$l(u) = \kappa(u, \dot{u}) - \pi(u, \dot{u}) = \frac{1}{2} m \dot{u}^2 - \frac{1}{2} k u^2 + pu. \tag{4.1.1}$$

We have added the phrase "in mechanics" to distinguish it from the Lagrange

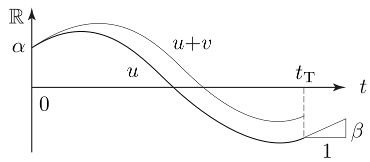


Fig. 4.2: Expanded Hamilton's principle displacement u and arbitrary variation v.

function used until Chap. 3. Furthermore, the action integral is given by

$$a(u) = \int_0^{t_\mathrm{T}} l(u)\,\mathrm{d}t. \tag{4.1.2}$$

Hamilton's principle forces u to be determined so that $a(u)$ of Eq. (4.1.2) becomes stationary while the displacements $u(0)$ and $u(t_\mathrm{T})$ for times $t=0$ and $t=t_\mathrm{T}$ are fixed with given values. Let us think about the meaning of Hamilton's principle within the solution of the following problem. In this next problem, however, the terminal condition of Hamilton's principle has been changed for future use.

Problem 4.1.1 (Extended Hamilton's principle) Let α and β be fixed constants, U be the set of $u : (0, t_\mathrm{T}) \to \mathbb{R}$ satisfying $u(0) = \alpha$ and $l(u)$ be Eq. (4.1.1). Moreover, let the expanded action integral be

$$f(u) = \int_0^{t_\mathrm{T}} l(u)\,\mathrm{d}t - m\beta u(t_\mathrm{T}).$$

When u varies arbitrarily within the set U, find the conditions at which f becomes stationary. □

In Problem 4.1.1, u is an element of the set U of functions satisfying the condition $u(0) = \alpha$. Hence if the function representing an arbitrary variation from u is written as $v : (0, t_\mathrm{T}) \to \mathbb{R}$, there is a need for v to satisfy the condition $v(0) = 0$. Figure 4.2 shows an example of u and v. A set of such v will be expressed as V here. In this case, we have

$$\begin{aligned}
f(u+v) &= \int_0^{t_\mathrm{T}} \left\{ \frac{1}{2} m (\dot{u}+\dot{v})^2 - \frac{1}{2} k (u+v)^2 + p(u+v) \right\} \mathrm{d}t \\
&\quad - m\beta \left(u(t_\mathrm{T}) + v(t_\mathrm{T}) \right) \\
&= f(u) + \left\{ \int_0^{t_\mathrm{T}} (m\dot{u}\dot{v} - kuv + pv)\,\mathrm{d}t - m\beta v(t_\mathrm{T}) \right\} \\
&\quad + \int_0^{t_\mathrm{T}} \left(\frac{1}{2} m\dot{v}^2 - \frac{1}{2} k v^2 \right) \mathrm{d}t \\
&= f(u) - \left\{ \int_0^{t_\mathrm{T}} (m\ddot{u} + ku - p)\, v\,\mathrm{d}t - m\left(\dot{u}(t_\mathrm{T}) - \beta\right) v(t_\mathrm{T}) \right\}
\end{aligned}$$

$$+\int_0^{t_{\mathrm{T}}} \left(\frac{1}{2}m\dot{v}^2 - \frac{1}{2}kv^2\right) \mathrm{d}t \tag{4.1.3}$$

for every $v \in V$. Here $v(0) = 0$ and partial integration with respect to the integral of $m\dot{u}\dot{v}$ were used in the last equation. Let us rewrite the right-hand side of this equation collating each term in terms of the order of v:

$$f(u+v) = f(u) + f'(u)[v] + \frac{1}{2}f''(u)[v,v]. \tag{4.1.4}$$

The terms $f'(u)[v]$ and $f''(u)[v,v]$ in Eq. (4.1.4) are referred to as the first variation and second variation of f at u in calculus of variations. The condition for f to be stationary is defined as the condition under which the first variation is zero with respect to an arbitrary $v \in V$. In this problem, this condition is given as

$$f'(u)[v] = -\int_0^{t_{\mathrm{T}}} (m\ddot{u} + ku - p)\, v \,\mathrm{d}t + m(\dot{u}(t_{\mathrm{T}}) - \beta)\, v(t_{\mathrm{T}}) = 0. \tag{4.1.5}$$

This condition that holds for an arbitrary $v \in V$ is equivalent to the validity of the following equations:

$$m\ddot{u} + ku = p \quad \text{in } (0, t_{\mathrm{T}}), \tag{4.1.6}$$

$$\dot{u}(t_{\mathrm{T}}) = \beta. \tag{4.1.7}$$

Equations (4.1.6) and (4.1.7) are called the motion equation and terminal condition of velocity, respectively. In Example 4.5.5, it can be verified that $f'(u)[v]$ and $f''(u)[v,v]$ satisfy the definitions of the Fréchet derivative and second-order Fréchet derivative (Definition 4.5.4), respectively. In Sect. 4.6.1, we will look in detail at what sort of function space U is and what type of function space is prepared with respect to p.

4.1.2 Minimum Principle of Potential Energy

Let us consider a one-dimensional linear elastic body such as the one in Fig. 4.3. Let l be a positive constant representing the length, $a_{\mathrm{S}} : (0,l) \to \mathbb{R}$ and $e_{\mathrm{Y}} : (0,l) \to \mathbb{R}$ be functions taking positive values representing the cross-section and Young's modulus respectively. Also, let $b : (0,l) \to \mathbb{R}$, $p_{\mathrm{N}} \in \mathbb{R}$ and $u : (0,l) \to \mathbb{R}$ denote the volume force (force per unit volume), the traction (force per unit area) at $x = l$ and displacement, respectively. Here, let us show that when something other than u is given, the minimum principle of potential energy can be used to obtain the equilibrium equation of forces that determines u.

As seen in Example 1.1.1, setting $\pi_{\mathrm{I}}(u)$ and $\pi_{\mathrm{E}}(u)$ as internal potential energy (elastic potential energy) and external potential energy (potential energy of external force) respectively, the potential energy of the whole system when $u = 0$ is the baseline is defined by

$$\pi(u) = \pi_{\mathrm{I}}(u) + \pi_{\mathrm{E}}(u)$$

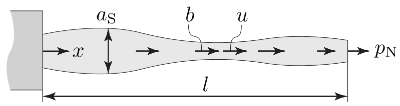


Fig. 4.3: One-dimensional linear elastic body.

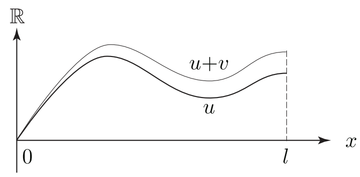


Fig. 4.4: Potential energy minimum principle displacement u and arbitrary variation v.

$$= \int_0^l \frac{1}{2}\sigma(u)\,\varepsilon(u)\,a_S\,\mathrm{d}x - \int_0^l b u a_S\,\mathrm{d}x - p_N u(l)\,a_S(l)\,. \tag{4.1.8}$$

In this case, the strain and stress are set as

$$\varepsilon(u) = \frac{\mathrm{d}u}{\mathrm{d}x} = \nabla u,$$
$$\sigma(u) = e_Y \varepsilon(u)\,,$$

respectively.

Let us obtain the conditions of u from the minimum principle of potential energy with respect to Fig. 4.3.

Problem 4.1.2 (Minimum principle of potential energy) Let U denote the set of all $u : (0,l) \to \mathbb{R}$ such that $u(0) = 0$ and $\pi(u)$ as Eq. (4.1.8). Find the conditions for u such that

$$\min_{u \in U} \pi(u)\,.$$

□

In Problem 4.1.2, u is an element of the set U of functions satisfying $u(0) = 0$. Denote the function expressing the arbitrary variation from u as $v : (0,l) \to \mathbb{R}$. In this case, there is a need for v to satisfy $v(0) = 0$. Figure 4.4 shows examples of such u and v. Hence, the set of v is the same as U. Here, we have

$$\begin{aligned}\pi(u+v) &= \int_0^l \frac{1}{2} e_Y (\nabla u + \nabla v)^2 a_S\,\mathrm{d}x - \int_0^l b(u+v)\,a_S\,\mathrm{d}x \\ &\quad - p_N (u(l) + v(l))\,a_S(l) \\ &= \pi(u) + \left\{ \int_0^l (e_Y \nabla u \nabla v - b v)\,a_S\,\mathrm{d}x - p_N v(l)\,a_S(l) \right\}\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^l \frac{1}{2} e_{\mathrm{Y}} (\nabla v)^2 a_{\mathrm{S}} \, \mathrm{d}x \\
&= \pi(u) \\
&\quad + \left[\int_0^l \{ -\nabla (e_{\mathrm{Y}} \nabla u) - b \} v \, a_{\mathrm{S}} \, \mathrm{d}x + (e_{\mathrm{Y}} \nabla u(l) - p_{\mathrm{N}}) v(l) a_{\mathrm{S}}(l) \right] \\
&\quad + \int_0^l \frac{1}{2} e_{\mathrm{Y}} (\nabla v)^2 a_{\mathrm{S}} \, \mathrm{d}x
\end{aligned}
\tag{4.1.9}
$$

for every $v \in U$, where $v(0) = 0$ and partial integrals relating to the integral of $e_{\mathrm{Y}} \nabla u \nabla v$ were used for the last equation. The right-hand side of this equation is written as

$$
\pi(u + v) = \pi(u) + \pi'(u)[v] + \frac{1}{2} \pi''(u)[v, v] \tag{4.1.10}
$$

by collating for each order of v. Here the stationary condition of $\pi(u)$ is that the first variation of π,

$$
\pi'(u)[v] = \int_0^l (-\nabla (e_{\mathrm{Y}} \nabla u) - b) v \, a_{\mathrm{S}} \, \mathrm{d}x + (e_{\mathrm{Y}} \nabla u(l) - p_{\mathrm{N}}) v(l) a_{\mathrm{S}}(l),
$$

is zero with respect to an arbitrary $v \in U$. This condition is equivalent to

$$
-\nabla (e_{\mathrm{Y}} \nabla u) = -\nabla \sigma(u) = b \quad \text{in } (0, l), \tag{4.1.11}
$$

$$
\sigma(u(l)) = e_{\mathrm{Y}} \nabla u(l) = p_{\mathrm{N}}. \tag{4.1.12}
$$

Furthermore, with respect to the second variation of π, we have

$$
\pi''(u)[v, v] = \int_0^l \frac{1}{2} e_{\mathrm{Y}} (\nabla v)^2 a_{\mathrm{S}} \, \mathrm{d}x \geq \alpha \int_0^l (\nabla v)^2 \, \mathrm{d}x \tag{4.1.13}
$$

for every $v \in U$. Here $\alpha = \min_{x \in (0,l)} e_{\mathrm{Y}}(x) \min_{x \in (0,l)} a_{\mathrm{S}}(x) / 2 > 0$. From this, the stationary condition of Eq. (4.1.11) and Eq. (4.1.12) express the minimum condition.

Equation (4.1.13) corresponds to the fact that the Hessian matrix of the potential energy was positive definite in Example 2.4.8. In a function optimization problem, the fact that Eq. (4.1.13) is satisfied is expressed as $\pi''(u)[v, v]$ is coercive (Definition 5.2.1).

4.1.3 Pontryagin's Minimum Principle

Finally, let us consider an optimum control problem as an example of using constraints. This section provides suggestions for considering an optimum design problem in the case when time-evolution problems or non-linear problems are set as state determination problems but those in a hurry should skip this section.

Firstly, let us consider the state equations of the system. The motion equation for a system with $n \in \mathbb{N}$ degrees of freedom can be written generally as

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{\xi}. \tag{4.1.14}$$

Here $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are seen as matrices in $\mathbb{R}^{n\times n}$ which express the mass, damping and stiffness, respectively. Moreover, $\boldsymbol{\xi} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ and $\boldsymbol{u} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ are seen as the control force and displacement, respectively. Let $\boldsymbol{v} = \dot{\boldsymbol{u}}$. In this case, Eq. (4.1.14) can be rewritten as

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0}_{\mathbb{R}^{n\times n}} \\ \boldsymbol{0}_{\mathbb{R}^{n\times n}} & \boldsymbol{M} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{v}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0}_{\mathbb{R}^{n\times n}} & -\boldsymbol{I} \\ \boldsymbol{K} & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{\mathbb{R}^{n}} \\ \boldsymbol{\xi} \end{pmatrix}. \tag{4.1.15}$$

In this way the second-order ordinary differential equation with constant coefficient such as Eq. (4.1.14) can be rewritten as a first-order ordinary differential equation with the variable doubled as Eq. (4.1.15). A similar expression also holds for higher-orders.

Let us redefine the symbols, take the control force to be of $d \in \mathbb{N}$ dimensions and define the state determination problem in optimum control problem as follows.

Problem 4.1.3 (Linear system with control) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times d}$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ and control force $\boldsymbol{\xi} : (0, t_{\mathrm{T}}) \to \mathbb{R}^d$ be given. Find the system status $\boldsymbol{u} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ which satisfies

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{\xi} \quad \text{in } (0, t_{\mathrm{T}}), \tag{4.1.16}$$

$$\boldsymbol{u}(0) = \boldsymbol{\alpha}. \tag{4.1.17}$$

□

Having defined the Lagrange function $\mathscr{L}_{\mathrm{S}}$ for the stepped one-dimensional linear elastic problem (Problem 1.1.3) as Eq. (1.1.12), let

$$\mathscr{L}_{\mathrm{S}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}) = \int_0^{t_{\mathrm{T}}} -(\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}) \cdot \boldsymbol{z}\, \mathrm{d}t \tag{4.1.18}$$

be the Lagrange function for Problem 4.1.3, where $\boldsymbol{z} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ has been introduced as a Lagrange multiplier with respect to Problem 4.1.3.

Use control force $\boldsymbol{\xi}$ and the solution $\boldsymbol{u}$ of Problem 4.1.3 to set the cost function of optimum control to be

$$f_0(\boldsymbol{\xi}, \boldsymbol{u}) = \frac{1}{2}\int_0^{t_{\mathrm{T}}} \left(\|\boldsymbol{u}\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 \right) \mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}(t_{\mathrm{T}})\|_{\mathbb{R}^n}^2. \tag{4.1.19}$$

Moreover, the constraint of the control force is set as

$$\frac{1}{2}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 - 1 \le 0 \quad \text{in } (0, t_{\mathrm{T}}). \tag{4.1.20}$$

Here the optimum control problem is constructed as follows.

Problem 4.1.4 (Optimum control problem of linear system) Let Ξ be the set of functions $\boldsymbol{\xi} : (0, t_{\mathrm{T}}) \to \mathbb{R}^d$ and U be the set of functions $\boldsymbol{u} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$. Let f_0 be given by Eq. (4.1.19). In this case, obtain the KKT conditions with respect to $\boldsymbol{\xi}$ satisfying

$$\min_{(\boldsymbol{\xi},\boldsymbol{u})\in \Xi\times U} \{ f_0 (\boldsymbol{\xi}, \boldsymbol{u}) \mid \text{Eq. (4.1.20)},\ \text{Problem 4.1.3} \}.$$

□

Problem 4.1.4 is an optimization problem with equality and inequality constraints and has the same structure as Problem 2.8.1. In Problem 2.8.1, Ξ and U were defined as finite-dimensional vector spaces. Here let us extend Ξ and U to an infinite vector space and formally obtain the KKT conditions.

Define the Lagrange function of Problem 4.1.4 as follows. Let $\boldsymbol{z}_0 : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ $(\boldsymbol{z}_0 (t_{\mathrm{T}}) = \boldsymbol{u} (t_{\mathrm{T}}))$ be the Lagrange multiplier with respect to Problem 4.1.3 for f_0 and the set of them be expressed as Z. Moreover, let $p : (0, t_{\mathrm{T}}) \to \mathbb{R}$ be the Lagrange multiplier with respect to Eq. (4.1.20) and the set of these be P. In this case, with respect to an arbitrary $(\boldsymbol{z}_0, p) \in Z \times P$, let

$$\mathscr{L} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) = \mathscr{L}_0 (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) + \mathscr{L}_1 (\boldsymbol{\xi}, p) \tag{4.1.21}$$

be the Lagrange function of Problem 4.1.4, where

$$\begin{aligned} \mathscr{L}_0 (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) &= f_0 (\boldsymbol{\xi}, \boldsymbol{u}) + \mathscr{L}_{\mathrm{S}} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) \\ &= \int_0^{t_{\mathrm{T}}} \left\{ \frac{\|\boldsymbol{u}\|^2_{\mathbb{R}^n}}{2} + \frac{\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d}}{2} - (\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}) \cdot \boldsymbol{z}_0 \right\} \mathrm{d}t \\ &\quad + \frac{1}{2} \|\boldsymbol{u} (t_{\mathrm{T}})\|^2_{\mathbb{R}^n}, \end{aligned} \tag{4.1.22}$$

$$\mathscr{L}_1 (\boldsymbol{\xi}, p) = \int_0^{t_{\mathrm{T}}} \left(\frac{\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d}}{2} - 1 \right) p \, \mathrm{d}t \tag{4.1.23}$$

are the Lagrange functions with respect to $f_0 (\boldsymbol{\xi}, \boldsymbol{u})$ and Eq. (4.1.20), respectively. Let an arbitrary variation of $(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) \in \Xi \times U \times Z \times P$ be $\{\boldsymbol{\eta}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{z}}_0, \hat{p}\} \in \Xi \times V \times W \times P$, where V is the set of $\hat{\boldsymbol{u}} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ satisfying $\hat{\boldsymbol{u}} (0) = \boldsymbol{0}_{\mathbb{R}^n}$ and W is the set of $\hat{\boldsymbol{z}}_0 : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ satisfying $\hat{\boldsymbol{z}}_0 (t_{\mathrm{T}}) = \boldsymbol{0}_{\mathbb{R}^n}$. Then, the first variation of $\mathscr{L}$ with respect to an arbitrary $\{\boldsymbol{\eta}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{z}}_0, \hat{p}\} \in \Xi \times V \times W \times P$ is

$$\begin{aligned} \mathscr{L}' (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) [\boldsymbol{\eta}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{z}}_0, \hat{p}] &= \mathscr{L}_{0\boldsymbol{\xi}} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) [\boldsymbol{\eta}] + \mathscr{L}_{1\boldsymbol{\xi}} (\boldsymbol{\xi}, p) [\boldsymbol{\eta}] \\ &\quad + \mathscr{L}_{0\boldsymbol{u}} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) [\hat{\boldsymbol{u}}] + \mathscr{L}_{0\boldsymbol{z}_0} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) [\hat{\boldsymbol{z}}_0] + \mathscr{L}_{1p} (\boldsymbol{\xi}, p) [\hat{p}]. \end{aligned} \tag{4.1.24}$$

The fourth and fifth terms on the right-hand side of Eq. (4.1.24) are

$$\mathscr{L}_{0\boldsymbol{z}_0} (\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) [\hat{\boldsymbol{z}}_0] = - \int_0^{t_{\mathrm{T}}} (\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}) \cdot \hat{\boldsymbol{z}}_0 \, \mathrm{d}t,$$

$$\mathscr{L}_{1p}\left(\boldsymbol{\xi}, p\right)\left[\hat{p}\right] = \int_0^{t_{\mathrm{T}}} \left(\frac{\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2}{2} - 1 \right) \hat{p}\,\mathrm{d}t,$$

respectively. When $\boldsymbol{u}$ is the solution of the state determination problem (Problem 4.1.3) and Eq. (4.1.20) is satisfied, these terms are zero. Moreover, the third term on the right-hand side of Eq. (4.1.24) is

$$\begin{aligned}
&\mathscr{L}_{0\boldsymbol{u}}\left(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0\right)\left[\hat{\boldsymbol{u}}\right] \\
&\quad = \int_0^{t_{\mathrm{T}}} \left\{ \boldsymbol{u} \cdot \hat{\boldsymbol{u}} - \left(\dot{\hat{\boldsymbol{u}}} - \boldsymbol{A}\hat{\boldsymbol{u}} \right) \cdot \boldsymbol{z}_0 \right\} \mathrm{d}t + \boldsymbol{u}\left(t_{\mathrm{T}}\right) \cdot \hat{\boldsymbol{u}}\left(t_{\mathrm{T}}\right) \\
&\quad = \int_0^{t_{\mathrm{T}}} \left(\boldsymbol{u} + \dot{\boldsymbol{z}}_0 + \boldsymbol{A}^{\top}\boldsymbol{z}_0 \right) \cdot \hat{\boldsymbol{u}}\,\mathrm{d}t + \left(\boldsymbol{u}\left(t_{\mathrm{T}}\right) - \boldsymbol{z}_0\left(t_{\mathrm{T}}\right)\right) \cdot \hat{\boldsymbol{u}}\left(t_{\mathrm{T}}\right),
\end{aligned}$$

where $\hat{\boldsymbol{u}}(0) = \boldsymbol{0}_{\mathbb{R}^n}$ was used. This term is zero when $\boldsymbol{z}_0$ is the solution to the following adjoint problem.

Problem 4.1.5 (Adjoint problem with respect to f_0) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be as per Problem 4.1.3. Find $\boldsymbol{z}_0 : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ such that

$$\dot{\boldsymbol{z}}_0 = -\boldsymbol{A}^{\top}\boldsymbol{z}_0 - \boldsymbol{u} \quad \text{in } \left(0, t_{\mathrm{T}}\right), \tag{4.1.25}$$

$$\boldsymbol{z}_0\left(t_{\mathrm{T}}\right) = \boldsymbol{u}\left(t_{\mathrm{T}}\right). \tag{4.1.26}$$

□

The first and the second term on the right-hand side of Eq. (4.1.24) become

$$\mathscr{L}_{0\boldsymbol{\xi}}\left(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0\right)\left[\boldsymbol{\eta}\right] = \int_0^{t_{\mathrm{T}}} \left(\boldsymbol{\xi} + \boldsymbol{B}^{\top}\boldsymbol{z}_0 \right) \cdot \boldsymbol{\eta}\,\mathrm{d}t = \left\langle \boldsymbol{g}_0, \boldsymbol{\eta} \right\rangle,$$

$$\mathscr{L}_{1\boldsymbol{\xi}}\left(\boldsymbol{\xi}, p\right)\left[\boldsymbol{\eta}\right] = \int_0^{t_{\mathrm{T}}} p\boldsymbol{\xi} \cdot \boldsymbol{\eta}\,\mathrm{d}t = \left\langle \boldsymbol{g}_1, \boldsymbol{\eta} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes a dual space (Definition 4.4.5). In this case, it is seen as equivalent to the inner product in a finite-dimensional vector space. In addition, in view of the fact that the KKT conditions of Problem 2.8.2 are given by Eq. (2.8.5) to Eq. (2.8.8), the KKT conditions with respect to the solution $\boldsymbol{\xi}$ to Problem 4.1.4 become

$$\boldsymbol{g}_0 + \boldsymbol{g}_1 = (1 + p)\,\boldsymbol{\xi} + \boldsymbol{B}^{\top}\boldsymbol{z}_0 = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{in } \left(0, t_{\mathrm{T}}\right), \tag{4.1.27}$$

$$\frac{1}{2}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 \le 1 \quad \text{in } \left(0, t_{\mathrm{T}}\right), \tag{4.1.28}$$

$$\left(\frac{1}{2}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 - 1 \right) p = 0 \quad \text{in } \left(0, t_{\mathrm{T}}\right), \tag{4.1.29}$$

$$p \ge 0 \quad \text{in } \left(0, t_{\mathrm{T}}\right). \tag{4.1.30}$$

Let us rewrite the KKT condition obtained here into a different expression. Suppose $\boldsymbol{\xi}$, $\boldsymbol{u}$, $\boldsymbol{z}_0$ satisfies Eq. (4.1.27) to Eq. (4.1.30) and $\boldsymbol{\zeta} \in \mathbb{R}^d$ is taken to be an arbitrary vector satisfying $\|\boldsymbol{\zeta}\|_{\mathbb{R}^d}^2/2 \leq 1$. In this case, if $\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2/2 < 1$, $p = 0$ is true, then Eq. (4.1.27) implies that

$$\langle \boldsymbol{g}_0, \boldsymbol{\zeta} - \boldsymbol{\xi} \rangle = \left(\boldsymbol{\xi} + \boldsymbol{B}^\top \boldsymbol{z}_0 \right) \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) = 0 \quad \text{in } (0, t_{\mathrm{T}}) .$$

Moreover, if $\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2/2 = 1$, then we get $p > 0$, $\boldsymbol{\xi} \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) \leq 0$ and

$$\langle \boldsymbol{g}_0 + \boldsymbol{g}_1, \boldsymbol{\zeta} - \boldsymbol{\xi} \rangle = \left(\boldsymbol{\xi} + \boldsymbol{B}^\top \boldsymbol{z}_0 \right) \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) + p \boldsymbol{\xi} \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) = 0 \quad \text{in } (0, t_{\mathrm{T}}) .$$

Therefore, in any case, the inequality

$$\left(\boldsymbol{\xi} + \boldsymbol{B}^\top \boldsymbol{z}_0 \right) \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) = \boldsymbol{g}_0 \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) \geq 0 \quad \text{in } (0, t_{\mathrm{T}}) \tag{4.1.31}$$

holds. Eq. (4.1.31) shows the fact that $\boldsymbol{\xi}$ is a local minimizer of the Lagrange function $\mathscr{L}_0$ of the cost function f_0. This condition is called Pontryagin's local minimum condition.

Furthermore, when the Hamilton function is defined as

$$\mathscr{H}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}) = (\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{\xi}) \cdot \boldsymbol{z} + \frac{1}{2} \left(\|\boldsymbol{u}\|_{\mathbb{R}^n}^2 + \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2 \right),$$

the adjoint problem (Problem 4.1.5) can be written as

$$\begin{aligned} \dot{\boldsymbol{z}}_0 \cdot \hat{\boldsymbol{u}} &= -\mathscr{H}_{\boldsymbol{u}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0)[\hat{\boldsymbol{u}}] \quad \text{in } (0, t_{\mathrm{T}}), \\ \boldsymbol{z}_0(t_{\mathrm{T}}) &= \boldsymbol{u}(t_{\mathrm{T}}) \end{aligned}$$

with respect to an arbitrary $\hat{\boldsymbol{u}} \in U$. Hence, Eq. (4.1.31) can be written as

$$\mathscr{H}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0)[\boldsymbol{\zeta} - \boldsymbol{\xi}] \geq 0 \quad \text{in } (0, t_{\mathrm{T}}) .$$

This relationship shows that

$$\mathscr{H}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) \leq \mathscr{H}(\boldsymbol{\zeta}, \boldsymbol{v}, \boldsymbol{w}_0) \quad \text{in } (0, t_{\mathrm{T}})$$

with respect to an arbitrary vector $\boldsymbol{\zeta}$ satisfying $\|\boldsymbol{\zeta}\|_{\mathbb{R}^d}^2/2 \leq 1$. In addition, we see that $\boldsymbol{v}$ and $\boldsymbol{w}_0$ are solutions to the state determination problem (Problem 4.1.3) and the adjoint problem (Problem 4.1.5), respectively. In this way, $\boldsymbol{\xi}$ which satisfies the KKT conditions of Problem 4.1.4 means that the Hamilton function is minimized. This minimum condition is called Pontryagin's minimum principle.

A similar result can also be obtained for non-linear systems (see [6, Section 5.4, p. 140], where t_{T} is assumed to be a variable, while being fixed here). The state determination problem with respect to an optimum control problem of non-linear system is defined as follows.

Problem 4.1.6 (Non-linear system with control) Let $\boldsymbol{b} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ and control force $\boldsymbol{\xi} : (0, t_{\mathrm{T}}) \to \mathbb{R}^d$ be given. Find the system status $\boldsymbol{u} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ which satisfies

$$\dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{u}^{\top}} (\boldsymbol{\xi}, \boldsymbol{u}) \, \boldsymbol{u} + \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\xi}^{\top}} (\boldsymbol{\xi}, \boldsymbol{u}) \, \boldsymbol{\xi} \quad \text{in } (0, t_{\mathrm{T}}) , \tag{4.1.32}$$

$$\boldsymbol{u}(0) = \boldsymbol{\alpha}. \tag{4.1.33}$$

□

Use the control force $\boldsymbol{\xi}$ and solution $\boldsymbol{u}$ of the state determination problem (Problem 4.1.3) to express the cost function of optimum control as

$$f_0 (\boldsymbol{\xi}, \boldsymbol{u}) = \int_0^{t_{\mathrm{T}}} h (\boldsymbol{\xi}, \boldsymbol{u}) \, \mathrm{d}t + j (\boldsymbol{u} (t_{\mathrm{T}})) . \tag{4.1.34}$$

Here, $h : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ and $j : \mathbb{R}^n \to \mathbb{R}$ are given functions. Moreover, when a convex domain $\Omega \subset \mathbb{R}^d$ is given, the constraint of the control force is

$$\boldsymbol{\xi} \in \Omega \quad \text{in } (0, t_{\mathrm{T}}) . \tag{4.1.35}$$

The corresponding optimum control problem is constructed as follows.

Problem 4.1.7 (Optimum control problem of non-linear system) Let Ξ be a set of functions $\boldsymbol{\xi} : (0, t_{\mathrm{T}}) \to \mathbb{R}^d$ and U be a set of functions $\boldsymbol{u} : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$. Let f_0 be given by Eq. (4.1.34). In this case, obtain the KKT conditions with respect to the problem seeking $\boldsymbol{\xi}$ satisfying

$$\min_{(\boldsymbol{\xi}, \boldsymbol{u}) \in \Xi \times U} \{ f_0 (\boldsymbol{\xi}, \boldsymbol{u}) \mid \text{Eq. (4.1.35)}, \ \text{Problem 4.1.6} \} .$$

□

Define the Lagrange function in a similar way to Problem 4.1.4. Here the adjoint problem of this problem is given as follows.

Problem 4.1.8 (Adjoint problem with respect to f_0) With respect to the solution $\boldsymbol{u}$ to Problem 4.1.6 and f_0 of Eq. (4.1.34), obtain $\boldsymbol{z}_0 : (0, t_{\mathrm{T}}) \to \mathbb{R}^n$ which satisfies

$$\dot{\boldsymbol{z}}_0 = - \left(\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{u}^{\top}} (\boldsymbol{\xi}, \boldsymbol{u}) \right)^{\top} \boldsymbol{z}_0 - \frac{\partial h}{\partial \boldsymbol{u}} (\boldsymbol{\xi}, \boldsymbol{u}) ,$$

$$\boldsymbol{z}_0 (t_{\mathrm{T}}) = \frac{\partial j}{\partial \boldsymbol{u}} (\boldsymbol{u} (t_{\mathrm{T}})) .$$

□

When $\boldsymbol{u}$ and $\boldsymbol{z}_0$ are the respective solutions to the state determination problem (Problem 4.1.6) and adjoint problem (Problem 4.1.8), the Pontryagin local minimum condition is given as

$$\left(\frac{\partial h}{\partial \boldsymbol{\xi}} (\boldsymbol{\xi}, \boldsymbol{u}) + \left(\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\xi}^\top} (\boldsymbol{\xi}, \boldsymbol{u}) \right)^\top \boldsymbol{z}_0 \right) \cdot (\boldsymbol{\zeta} - \boldsymbol{\xi}) \geq 0 \quad \text{in } (0, t_{\mathrm{T}}) .$$

for an arbitrary $\boldsymbol{\zeta} \in \Omega$.

Problems 4.1.4 and 4.1.7 are good examples of how an adjoint problem is constructed when considering the solution of function optimization problem with respect to time-evolution problems.

4.2 Abstract Spaces

In Sect. 4.1, we saw that variational principles have become function optimization problems with functions of time and location as the design variables. In this section and Sect. 4.3, we will look at linear spaces containing design variables of function optimization problems. In this section, we shall start with the definition of a linear space and define abstract spaces that will be needed in future dscussions. Here, abstract spaces refer to a set such that rules for the operation or definition for the proximity between all elements are determined.

In this section, a linear space is defined as an abstract space in which linear operators can be applied, and then a metric space is introduced in which metric (distance) can be used. In a metric space, completeness which secures the possibility of limit operations is defined. After that, we return to linear spaces, and linear spaces in which a norm is defined (norm spaces) or linear spaces in which inner products can be applied (inner product spaces) are defined. Norm spaces with completeness (Banach spaces) and inner product space with completeness (Hilbert spaces) are important abstract spaces that will be used in future discussions.

4.2.1 Linear Space

Let us give a definition of a linear space which is one of the more basic abstract spaces in this book. The so-called vector space is another name for a linear space. A linear space is defined as follows.

Definition 4.2.1 (Linear spaces) With respect to arbitrary elements $\boldsymbol{x}$ and $\boldsymbol{y}$ from set X, let addition $\boldsymbol{x} + \boldsymbol{y} \in X$ be defined. With respect to an arbitrary element α in the set K representing $\mathbb{R}$ or $\mathbb{C}$ and an arbitrary element $\boldsymbol{x}$ in X, let a scalar product $\alpha \boldsymbol{x} \in X$ be defined. In this case, if:

(1) commutative law of addition $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$,

(2) associative law of addition $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$,

(3) there exists a zero element $\boldsymbol{e} \in X$ satisfying $\boldsymbol{e} + \boldsymbol{x} = \boldsymbol{x}$,

(4) there exists an inverse element $-\boldsymbol{x} \in X$ satisfying $(-\boldsymbol{x}) + \boldsymbol{x} = \boldsymbol{e}$,

(5) there exists a unit element $1 \in K$ satisfying $1\boldsymbol{x} = \boldsymbol{x}$,

(6) scalar product associative law $\alpha(\beta\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$,

(7) partition law of scalars $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$,

(8) partition law of vectors $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$

holds for every $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in X$ and α, $\beta \in K$, then X is called a linear space on K. An element of X is called a vector or a point. An element of K is called a scalar. Furthermore, X is called a real linear space when $K = \mathbb{R}$. □

With respect to an element $\boldsymbol{x}$ of linear space X, $\alpha\boldsymbol{x} + \beta\boldsymbol{y}$ of an arbitrary α, $\beta \in K$ is called a linear operation or linear combination. Moreover, the direct product of linear spaces X and Y, $X \times Y$ is a linear space by addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and scalar product $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$ with respect to arbitrary $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $\alpha \in K$.

Linear spaces have been defined, so let us now provide an example of one. When d is a natural number, it is easy to see that $\mathbb{R}^d$ satisfies the definition of a real linear space. Here, the zero element $\boldsymbol{e}$ is $\boldsymbol{0}_{\mathbb{R}^d}$ and the inverse element of $\boldsymbol{x} \in \mathbb{R}^d$ is the minus element $-\boldsymbol{x}$.

Set of all continuous functions

Next let us look at the fact that the set of all continuous functions with the range of real numbers is a real linear space. From the fact that the set of all continuous functions is a function space, it is appropriate within the contents of this book to provide an explanation that can be found in Sect. 4.3. However, in order to have a concrete image of a linear space, we will provide here a definition of the set of all continuous functions.

From here on, d is taken to be a natural number and k a non-negative integer. Firstly, we will explain a rule known as the multi-index used when expressing partial differentials of continuous functions. With respect to a function which is k-th order partially differentiable $f : \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^\top \in \{0, \ldots, k\}^d$ which satisfies $\sum_{i \in \{1,\ldots,d\}} \beta_i \le k$ is given, $\nabla^{\boldsymbol{\beta}} f$ and $|\boldsymbol{\beta}|$ are defined as

$$\nabla^{\boldsymbol{\beta}} f = \frac{\partial^{\beta_1} \partial^{\beta_2} \cdots \partial^{\beta_d} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_d^{\beta_d}}, \quad |\boldsymbol{\beta}| = \sum_{i \in \{1,\ldots,d\}} \beta_i \le k$$

respectively. $\boldsymbol{\beta}$ in this case is called a multi-index. Moreover, the closure of subset $\left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; f(\boldsymbol{x}) \neq 0 \right\}$ of $\mathbb{R}^d$ is called the support of f and written as $\operatorname{supp} f$.

The set of all real number functions which is continuous (Section A.1.2) up to the k-th order partial derivative is written as follows. Hereinafter, Ω is taken to be $\mathbb{R}^d$ or a connected open subset of $\mathbb{R}^d$ and called a domain (Section A.5). When the domain is bounded, its boundary $\partial\Omega$ is assumed to be a Lipschitz boundary (Section A.5) and Ω in that case is the Lipschitz domain. Moreover, $\bar{\Omega}\,(=\Omega\cup\partial\Omega)$ expresses the closure of Ω (Section A.1.1).

Definition 4.2.2 (Set of all continuous functions) Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. The set of continuous functions $f:\Omega\to\mathbb{R}$ is defined as follows with respect to $k\in\mathbb{N}\cup\{0\}$:

(1) The set of all f is written as $C(\Omega;\mathbb{R})$. The set of all f for which $\nabla^{\boldsymbol{\beta}} f : \Omega\to\mathbb{R}$ ($|\boldsymbol{\beta}|\le k$) is continuous is written as $C^k(\Omega;\mathbb{R})$. Furthermore, when the domain is $\bar{\Omega}$, we write $C^k\left(\bar{\Omega};\mathbb{R}\right)$.

(2) The set of all bounded functions f is denoted by $C_{\mathrm{B}}(\Omega;\mathbb{R})$ and the set of such functions $\nabla^{\boldsymbol{\beta}} f:\Omega\to\mathbb{R}$ ($|\boldsymbol{\beta}|\le k$) by $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$.

(3) The set of all f such that $\operatorname{supp} f$ is a compact set in Ω (see Proposition 4.2.12) is written as $C_0(\Omega;\mathbb{R})$. The set $C^k(\Omega;\mathbb{R})\cap C_0(\Omega;\mathbb{R})$ is expressed as $C_0^k(\Omega;\mathbb{R})$.

□

The difference between the sets $C(\Omega;\mathbb{R})$ and $C_{\mathrm{B}}(\Omega;\mathbb{R})$ of functions defined in (1) and (2) of Definition 4.2.2 is the boundedness. The set Ω is an open set, hence $C(\Omega;\mathbb{R})$ includes continuous functions which become infinite at the boundary. For example, with respect to $x\in(0,\infty]$, $f(x)=1/x$ is continuous but is not bounded. In contrast, elements of $C_{\mathrm{B}}(\Omega;\mathbb{R})$ will not include continuous functions which become infinite at the boundary. Thus, we see that

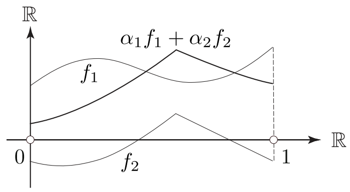
$$C_{\mathrm{B}}(\Omega;\mathbb{R})\subset C(\Omega;\mathbb{R})\,. \tag{4.2.1}$$

Moreover, the difference between $C\left(\bar{\Omega};\mathbb{R}\right)$ and $C_{\mathrm{B}}(\Omega;\mathbb{R})$ comes from the difference between the defined domain being a closed set $\bar{\Omega}$ or an open set Ω. If the defined domain is a bounded closed set, continuous functions can be shown to be uniformly continuous (Section A.1.2) and bounded. On the other hand, if the defined domain is an open set there are examples of them being bounded but not uniformly continuous. For example, for $x\in(0,\infty]$, the function $f(x)=\sin(1/x)$ is continuous and bounded but is not uniformly continuous. Here, the inclusion

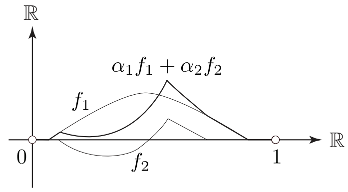
$$C\left(\bar{\Omega};\mathbb{R}\right)\subset C_{\mathrm{B}}(\Omega;\mathbb{R}) \tag{4.2.2}$$

is established. However, in a later discussion, a result stating that the norms of $C\left(\bar{\Omega};\mathbb{R}\right)$ and $C_{\mathrm{B}}(\Omega;\mathbb{R})$ are the same will be shown (Proposition 4.2.15).

Furthermore, the set $C_0(\Omega;\mathbb{R})$ defined in Definition 4.2.2 (3) shows that it is a set of functions such that $f=0$ in the neighborhood of the boundary $\partial\Omega$ of Ω (infinity when $\Omega=\mathbb{R}^d$). Here, $C_0^k(\Omega;\mathbb{R})$ is defined in Definition 4.2.2 (3) as a set of functions such that $\nabla^{\boldsymbol{\beta}} f=0$ ($|\boldsymbol{\beta}|\le k$) in the neighborhood of $\partial\Omega$. Based



(a) $f_1, f_2 \in C((0,1);\mathbb{R})$



(b) $f_1, f_2 \in C_0((0,1);\mathbb{R})$

Fig. 4.5: Linear combinations of continuous functions.

on this definition, we need to emphasize that there is no point in replacing Ω with $\bar{\Omega}$ with respect to $C_0(\Omega;\mathbb{R})$ or $C_0^k(\Omega;\mathbb{R})$. This is because the fact that the support (defined as the closure of domain with non-zero values) of the functions being $\bar{\Omega}$ is inconsistent with the fact that the value of the functions is zero in the neighborhood of the boundary.

Meanwhile, the set $C_0^\infty(\Omega;\mathbb{R})$ defined in Definition 4.2.2 (3) will eventually be used as a test function of Schwartz distribution (Definition 4.3.7) when defining the derivative of a function f within a Sobolev space $W^{k,p}(\Omega;\mathbb{R})$ (Definition 4.3.10). The set $C^\infty(\Omega;\mathbb{R})$ can also be used as a test function when defining the function f which is in Sobolev space $W_0^{k,p}(\Omega;\mathbb{R})$ (Definition 4.3.10) (see Eq. (4.3.10)).

The following can be said about the sets $C^k(\Omega;\mathbb{R})$ and $C_0^k(\Omega;\mathbb{R})$ of all continuous functions.

Proposition 4.2.3 (Sets of all continuous functions) The sets $C^k(\Omega;\mathbb{R})$, $C^k(\bar{\Omega};\mathbb{R})$, $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$ and $C_0^k(\Omega;\mathbb{R})$ are real linear spaces with the zero element $f_0 = 0$ in Ω and inverse element $-f$ of f. Moreover, $C^k(\bar{\Omega};\mathbb{R})$, $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$ and $C_0^k(\Omega;\mathbb{R})$ are real linear subspaces (subsets and real linear spaces) of $C^k(\Omega;\mathbb{R})$. □

Proof The fact that a linear combination of continuous functions is a linear function needs to be confirmed. With respect to arbitrary $f_1, f_2 \in C^k(\Omega;\mathbb{R})$ and arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 f_1 + \alpha_2 f_2$ is an element of $C^k(\Omega;\mathbb{R})$ (see Fig. 4.5 (a)). Therefore, $C^k(\Omega;\mathbb{R})$ is a linear space. The same holds for $C(\bar{\Omega};\mathbb{R})$ and $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$. Furthermore, $\alpha_1 f_1 + \alpha_2 f_2$ with respect to arbitrary $f_1, f_2 \in C_0^k(\Omega;\mathbb{R})$ and arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 f_1 + \alpha_2 f_2 = 0$ holds in the neighborhood of $\partial\Omega$, hence $\alpha_1 f_1 + \alpha_2 f_2$ is an element of $C_0^k(\Omega;\mathbb{R})$ (see Fig. 4.5 (b)). Thus, $C_0^k(\Omega;\mathbb{R})$ is a real linear space. In addition, $C^k(\bar{\Omega};\mathbb{R}) \subset C^k(\Omega;\mathbb{R})$, $C_{\mathrm{B}}^k(\Omega;\mathbb{R}) \subset C^k(\Omega;\mathbb{R})$ and $C_0^k(\Omega;\mathbb{R}) \subset C^k(\Omega;\mathbb{R})$, hence $C^k(\bar{\Omega};\mathbb{R})$, $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$ and $C_0^k(\Omega;\mathbb{R})$ are real linear subspaces of $C^k(\Omega;\mathbb{R})$. □

Next let us consider the dimension of $C^k(\Omega;\mathbb{R})$. The dimension is defined as follows.

Definition 4.2.4 (Dimension) Let n be a natural number and when a linear space X includes n linear independent vectors but when $n+1$ vectors are selected they are always linear dependent, then the dimension of X is n. □

Here the following can be said.

Proposition 4.2.5 (Dimension of set of all continuous functions) The dimension of $C^k(\Omega; \mathbb{R})$ is infinite. □

Proof We can show that an infinite number of linear independent continuous functions can be found. If $\{f_n\}_{n\in\mathbb{N}} \in C((0,1); \mathbb{R})$ is selected as $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, … , $f_n(x) = x^{n-1}$, then these are linearly independent. This is because x^{n-1} cannot be expressed as a linear combination of 1, … , x^{n-2}. Thus, n is infinite since it can be chosen arbitrarily. □

As seen so far, it has become apparent that the set of all continuous functions is a real linear space (Proposition 4.2.3) of infinite dimension (Proposition 4.2.5).

4.2.2 Linear Subspaces

With the image of a linear space taking shape, we will introduce several definitions relating to the subspaces of linear spaces. First, we will define the linear subspace constructed by linear combination of elements of a finite number as follows.

Definition 4.2.6 (Linear span) Let m be a natural number and $V = \{x_1, \ldots, x_m\}$ be a bounded subset of X with respect to a linear space X in K ($\mathbb{R}$ or $\mathbb{C}$). Here

$$\operatorname{span} V = \{\alpha_1 x_1 + \cdots + \alpha_m x_m \mid \alpha_1, \ldots, \alpha_m \in K\}$$

is called a linear span of V, linear hull of V or a linear subspace of X spanned by V. □

The linear span of V, span V, can be shown to be a subset of X and a minimal linear subspace including V. In the Galerkin method shown as a numerical solution of boundary value problems of partial differential equations in Chap. 6, the set of approximation functions is constructed from a linear span of the given functions.

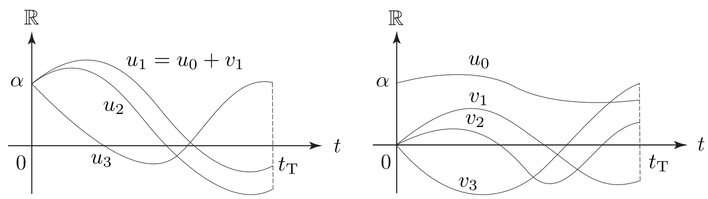
Moreover, a set constructed as the sum of an element of a linear space and a linear subspace not including the element is referred to as follows.

Definition 4.2.7 (Affine subspaces) Let X be a linear space and V be a linear subspace of X. With respect to $\boldsymbol{x}_0 \in X \setminus V$, we write

$$V(\boldsymbol{x}_0) = \{\boldsymbol{x}_0 + \boldsymbol{x} \mid \boldsymbol{x} \in V\}$$

and call $V(\boldsymbol{x}_0)$ an affine subspace of V. □

As an example of an affine subspace, the set U of functions in the expanded Hamiltonian principle (Problem 4.1.1) can be mentioned. As shown later on in Sect. 4.6.1, U is a set of functions u satisfying $u = \alpha$ at $t = 0$ such as in Fig. 4.6 (a) (see Eq. (4.6.3)). In this case U is not a linear space. This is because with



(a) $u_1, u_2, u_3 \in U$ (b) $u_0 \in X$, $v_1, v_2, v_3 \in V$

Fig. 4.6: Affine subspace $U = V(u_0)$ of $X = H^1((0, t_{\mathrm{T}}); \mathbb{R})$.

respect to $u_1, u_2 \in U$, $u_1 + u_2$ is $2\alpha \neq \alpha$ at $t = 0$. With respect to this, at $t = 0$ such as in Fig. 4.6 (b), the set of functions V (see Eq. (4.6.4)) satisfying $v = 0$ is a linear space. This is because, in reality, with respect to $v_1, v_2 \in V$, $v_1 + v_2$ is zero at $t = 0$. The condition such that it is zero on a boundary like this is called a homogeneous Dirichlet condition in boundary value problems of partial differential equations. On the other hand, the condition such that it is not zero is called inhomogeneous Dirichlet condition. Here, if $H^1((0, t_{\mathrm{T}}); \mathbb{R})$ (see Definition 4.3.10) which will be defined later is set to be the linear space X, and a function $u_0 \in X$ (for example u_0 in Fig. 4.6 (b)) which is $u_0 = \alpha$ at $t = 0$ is chosen and fixed, U matches the affine subspace $V(u_0)$ of V. Moreover, $u \in V(u_0)$ agrees with

$$u - u_0 \in V. \tag{4.2.3}$$

In this book, from the fact that importance is given to linear spaces, if a boundary value problem for partial differential equations is to be defined from Chap. 5 onward, the expression Eq. (4.2.3) will mainly be used.

4.2.3 Metric Space

Next, let us think about the conditions which enable limit operation in a set of functions. In this regard, the metric and metric space are defined as follows.

Definition 4.2.8 (Metric space) With respect to a set X, when a function $d : X \times X \to \mathbb{R}$ satisfies the following property:

(1) non-negativity $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$,

(2) identity $d(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(3) symmetry $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$,

(4) triangle inequality $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$

for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$, then d is called a metric on X. Moreover, the set X is called metric space with metric d. □

As can be seen from this definition, a metric space does not need to be a linear space. The characteristics equivalent to the continuity of function spaces should be defined in the metric space. Hence, these definitions will be shown in this section on the metric space.

Denseness

Let us think about the characteristics relating to a subset of a metric space. Let X be a metric space and V be its subset, and $\bar{V}$ be its closure (Section A.1.1). In this case, if $X = \bar{V}$, V is referred to as being dense in X. This is equivalent to the fact that there exists at least a point of V in the neighborhood of an arbitrary $\boldsymbol{x} \in X$.

This is explained as followed by using the relationship between the set of all real numbers $\mathbb{R}$ and the set of all rational numbers $\mathbb{Q}$. When the absolute value $|x - y|$ is taken as its metric with respect to an arbitrary $x, y \in \mathbb{R}$, there exists at least an element of $\mathbb{Q}$ in the neighborhood of an arbitrary $\boldsymbol{x} \in \mathbb{R}$.

Separability

Next, let us think about the quality that an infinite sequence of points can be selected in order to investigate the continuity or closure of metric spaces. Here, infinite sequence of points expresses the set with an infinite number of points from a metric space lined up. When X is a metric space and X has a dense subset constructed of at most a countable number of points (infinite number about the same level as the number of elements in the total set of natural numbers), X is called separable. The set of all rational numbers $\mathbb{Q}$ (because it is a set of fractions with natural numbers as the denominator and the numerator) is a countable set. The set $\mathbb{R}$ contains $\mathbb{Q}$, hence $\mathbb{R}$ is separable. This type of separability is a characteristic which is a prerequisite when discussing convergence using an infinite sequence of points.

Completeness

The concept of continuity in a metric space is called completeness and is defined using a Cauchy sequence. Firstly, a Cauchy sequence is defined as follows.

Definition 4.2.9 (Cauchy sequence) If the infinite sequence of points $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ in metric space X satisfies

$$\lim_{n,m\to\infty} d\left(\boldsymbol{x}_n, \boldsymbol{x}_m\right) = 0,$$

then $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence. □

The Cauchy sequence is used to define completeness as follows.

Definition 4.2.10 (Completeness) When any Cauchy sequence of a metric space X converges to a point within X, X is called complete. □

In what follows we will explain the meaning of infinite in an infinite sequence of points. It is not conceivable that the meaning of being an infinite number of elements in the set of all natural numbers $\mathbb{N}$ is the same as the meaning of there being an infinite number of elements in the set of all real numbers $\mathbb{R}$. It is Cantor's diagonal argument which answered such a simple question (for example, [11, Section 3.4, p. 65]). The concept of cardinality (size) of an infinite set was defined and this showed that the cardinality of $\mathbb{R}$ (cardinality of the continuum) is higher than the cardinality of $\mathbb{N}$ (cardinality of countable set). However, the infinite sequence of points used in the definition of Cauchy sequence is constructed of countable infinity meaning countable cardinality. This can be interpreted that even if the cardinality of $\mathbb{R}$ is cardinality of the continuum, it is sufficient to use a countable infinite number of Cauchy sequences to investigate the completeness in metric spaces.

This can be seen from the fact that the continuity of the set of all real numbers $\mathbb{R}$ is covered by the following axiom.

Axiom 4.2.11 (Completeness of $\mathbb{R}$) For arbitrary elements $x, y \in \mathbb{R}$, if the the absolute value $|x - y|$ is taken as a metric, then every Cauchy sequence in $\mathbb{R}$ converges to a point within $\mathbb{R}$. □

In this axiom every Cauchy sequence in $\mathbb{R}$ can be constructed of just elements of $\mathbb{Q}$. For example, $\sqrt{2}$ can be defined as a convergent point of an infinite sequence generated by $x_1 = 1$ and $x_{n+1} = x_n/2 + 1/x_n$. The infinite matrix is a Cauchy sequence. In reality, when $n \to \infty$, $|x_{n+1} - x_n| = |1/x_n - x_n/2| \to 0$. Therefore, if a set including all the convergence points of Cauchy sequences of $\mathbb{Q}$ is considered, such a set is complete. It is agreed that such a set is seen as $\mathbb{R}$. In this regard, $\mathbb{R}$ can be said to be a set in which $\mathbb{Q}$ has undergone completion.

The relationship between completeness and subsets of real numbers is as follows. The open interval $(0, 1)$ is not complete. However, the closed interval $[0, 1]$ is complete. This is because the Cauchy sequence $\{1/2, 1/3, 1/4, 1/5, \ldots\}$ does not converge within $(0, 1)$, but converges in $[0, 1]$.

Compactness

Denseness indicates the characteristic that even within a subset of a metric space, the closure of the subset is the metric space. In contrast, the characteristic that an infinite sequence from a subset of metric space converging to within that subset is called compactness. Let X be a complete metric space and V its subset. When an arbitrary infinite sequence of points of V includes a partial infinite sequence of points which converges within V, V is said to be compact. When the closure of V contains a convergent infinite subsequence, V is said to be a relative compact set. Here the following proposition is established.

Proposition 4.2.12 (Boundedness of a compact set) Let X be a complete metric space and V be a subset of X. If V is compact, then V is a bounded closed set. □

Proof From the definition of compactness, V is a closed set. We will show the boundedness of V by contradiction. If V is not bounded, with respect to a fixed point $\boldsymbol{x}$ in X, there exists an infinite sequence of points $\{\boldsymbol{y}_n\}_{n\in\mathbb{N}}$ that is $d(\boldsymbol{x}, \boldsymbol{y}_n) \to \infty$. An infinite subsequence of points which converges cannot be selected from within $\{\boldsymbol{y}_n\}_{n\in\mathbb{N}}$. This is because a convergent infinite sequence of points is bounded. Hence, V needs to be bounded. □

The reverse proposition of Proposition 4.2.12 "If V is a bounded closed set, then V is compact" is true when X is a finite-dimensional vector space, while it is not true when X is an infinite-dimensional space (see the upper part of Proposition 4.4.11).

4.2.4 Normed Space

Although we have been looking at completeness in metric spaces, there was no need for a metric space to be a linear space. Here we will define a linear space with a defined metric and a linear space with the completeness property.

Let X be a linear space on K. If the function $\|\cdot\| : X \to \mathbb{R}$ ($\|\boldsymbol{x}\| : X \ni \boldsymbol{x} \mapsto \|\boldsymbol{x}\| \in \mathbb{R}$) has the following properties:

(1) positivity $\|\boldsymbol{x}\| \geq \boldsymbol{0}$,

(2) equivalence of $\|\boldsymbol{x}\| = 0$ and $\boldsymbol{x} = \boldsymbol{0}$,

(3) homogeneity or proportionality $\|\alpha\boldsymbol{x}\| = |\alpha|\,\|\boldsymbol{x}\|$,

(4) triangle inequality $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

for every $\boldsymbol{x}, \boldsymbol{y} \in X$ and arbitrary $\alpha \in K$, then $\|\cdot\|$ is called a norm on X and can also be written as $\|\cdot\|_X$.

In this case, a normed space is defined as follows.

Definition 4.2.13 (normed space) A linear space in which a norm is defined is called a normed space. When a set K of scalars is $\mathbb{R}$, it is called a real normed space. □

A normed space becomes a metric space if $\|\boldsymbol{x}-\boldsymbol{y}\|$ is set as the metric $d(\boldsymbol{x}, \boldsymbol{y})$. Therefore, a Cauchy sequence can be defined and completeness can be studied.

Banach space

A complete normed space is defined as follows.

Definition 4.2.14 (Banach space) A complete normed space X is called a Banach space. When a set K of scalars is $\mathbb{R}$, it is called a real Banach space. □

We will give a specific example. Let the absolute value $|x|$ be a norm with respect to an element x of $\mathbb{R}$. In this case $\mathbb{R}$ is a Banach space. However, the closed interval $[0,1]$ is not a Banach space. This is because $[0,1]$ is not a linear space.

Next, let us consider the set $\mathbb{R}^d$. For an element $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$, the expression

$$\|\boldsymbol{x}\|_{\mathbb{R}^d} = \sqrt{|x_1|^2 + \cdots + |x_d|^2}$$

is called a Euclidean norm. This norm has the same definition as when norm is defined by $\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ using the inner product in $\mathbb{R}^d$. There are others which satisfy the definition of the norm. For $p \in [1, \infty)$,

$$\|\boldsymbol{x}\|_p = \left(|x_1|^p + \cdots + |x_d|^p\right)^{1/p}$$

is called the p-norm. Meanwhile, for $p = \infty$,

$$\|\boldsymbol{x}\|_\infty = \max\left\{|x_1|, \ldots, |x_d|\right\}$$

is called the maximum norm or Chebyshev norm. With respect to these norms, $\mathbb{R}^d$ is a Banach space.

Let us now turn our attention to the set of all continuous functions (Definition 4.2.2). The fact that $C^k(\Omega; \mathbb{R})$ is a real linear space was confirmed by Proposition 4.2.3. However, an element of $C^k(\Omega; \mathbb{R})$ has the possibility of becoming infinite. Hence, when considering a linear space with completeness, $C^k(\Omega; \mathbb{R})$ needs to be excluded. In fact, $C^k\left(\bar{\Omega}; \mathbb{R}\right)$ and $C^k_{\mathrm{B}}(\Omega; \mathbb{R})$ has the possibility of being a Banach space. Now we will define their corresponding norms and use them to see how completeness (Cauchy sequence of continuous functions converging to continuous functions) can be shown.

In relation to this, it would be good to first investigate how the set of all continuous functions is separable (the fact that Cauchy sequence of continuous functions can be constructed). Continuous functions include the set of all polynomials whose coefficients take rational numbers. There are at most a countable number of these sets. Hence, the set of all polynomials with rational coefficients is a dense subset of the set of all polynomials with real coefficients. Furthermore, from Weierstrass's approximation theorem, the set of all polynomials with real coefficients can be said to be a dense subset of the set of all continuous functions. So it confirms that the set of all continuous functions is separable. Completeness can be verified as follows.

Proposition 4.2.15 (Set of all continuous functions) The sets $C^k\left(\bar{\Omega}; \mathbb{R}\right)$ and $C^k_{\mathrm{B}}(\Omega; \mathbb{R})$ (Definition 4.2.2) are real Banach spaces with

$$\|f\|_{C^k\left(\bar{\Omega}; \mathbb{R}\right)} = \|f\|_{C^k_{\mathrm{B}}(\Omega; \mathbb{R})} = \max_{|\boldsymbol{\beta}| \le k} \sup_{\boldsymbol{x} \in \Omega} \left|\nabla^{\boldsymbol{\beta}} f(\boldsymbol{x})\right|$$

as the norm. □

Proof The key point of the proof is that although the Cauchy sequence of continuous functions converges at each point, it is whether the functions converging at each point are uniformly continuous or not.

Firstly, consider the case when $k = 0$. Suppose $\{f_n\}_{n\in\mathbb{N}} \in C\left(\bar{\Omega};\mathbb{R}\right)$ is a Cauchy sequence. From the fact that when fixing on a chosen arbitrary $\boldsymbol{x} \in \bar{\Omega}$,

$$|f_n(\boldsymbol{x}) - f_m(\boldsymbol{x})| \le \|f_n - f_m\|_{C\left(\bar{\Omega};\mathbb{R}\right)} \to 0$$

with respect to $n, m \to \infty$ at each point $\boldsymbol{x} \in \bar{\Omega}$, then there is convergence by $\mathbb{R}$ norm (absolute value). Write this as $f(\boldsymbol{x})$.

Next, choose n_0 such that, for every $\epsilon > 0$ and $n, m > n_0$, the following holds true:

$$\|f_n - f_m\|_{C\left(\bar{\Omega};\mathbb{R}\right)} \le \frac{\epsilon}{2}.$$

Then, for every $n > n_0$, we have

$$\begin{aligned}|f_n(\boldsymbol{x}) - f(\boldsymbol{x})| &\le |f_n(\boldsymbol{x}) - f_m(\boldsymbol{x})| + |f_m(\boldsymbol{x}) - f(\boldsymbol{x})| \\ &\le \|f_n - f_m\|_{C\left(\bar{\Omega};\mathbb{R}\right)} + |f_m(\boldsymbol{x}) - f(\boldsymbol{x})|.\end{aligned}$$

In this case, if m is taken to be large enough so that $|f_m(\boldsymbol{x}) - f(\boldsymbol{x})| \le \epsilon/2$ is true, $|f_n(\boldsymbol{x}) - f(\boldsymbol{x})| \le \epsilon$ holds and $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f. If a continuous function uniformly converges, because of its limit being continuous too, $f \in C\left(\bar{\Omega};\mathbb{R}\right)$ and $C\left(\bar{\Omega};\mathbb{R}\right)$ is complete.

Now, let $k > 0$. From the definition of the norm of $C^k\left(\bar{\Omega};\mathbb{R}\right)$ and with respect to all $|\boldsymbol{\beta}| \le k$, in the sense of uniform convergence, we have that $\nabla^{\boldsymbol{\beta}} f_n \to \nabla^{\boldsymbol{\beta}} f$ and $\|f_n - f\|_{C^k\left(\bar{\Omega};\mathbb{R}\right)} \to 0$, as $n \to \infty$, are equivalent. Therefore, $C^k\left(\bar{\Omega};\mathbb{R}\right)$ is complete.

Since the elements of $C_{\mathrm{B}}^k(\Omega;\mathbb{R})$ are bounded, the same can be said even when $\bar{\Omega}$ is changed to Ω. □

The direct product space $X \times Y$ of Banach spaces X and Y becomes a Banach space if the norm $\|(x,y)\|_{X\times Y}$ of $(x,y) \in X \times Y$ is set to be $(\|x\|_X^p + \|y\|_Y^p)^{1/p}$ with respect to $p \in [1,\infty)$ or $\max\{\|x\|_X, \|y\|_Y\}$. Hence, with r as a natural number, the set $C^k\left(\bar{\Omega};\mathbb{R}^r\right)$ of all functions $\boldsymbol{f} = (f_1, \ldots, f_r)^\top : \bar{\Omega} \to \mathbb{R}^r$ which are k-th order differentiable with $\mathbb{R}^r$ as a range becomes a direct product space $\left(C^k\left(\bar{\Omega};\mathbb{R}\right)\right)^r$ and if the norm $\|\boldsymbol{f}\|_{C^k\left(\bar{\Omega};\mathbb{R}^r\right)}$ is $\left(\sum_{i\in\{1,\ldots,r\}} \|f_i\|_{C^k\left(\bar{\Omega};\mathbb{R}\right)}^p\right)^{1/p}$ or $\max_{i\in\{1,\ldots,r\}} \|f_i\|_{C^k\left(\bar{\Omega};\mathbb{R}\right)}$, $C^k\left(\bar{\Omega};\mathbb{R}^r\right)$ becomes a Banach space.

Let us confirm here about the need for Banach spaces in optimization problems. If a linear space for which design variables are defined is chosen in a Banach space, the following can be said due to the completeness of Banach spaces. The convergence point when trial points are repeatedly found via iterative methods such as those looked at in Chap. 3 guarantees the existence as an element of a Banach space. Furthermore, in order to use the gradient method, there is a need to generalize the derivative of cost function or gradient method. A generalized derivative is defined for a Banach space in Sect. 4.5 as the Fréchet derivative. A topic of Chap. 7 is the generalization of the gradient method. In this chapter, a Hilbert space which is a complete inner space shown below is required.

4.2.5 Inner Product Space

We will introduce an inner product in a finite-dimensional vector space to an abstract linear space. An inner product can be defined as follows. Let X be a linear space on K ($\mathbb{R}$ or $\mathbb{C}$). When a function $(\,\cdot\,,\,\cdot\,) : X \times X \to K$ satisfies:

(1) equivalence of $(\boldsymbol{x}, \boldsymbol{x}) = 0$ and $\boldsymbol{x} = \boldsymbol{0}_X$,

(2) linearity in a vector $(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{z}) + (\boldsymbol{y}, \boldsymbol{z})$,

(3) linearity in a scalar $(\alpha \boldsymbol{x}, \boldsymbol{y}) = \alpha\,(\boldsymbol{x}, \boldsymbol{y})$,

(4) symmetry $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})$ or Hermitian symmetry $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})^*$ ($(\,\cdot\,)^*$ indicates a complex conjugate),

(5) positivity $(\boldsymbol{x}, \boldsymbol{x}) > 0$ for $\boldsymbol{x} \in X \setminus \{\boldsymbol{0}\}$

with respect to arbitrary $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$ and $\alpha \in K$, $(\,\cdot\,,\,\cdot\,)$ is called an inner product or a scalar product on X. Moreover, $(\,\cdot\,,\,\cdot\,)$ can be written as $(\,\cdot\,,\,\cdot\,)_X$. An inner product space is defined as follows.

Definition 4.2.16 (Inner product space) A linear space for which an inner product is defined is called an inner product space. When a set K of scalars is $\mathbb{R}$, it is called a real inner product space. □

If an inner product is defined, $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})}$ satisfies the definition of the norm. Therefore, an inner product space is also a normed space and completeness can be studied.

Hilbert space

The following can be said about a complete inner product space.

Definition 4.2.17 (Hilbert space) When an inner product space is complete with respect to the norm

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})},$$

it is called a Hilbert space. When the set K of scalars is $\mathbb{R}$, it is called a real Hilbert space. □

A finite-dimensional vector space $\mathbb{R}^d$ is a real Hilbert space of d dimensions. We will look at the fact that there are Hilbert spaces within function spaces in Sect. 4.3. In fact, the most important abstract space in this book is the Hilbert space. One of the reasons for its importance is that most of the variational principles looked at in Sect. 4.1 have become optimization problems in function spaces where the definition of Hilbert spaces is satisfied. This is confirmed in Sect. 4.6. Moreover, optimum design problems explained in Chap. 7 and beyond will also be defined in a function space in which real Hilbert spaces are defined. Furthermore, the Hilbert space is also used in Chap. 7 when generalizing the gradient method shown in Chap. 3.

4.3 Function Spaces

We have been looking at abstract spaces with completeness and inner products with linear spaces and metric spaces as the base. For these, the set of all continuous functions $C^k(\Omega; \mathbb{R})$ was defined as a set of all continuous functions with value $\mathbb{R}$ defined in Ω. This is to say that it represents the specific set of all functions. A set of all functions whose domain and range satisfy certain conditions such as these is called a function space. Here the function spaces other than the set of all the continuous functions are defined. After all of this we will summarize the definition of norms or inner products if they satisfy the requirements of Banach space or Hilbert space.

Here the domain of the function is written as Ω. However, we mention that this definition can change depending on the function. If a function is continuous, Ω is assumed to be a Lipschitz domain (Section A.5) explained prior to Definition 4.2.2. On the other hand, when considering a function for which attention is drawn only to the integral being bounded (integrable), Ω is seen as a measurable set on Ω excluding the subset of $\mathbb{R}^d$ whose Lebesgue measure is zero. However, with respect to $d = 1, 2, 3$, the Lebesgue measure in $\mathbb{R}^d$ will indicate length, area and volume. Hence, the set for which the Lebesgue measure is zero will mean the set of points, length and area respectively with respect to $d = 1, 2, 3$. In equations established on this type of measurable set, a.e. in the sense "almost everywhere" is added.

The proofs relating to theorems and propositions go beyond the level of this book so we refer the interested readers to the literature referenced as an example.

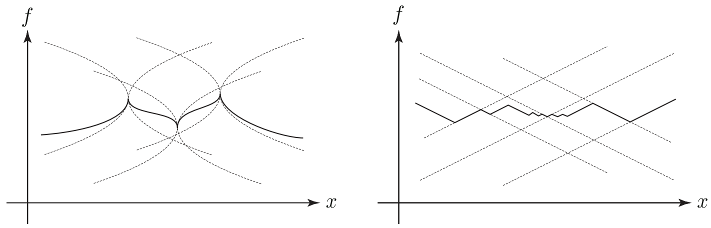
4.3.1 Hölder Space

Firstly, let us define a linear subspace of $C^k\left(\bar{\Omega}; \mathbb{R}\right)$ where the definition of continuity is even more strict. Lipschitz continuity shown here is used when defining smoothness with respect to domain boundaries (Section A.5). In shape optimization problems shown in Chaps. 8 and 9, the fact that design variables vary in order to maintain the smoothness is a topic.

Definition 4.3.1 (Hölder space) Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $f : \bar{\Omega} \to \mathbb{R}$. If for a $\sigma \in (0, 1]$ there exists some $\beta > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq \beta \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^d}^{\sigma}, \tag{4.3.1}$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \bar{\Omega}$, then f is called Hölder continuous. Here, σ is called the Hölder index and β in this case is called the Lipschitz constant. Moreover, with respect to $k \in \mathbb{N} \cup \{0\}$, the set of f for which $\nabla^{\boldsymbol{\beta}} f$ ($|\boldsymbol{\beta}| \leq k$) is Hölder continuous is written as $C^{k,\sigma}\left(\bar{\Omega}; \mathbb{R}\right)$ and called the Hölder space.[1] In particular, when $k = 0$ and $\sigma = 1$, f is said to be Lipschitz continuous and $C^{0,1}\left(\bar{\Omega}; \mathbb{R}\right)$ is called the Lipschitz space. □



(a) $\sigma = 0.5$ (b) $\sigma = 1$ (Lipschitz continuous)

Fig. 4.7: Hölder continuous functions.

Figure 4.7 shows cases when $f : \mathbb{R} \to \mathbb{R}$ is Hölder continuous and Lipschitz continuous.

The following results can be obtained with respect to $C^{k,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ (cf. [3, Theorem 1, p. 241]).

Proposition 4.3.2 (Hölder space) The space $C^{k,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ in Definition 4.3.1 is a real Banach space with

$$\left|\nabla^{\boldsymbol{\beta}} f\right|_{C^{0,\sigma}\left(\bar{\Omega};\mathbb{R}\right)} = \sup_{\boldsymbol{x},\boldsymbol{y}\in\bar{\Omega},\ \boldsymbol{x}\neq\boldsymbol{y}} \frac{\left|\nabla^{\boldsymbol{\beta}} f\left(\boldsymbol{x}\right) - \nabla^{\boldsymbol{\beta}} f(\boldsymbol{y})\right|}{\|\boldsymbol{x}-\boldsymbol{y}\|_{\mathbb{R}^d}^{\sigma}}$$

as the semi-norm and

$$\|f\|_{C^{k,\sigma}\left(\bar{\Omega};\mathbb{R}\right)} = \|f\|_{C^{k}\left(\bar{\Omega};\mathbb{R}\right)} + \max_{|\boldsymbol{\beta}|=k} \left|\nabla^{\boldsymbol{\beta}} f\right|_{C^{0,\sigma}\left(\bar{\Omega};\mathbb{R}\right)} \tag{4.3.2}$$

as the norm. Here, $\|f\|_{C^{k}\left(\bar{\Omega};\mathbb{R}\right)}$ is defined by Proposition 4.2.15. □

In Eq. (4.3.2), the norm of $C^{k,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ includes the norm of $C^{k}\left(\bar{\Omega};\mathbb{R}\right)$. Therefore, the inclusion

$$C^{k,\sigma}\left(\bar{\Omega};\mathbb{R}\right) \subset C^{k}\left(\bar{\Omega};\mathbb{R}\right) \tag{4.3.3}$$

holds.

4.3.2 Lebesgue Space

Next we will define the set of all functions for which attention is drawn to the characteristic that integration is defined (integrability) without continuity. Note that integration is defined even for functions such as Fig. 4.8. This sort of integrability is related to energy being defined in variational principles. This can be confirmed in Sect. 4.6.

In Lebesgue spaces and Sobolev spaces defined in this and the next section, the functions $f_1(x)$ and $f_2(x)$ which are the same apart from the set of Lebesgue measure zero are seen as the same functions. This is expressed as $f_1(x) = f_2(x)$ for a.e. $x \in \Omega$. For details see textbooks on Lebesgue integrals.

[1] According to a book, there are cases when $C^{0,1}\left(\bar{\Omega};\mathbb{R}\right)$ is not included in Hölder space.

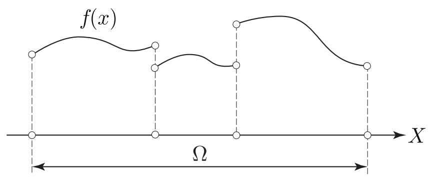


Fig. 4.8: Integrable functions $f : \mathbb{R} \to \mathbb{R}$.

Definition 4.3.3 (Lebesgue space) Let $\Omega \subset \mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$. For $p \in [1, \infty)$, if

$$\int_\Omega |f(\boldsymbol{x})|^p \, \mathrm{d}x < \infty \tag{4.3.4}$$

is satisfied in the sense of a Lebesgue integral (integral of measurable set), f is said to be a p-th order Lebesgue integrable. For $p = \infty$, if

$$\operatorname*{ess\,sup}_{\text{a.e. } \boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| < \infty \tag{4.3.5}$$

is satisfied, f is said to be essentially bounded. Such a set of all f is called a Lebesgue space and is written as $L^p(\Omega; \mathbb{R})$. □

The following result can be obtained for $L^p(\Omega; \mathbb{R})$.

Proposition 4.3.4 (Lebesgue space) The space $L^p(\Omega; \mathbb{R})$ in Definition 4.3.3 is a real Banach space with

$$\|f\|_{L^p(\Omega;\mathbb{R})} = \begin{cases} \left(\int_\Omega |f(\boldsymbol{x})|^p \, \mathrm{d}x \right)^{1/p} & \text{for } p \in [1, \infty), \\ \operatorname*{ess\,sup}_{\text{a.e. } \boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| & \text{for } p = \infty \end{cases}$$

as the norm. □

In the proof of completeness of $L^1(\Omega; \mathbb{R})$ of this proposition, Lebesgue's convergence theorem is used (cf. [9, Theorem 2.5, p. 38]). Moreover, the proof that $L^p(\Omega; \mathbb{R})$ when $p \in (1, \infty)$ is a linear space uses Hölder's inequality (Theorem A.9.1) and Minkowski's inequality (Theorem A.9.2) (cf. [9, Theorem 2.10, p. 42]). Furthermore, the linearity and completeness of $L^\infty(\Omega; \mathbb{R})$ can be shown by making the essential supremum $\operatorname{ess\,sup}_{\text{a.e. } \boldsymbol{x} \in \Omega} |f(\boldsymbol{x})|$ a norm (cf. [9, Theorem 2.20, p. 46]). Here the fact that the norm of $L^\infty(\Omega; \mathbb{R})$ becomes the upper limit value of the absolute value of the function can be understood if it is thought of as an expansion of a maximum value with respect to finite-dimensional vectors to function.

When $p = 2$, $L^2(\Omega; \mathbb{R})$ indicates a set of all functions that are square integrable and is a real Hilbert space with

$$(f, g)_{L^2(\Omega;\mathbb{R})} = \int_\Omega f(\boldsymbol{x}) \, g(\boldsymbol{x}) \, \mathrm{d}x \tag{4.3.6}$$

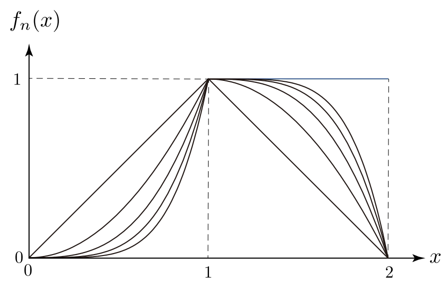


Fig. 4.9: Function sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ on $[0,2]$.

as inner product. This function space is one of the important function spaces in future development.

Here let us look at an example for which a Cauchy sequence of continuous functions can be taken within $L^2(\Omega;\mathbb{R})$ (by $L^2(\Omega;\mathbb{R})$ norm) such that it converges to a discontinuous function which is an element of $L^2(\Omega;\mathbb{R})$.

Exercise 4.3.5 (Cauchy sequence of continuous functions by L^2 norm) Consider a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ generated from $C([0,2];\mathbb{R})$ by

$$f_n(x) = \begin{cases} x^n & \text{in } (0,1), \\ 1-(x-1)^n & \text{in } (1,2). \end{cases}$$

Show that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with

$$\|f\|_{L^2((0,2);\mathbb{R})} = \left(\int_0^2 |f(x)|^2 \, \mathrm{d}x\right)^{1/2}$$

as the norm. Moreover, find the function for which its Cauchy sequence converges. □

Answer With respect to a function sequence $\{f_n\}_{n\in\mathbb{N}}$ such as the one in Fig. 4.9, when $m,n\to\infty$, we have

$$\begin{aligned}(f_m,f_n)_{L^2((0,2);\mathbb{R})} &= \int_0^1 x^{m+n}\,\mathrm{d}x + \int_1^2 \{1-(x-1)^m\}\{1-(x-1)^n\}\,\mathrm{d}x \\ &= \frac{1}{1+m+n} + \frac{m}{1+m} - \frac{1}{1+n} - \frac{1}{1+m+n} \to 1.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\|f_m - f_n\|^2_{L^2((0,2);\mathbb{R})} \\ &\quad = (f_m,f_m)_{L^2((0,2);\mathbb{R})} - 2(f_n,f_m)_{L^2((0,2);\mathbb{R})} + (f_n,f_n)_{L^2((0,2);\mathbb{R})} \to 0.\end{aligned}$$

Thus, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $L^2(\Omega;\mathbb{R})$ norm. Moreover, this Cauchy sequence converges to

$$f = \begin{cases} 0 & \text{in } [0,1), \\ 1 & \text{in } [1,2]. \end{cases}$$

In fact, as $n \to \infty$, we have

$$\|f_n - f\|^2_{L^2((0,1);\mathbb{R})} = \int_0^1 x^{2n} \mathrm{d}x = \frac{1}{1+2n} \to 0,$$
$$\|f_n - f\|^2_{L^2((1,2);\mathbb{R})} = \int_1^2 \{-(x-1)^n\}^2 \mathrm{d}x = \frac{1}{1+2n} \to 0.$$

As a result, we get $\|f_n - f\|_{L^2((0,2);\mathbb{R})} \to 0$ as $n \to \infty$. □

Based on this fact, it can be interpreted that the completed set by including all functions to which the Cauchy sequence due to $L^2(\Omega;\mathbb{R})$ of $C(\bar{\Omega};\mathbb{R})$ converges is taken to be $L^2(\Omega;\mathbb{R})$. This is similar to setting the set including all of the convergence points of Cauchy sequence of $\mathbb{Q}$ due to $\mathbb{R}$ norm as $\mathbb{R}$. Furthermore, it can be also shown that $L^2(\Omega;\mathbb{R})$ is the completion of $C^\infty(\bar{\Omega};\mathbb{R})$ or $C_0^\infty(\Omega;\mathbb{R})$. These examples are formed by using the Friedrichs mollifier. On the other hand, these facts show that $C(\bar{\Omega};\mathbb{R})$ becomes a dense subspace of $L^2(\Omega;\mathbb{R})$. From this it can be said that $L^2(\Omega;\mathbb{R})$ is separable (Cauchy sequence can be taken). Separability can be expanded to $p \in [1,\infty)$ and the separability of $L^p(\Omega;\mathbb{R})$ can be shown by using the fact that $C(\bar{\Omega};\mathbb{R})$ is a dense subspace of $L^p(\Omega;\mathbb{R})$.

4.3.3 Sobolev Space

In what follows, we will define the set of all integrable functions including derivatives. Variational principles seen in Sect. 4.1 defined energy using integral functions which are displacement differentiated with respect to time or location. Some of those in function space that appear below genuinely have the characteristics required with respect to displacement. This is confirmed in Sect. 4.6. Here we will define the derivative of functions using integrability and then proceed to the main subject.

Schwartz distribution

Differentiation of functions which are integrable but discontinuous (see Fig. 4.8) utilizes a definition using the Schwartz distribution. Here we will take a look at the definition and the differentiation of discontinuous functions using such a definition.

The definition of the Schwartz distribution uses bounded linear functionals. In this book a bounded linear functional is defined as an operator in Sect. 4.4.5. We will first define it as follows. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function which is Lebesgue integrable and ϕ be an arbitrary function within $C_0^\infty(\mathbb{R}^d;\mathbb{R})$ (Definition 4.2.2). Here $\langle f, \cdot \rangle : C_0^\infty(\mathbb{R}^d;\mathbb{R}) \to \mathbb{R}$ defined by

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} f \phi \, \mathrm{d}x < \infty \tag{4.3.7}$$

is called a bounded linear functional determined by f. The arbitrary function $\phi \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ used here is called a test function since it is a function used experimentally in defining a bounded linear functional. Moreover, the function space of the test function is generally written as $\mathscr{D}(\Omega)$. Hence, hereinafter, $C_0^\infty(\Omega; \mathbb{R})$ will be written as $\mathscr{D}(\Omega)$.

Definition 4.3.6 (Schwartz distribution) Let $\Omega \subset \mathbb{R}^d$. For the function sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of $\mathscr{D}(\Omega)$ such that when $n \to \infty$ all the partial derivatives converge uniformly to zero on the compact set in Ω, the bounded linear functional $\langle f, \cdot \rangle : \mathscr{D}(\Omega) \to \mathbb{R}$ satisfying $\langle f, \phi_n \rangle \to 0$ $(n \to \infty)$ is called a Schwartz distribution or distribution determined from f. In this book the same symbol f is used for it as long as there is no confusion. The set of Schwartz distributions f with Ω as the domain is written as $\mathscr{D}'(\Omega)$. □

From Definition 4.3.6, a distribution attempts to define functions which cannot be defined in the ordinary way as a mapping from domain to range in the same way as bounded linear functionals defined by integrals using test functions with good characteristics.

This type of differentials with respect to the Schwartz distribution is defined below.

Definition 4.3.7 (Partial derivatives of Schwartz distributions) Let $\langle f, \cdot \rangle : \mathscr{D}(\Omega) \to \mathbb{R}$ be a Schwartz distribution determined from f. When

$$\left\langle \frac{\partial f}{\partial x_i}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_i} \right\rangle, \quad \text{for } i \in \{1, \ldots, d\}$$

holds true with respect to an arbitrary $\phi \in \mathscr{D}(\Omega)$, $\langle \partial f/\partial x_i, \cdot \rangle$ is called a partial derivative of a Schwartz distribution or partial derivative of a distribution determined from f. In this book, as long as there is no confusion, the same symbol $\partial f/\partial x_i$ will be used. □

The fact that $\langle \partial f/\partial x_i, \cdot \rangle$ of Definition 4.3.7 becomes an element of $\mathscr{D}'(\Omega)$ can be said from the fact that even when the function sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of $\mathscr{D}(\Omega)$ is changed to $\{\partial \phi_n/\partial x_i\}_{n\in\mathbb{N}}$ in Definition 4.3.6, $\{\partial \phi_n/\partial x_i\}_{n\in\mathbb{N}}$ is a function sequence of $\mathscr{D}(\Omega)$ (cf. [1, Section 1.60, p. 21], [10, Proposition 2.8, p. 30]). If Definition 4.3.7 is repeatedly used, a higher-order partial derivative in the sense of a Schwartz distribution can be defined. When $\boldsymbol{\beta}$ is a multi-index and $\nabla^{\boldsymbol{\beta}}(\,\cdot\,) = \partial^{\beta_1}\partial^{\beta_2}\cdots\partial^{\beta_d}(\,\cdot\,)/\partial x_1^{\beta_1}\partial x_2^{\beta_2}\cdots\partial x_d^{\beta_d}$, then

$$\left\langle \nabla^{\boldsymbol{\beta}} f, \phi \right\rangle = (-1)^{|\boldsymbol{\beta}|} \left\langle f, \nabla^{\boldsymbol{\beta}} \phi \right\rangle$$

holds. The expression $\left\langle \nabla^{\boldsymbol{\beta}} f, \cdot \right\rangle$ is called a partial derivative of $|\boldsymbol{\beta}|$-th order in the sense of a Schwartz distribution. In this book, as far as there is no confusion, the same symbol $\nabla^{\boldsymbol{\beta}} f$ will be used.

Let us mention a specific example here. The function $\delta : \Omega^d \to \mathbb{R}$ for which

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}^d} \delta \phi \, \mathrm{d}x = \phi(\mathbf{0}_{\mathbb{R}^d}) \tag{4.3.8}$$

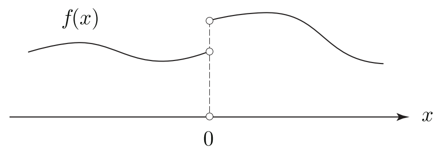


Fig. 4.10: Discontinuous function $f : \mathbb{R} \to \mathbb{R}$.

holds with respect to an arbitrary $\phi \in C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ is called the Dirac delta function or Dirac distribution. Let us think about the derivative of a step function.

Exercise 4.3.8 (Derivative of Heaviside step function) Show that the derivative of the Heaviside step function

$$h = \begin{cases} 0 & \text{in } (-\infty, 0)\,, \\ 1 & \text{in } (0, \infty) \end{cases}$$

as a Schwartz distribution is a Dirac delta function. □

Answer From the definition of the derivative of a Schwartz distribution, we have

$$\langle \nabla h, \phi \rangle = \int_{-\infty}^{\infty} \nabla h \phi \, \mathrm{d}x = -\int_{-\infty}^{\infty} h \nabla \phi \, \mathrm{d}x = -\int_{0}^{\infty} \nabla \phi \, \mathrm{d}x = \phi\,(0) = \langle \delta, \phi \rangle\,.$$

□

We have seen how the derivative of the Heaviside step function was a Dirac delta function. Using this relationship, the derivative of a discontinuity function can be written in the following way.

Exercise 4.3.9 (Derivative of discontinuity function)
Show the derivative as a Schwartz distribution of a function $f : \mathbb{R} \to \mathbb{R}$ being discontinuous at the origin, such as the one in Fig. 4.10. □

Answer From the definition of the Schwartz distribution, one obtains

$$\begin{aligned}
\langle \nabla f, \phi \rangle &= -\int_{-\infty}^{\infty} f \nabla \phi \, \mathrm{d}x = -\int_{-\infty}^{0} f \nabla \phi \, \mathrm{d}x - \int_{0}^{\infty} f \nabla \phi \, \mathrm{d}x \\
&= \left(f\,(0_+) - f\,(0_-)\right)\phi\,(0) + \int_{-\infty}^{\infty} \nabla f \phi \, \mathrm{d}x \\
&= \left(f\,(0_+) - f\,(0_-)\right)\langle \delta, \phi \rangle + \int_{-\infty}^{\infty} \nabla f \phi \, \mathrm{d}x,
\end{aligned}$$

where $f\,(0_-) = \lim_{\epsilon \to 0} f\,(-\epsilon)$ and $f\,(0_+) = \lim_{\epsilon \to 0} f\,(\epsilon)$ with respect to $\epsilon > 0$. □

Sobolev space

Since the derivative of an integrable function has been defined, let us now define the function space of integrable functions including derivatives (cf. [5, Definition 1.3.2.1 and Definition 1.3.2.2, p. 16], [10, Definition 9.10, p. 195]).

Definition 4.3.10 (Sobolev space) Let $\Omega \subset \mathbb{R}^d$. For $k \in \mathbb{N} \cup \{0\}$, $s = k + \sigma$ ($\sigma \in (0, 1)$) and $p \in [1, \infty]$, the entire set of $f : \Omega \to \mathbb{R}$ such as the following is called a Sobolev space.

(1) If $\nabla^{\boldsymbol{\beta}} f \in L^p(\Omega; \mathbb{R})$ holds for all $|\boldsymbol{\beta}| \leq k$, then we write the set of all such functions as $W^{k,p}(\Omega; \mathbb{R})$.

(2) If $f \in W^{k,p}(\Omega; \mathbb{R})$ holds with respect to $p \in (1, \infty)$ and

$$\int_{\Omega} \int_{\Omega} \frac{\left|\nabla^{\boldsymbol{\beta}} f(\boldsymbol{x}) - \nabla^{\boldsymbol{\beta}} f(\boldsymbol{y})\right|^p}{\|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^d}^{d+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y < \infty \tag{4.3.9}$$

holds with respect to $|\boldsymbol{\beta}| \leq k$, then we write the set of all such functions as $W^{s,p}(\Omega; \mathbb{R})$.

(3) The closure of $C_0^{\infty}(\Omega; \mathbb{R})$ (Definition 4.2.2) in $W^{k,p}(\Omega; \mathbb{R})$ is expressed as $W_0^{k,p}(\Omega; \mathbb{R})$.

(4) If $k = 0$, then we write the set of all such functions as $W_0^{0,p}(\Omega; \mathbb{R}) = L^p(\Omega; \mathbb{R})$, where $p \in [1, \infty)$ ([1, Corollary 2.30, p. 38]).

(5) When $p = 2$, we denote $W^{k,2}(\Omega; \mathbb{R})$ and $W_0^{k,2}(\Omega; \mathbb{R})$ as $H^k(\Omega; \mathbb{R})$ and $H_0^k(\Omega; \mathbb{R})$, respectively.

□

In Definition 4.3.10, with respect to the definition of the Schwartz distribution of $f \in W_0^{k,p}(\Omega; \mathbb{R})$, $C^{\infty}(\Omega; \mathbb{R})$ is selected as a test function (note that it is not $C_0^{\infty}(\Omega; \mathbb{R})$). In other words, it is defined via $\langle f, \cdot \rangle : C^{\infty}(\Omega; \mathbb{R}) \to \mathbb{R}$ which satisfies

$$\langle f, \phi \rangle = \int_{\Omega} f \phi \,\mathrm{d}x \tag{4.3.10}$$

with respect to an arbitrary $\phi \in C^{\infty}(\Omega; \mathbb{R})$, instead of Eq. (4.3.7).

The following results can be obtained with respect to $W^{k,p}(\Omega; \mathbb{R})$ (cf. [1, Theorem 3.3, p. 60]).

Proposition 4.3.11 (Sobolev space) The space $W^{k,p}(\Omega; \mathbb{R})$ in Definition 4.3.10 is a real Banach space with

$$\|f\|_{W^{k,p}(\Omega;\mathbb{R})} = \begin{cases} \left(\sum_{|\boldsymbol{\beta}| \leq k} \left\|\nabla^{\boldsymbol{\beta}} f\right\|_{L^p(\Omega;\mathbb{R})}^p \right)^{1/p} & \text{for } p \in [0, \infty), \\ \max_{|\boldsymbol{\beta}| \leq k} \left\|\nabla^{\boldsymbol{\beta}} f\right\|_{L^{\infty}(\Omega;\mathbb{R})} & \text{for } p = \infty \end{cases} \tag{4.3.11}$$

as the norm. □

In view of the norm given in Proposition 4.3.11,

$$|f|_{W^{k,p}(\Omega;\mathbb{R})} = \begin{cases} \left(\sum_{|\boldsymbol{\beta}|=k} \left\| \nabla^{\boldsymbol{\beta}} f \right\|_{L^p(\Omega;\mathbb{R})}^p \right)^{1/p} & \text{for } p \in [0,\infty), \\ \max_{|\boldsymbol{\beta}|=k} \left\| \nabla^{\boldsymbol{\beta}} f \right\|_{L^\infty(\Omega;\mathbb{R})} & \text{for } p = \infty \end{cases} \tag{4.3.12}$$

is called the semi-norm.

In this book, the space $H^1(\Omega;\mathbb{R})$ is the most important function space. This is because it is a Hilbert space in which the inner product can be used in the following way (cf. [9, Theorem 6.28, p. 134]).

Proposition 4.3.12 (Sobolev space $H^k(\Omega;\mathbb{R})$) The space $W^{k,2}(\Omega;\mathbb{R}) = H^k(\Omega;\mathbb{R})$ is a real Hilbert space with

$$(f,g)_{H^k(\Omega;\mathbb{R})} = \sum_{|\boldsymbol{\beta}|\le k} \int_\Omega \nabla^{\boldsymbol{\beta}} f \cdot \nabla^{\boldsymbol{\beta}} g \,\mathrm{d}x \tag{4.3.13}$$

as the inner product. □

Among the spaces $H^k(\Omega;\mathbb{R})$, the sets $H^1(\Omega;\mathbb{R})$ and $H^1(\Omega;\mathbb{R}^d)$ are the most important function spaces used as function spaces that the functions from shape or topology optimization problems or boundary value problems of partial differential equation can be described by. Hence, let us define the inner product for validation. The inner product of $f, g \in H^1(\Omega;\mathbb{R})$ can be defined by

$$(f,g)_{H^1(\Omega;\mathbb{R})} = \int_\Omega (fg + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g)\,\mathrm{d}x. \tag{4.3.14}$$

Moreover, the inner product of $\boldsymbol{f}$, $\boldsymbol{g} \in H^1(\Omega;\mathbb{R}^d)$ can be defined by

$$\begin{aligned} (\boldsymbol{f},\boldsymbol{g})_{H^1(\Omega;\mathbb{R}^d)} &= \int_\Omega \left\{ \boldsymbol{f}\cdot\boldsymbol{g} + \left(\boldsymbol{\nabla}\boldsymbol{f}^\top\right)\cdot\left(\boldsymbol{\nabla}\boldsymbol{g}^\top\right) \right\} \mathrm{d}x \\ &= \int_\Omega \left(\boldsymbol{f}\cdot\boldsymbol{g} + \sum_{(i,j)\in\{0,\dots,d\}^2} \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \left(\frac{\partial g_i}{\partial x_j}\right)_{ij} \right) \mathrm{d}x. \end{aligned} \tag{4.3.15}$$

Let us investigate what is and what is not included in $H^1((0,1);\mathbb{R})$ using a power function.

Exercise 4.3.13 (Power function in $H^1((0,1);\mathbb{R})$) Let $x \in (0,1)$. Determine the conditions on $\alpha \in \mathbb{R}$ such that the function

$$f = x^\alpha$$

is an element of $H^1((0,1);\mathbb{R})$. □

Answer Let the derivative of f with respect to x be written as f'. Here, in order for the following inequality to hold,

$$\|f\|_{H^1((0,1);\mathbb{R})} = \left\{\int_0^1 \left(f^2 + f'^2\right) \mathrm{d}x\right\}^{1/2} = \left\{\int_0^1 \left(x^{2\alpha} + \alpha^2 x^{2(\alpha-1)}\right) \mathrm{d}x\right\}^{1/2}$$
$$= \left(\left[\frac{x^{2\alpha+1}}{2\alpha+1} + \frac{\alpha^2 x^{2\alpha-1}}{2\alpha-1}\right]_0^1\right)^{1/2} < \infty,$$

it must be $2\alpha - 1 > 0$ or equivalently $\alpha > 1/2$. □

From Exercise 4.3.13, we can see that the singularity (Section 5.3) of $f = \sqrt{x}$ at $x = 0$ is not permissible within the space $H^1((0,1);\mathbb{R})$.

4.3.4 Sobolev Embedding Theorem

According to the definition of Sobolev spaces $W^{k,p}(\Omega;\mathbb{R})$ (Definition 4.3.10), many function spaces can be created depending on how $d \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty]$ are selected. Furthermore, the embedding relationships of various function spaces including the Hölder space $C^{k,\sigma}(\Omega;\mathbb{R})$ are summarized by the Sobolev embedding theorem, shown below.

Let us start by taking a general view on such an embedded relationship. Suppose $\Omega \subset \mathbb{R}^d$ and k is fixed. If $q < p$, then the inclusion $W^{k,p}(\Omega;\mathbb{R}) \subset W^{k,q}(\Omega;\mathbb{R})$ holds. Moreover, if p is fixed, then $W^{k+1,p}(\Omega;\mathbb{R}) \subset W^{k,p}(\Omega;\mathbb{R})$ also holds. These relationships are apparent from the definition of the norm. The Sobolev embedding theorem shows the embedding relationship between Sobolev spaces with different p and k. This shows that when $k - d/p$ is viewed as an order of the differentiability and

$$k + 1 - \frac{d}{p} \geq k - \frac{d}{q}$$

holds, then the inclusion

$$W^{k+1,p}(\Omega;\mathbb{R}) \subset W^{k,q}(\Omega;\mathbb{R})$$

is true. Furthermore, if $0 < \sigma = k - d/p < 1$, it shows that

$$W^{k,p}(\Omega;\mathbb{R}) \subset C^{0,\sigma}(\Omega;\mathbb{R})$$

holds. In this case, there is a need for Ω to be a Lipschitz domain.

With these relationships in mind, let us look at a detailed explanation of Sobolev embedding theorem (cf. [1, Theorem 4.12, p. 85]).

Theorem 4.3.14 (Sobolev embedding theorem) Let $\Omega \subset \mathbb{R}^d$. Let $k \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Then, the following results hold:

(1) if $k - d/p < 0$, with $p^* = d/\{(d/p) - k\}$, then

$$W^{k+j,p}(\Omega;\mathbb{R}) \subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [p, p^*], \tag{4.3.16}$$

(2) if $k - d/p = 0$, then

$$W^{k+j,p}(\Omega;\mathbb{R}) \subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [p,\infty), \tag{4.3.17}$$

(3) if $k - d/p = j + \sigma > 0$ ($\sigma \in (0,1)$), or $k = d$ and $p = 1$, then

$$W^{k+j,p}(\Omega;\mathbb{R}) \subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [p,\infty]. \tag{4.3.18}$$

Furthermore, if Ω is a Lipschitz domain, then

(4) if $k - d/p = j + \sigma > 0$ ($\sigma \in (0,1)$), or $k = d$ and $p = 1$, we have

$$W^{k+j,p}(\Omega;\mathbb{R}) \subset C^{j,\lambda}\left(\bar{\Omega};\mathbb{R}\right) \quad \text{for } \lambda \in (0,\sigma], \tag{4.3.19}$$

(5) if $k - 1 = d$ and $p = 1$, then we have

$$W^{k+j,p}(\Omega;\mathbb{R}) \subset C^{j,1}\left(\bar{\Omega};\mathbb{R}\right). \tag{4.3.20}$$

□

In order to understand the idea on how Theorem 4.3.14 (1) holds, we use a Sobolev inequality. Under the assumption of Theorem 4.3.14 (1), Sobolev inequality theorem is provided by

$$\|f\|_{L^q(\Omega;\mathbb{R})} \leq c\,|f|_{W^{k,p}(\Omega;\mathbb{R})} \tag{4.3.21}$$

with respect to an arbitrary $f \in C_0^\infty(\Omega;\mathbb{R})$, where c is a positive constant which is not dependent on f. Here, we pay attention to Eq. (4.3.21) when $k = 1$ (see, for example, [1, Theorem 4.31, p. 102] , [2, Théoréme IX.9, p. 162]).

Let $a > 0$. For an element $\boldsymbol{x} \in \Omega$, let $\boldsymbol{y} = a\boldsymbol{x} \in \hat{\Omega}$ and $f(\boldsymbol{x}) = \hat{f}(\boldsymbol{y})$. In this case, with respect to the left-hand side of Eq. (4.3.21), we have

$$\begin{aligned}\|f\|_{L^q(\Omega;\mathbb{R})} &= \left(\int_\Omega |f|^q \,\mathrm{d}x_1 \cdots \mathrm{d}x_d\right)^{1/q} = a^{-d/q}\left(\int_{\hat{\Omega}} |\hat{f}|^q \,\mathrm{d}y_1 \cdots \mathrm{d}y_d\right)^{1/q} \\ &= a^{-d/q}\|\hat{f}\|_{L^q(\hat{\Omega};\mathbb{R})}.\end{aligned} \tag{4.3.22}$$

On the other hand, with respect to $|f|_{W^{k,p}(\Omega;\mathbb{R})}$ in the right-hand side of Eq. (4.3.21), we get that

$$\begin{aligned}|f|_{W^{1,p}(\Omega;\mathbb{R})} &= \left(\int_\Omega \sum_{|\boldsymbol{\beta}|=1} \left|\frac{\partial^{|\boldsymbol{\beta}|} f}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}\right|^p \mathrm{d}x_1 \cdots \mathrm{d}x_d\right)^{1/p} \\ &= a^{(p-d)/p}\left(\int_{\hat{\Omega}} \sum_{|\boldsymbol{\beta}|=1} \left|\frac{\partial^{|\boldsymbol{\beta}|} \hat{f}}{\partial y_1^{\beta_1} \cdots \partial y_d^{\beta_d}}\right|^p \mathrm{d}y_1 \cdots \mathrm{d}y_d\right)^{1/p} \\ &= a^{1-d/p}|\hat{f}|_{W^{1,p}(\hat{\Omega};\mathbb{R})}\end{aligned} \tag{4.3.23}$$

holds. Here, note that the assumption on Theorem 4.3.14 (1) that $q \leq p^* = d/\{(d/p)-1\}$ is written as

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \leq \frac{1}{q},$$

and it can also be written as $1 - d/p + d/q \geq 0$. Hence, if Eq. (4.3.21) holds, then for an arbitrary $a > 0$, the inequality

$$\|\hat{f}\|_{L^q(\hat{\Omega};\mathbb{R})} \leq a^{1-d/p+d/q} c \left|\hat{f}\right|_{W^{1,p}(\hat{\Omega};\mathbb{R})} \tag{4.3.24}$$

holds. Since if $1 - d/p + d/q < 0$, then $a^{1-d/p+d/q} \to 0$ when $a \to \infty$, the assumption of Theorem 4.3.14 (1) is needed in order that Eq. (4.3.21) with respect to $\hat{f}$ holds.

Moreover, in Theorem 4.3.14 (4) and (5), the embedding relationship of the Sobolev space $W^{k,p}(\Omega;\mathbb{R})$ and Hölder space $C^{0,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ ($\sigma \in (0,1]$) is given. The relationship between the two needs to be explained. This is because in contrast to functions included in $C^{0,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ which are functions with values on all points of $\bar{\Omega}$, functions included in $W^{k,p}(\Omega;\mathbb{R})$ are functions defined on a measurable set of Ω (almost everywhere). If comparing both under these definitions, it is considered that $f \in W^{k,p}(\Omega;\mathbb{R})$ and an equivalent function $f^* \in C^{0,\sigma}\left(\bar{\Omega};\mathbb{R}\right)$ such that $f = f^*$ holds on the measurable set can be selected and

$$\|f^*\|_{C^{0,\sigma}(\bar{\Omega};\mathbb{R})} \leq c\,\|f\|_{W^{k,p}(\Omega;\mathbb{R})} \tag{4.3.25}$$

is established with respect to some $c > 0$ (cf. [1, Section 4.2, p. 79]).

Even among Hölder spaces, with respect to the embedding relationship between the Lipschitz space $C^{0,1}\left(\bar{\Omega};\mathbb{R}\right)$ and Sobolev space $W^{1,\infty}(\Omega;\mathbb{R})$, $W^{1,\infty}(\Omega;\mathbb{R}) = C^{0,1}\left(\bar{\Omega};\mathbb{R}\right)$ is established for the convex set Ω and real-valued (not $\mathbb{R}^n$) function f. In fact, the norms of $f \in W^{1,\infty}(\Omega;\mathbb{R})$ and its equivalent function $f^* \in C^{0,1}\left(\bar{\Omega};\mathbb{R}\right)$ are the same ([7, Proposition 1.39, p. 23]).

Furthermore, in Theorem 4.3.14, k and j were assumed to be integers. It is known that the following relationship holds with respect to embedding relationships when these are expanded to real numbers s and t (cf. [5, Eq. (1.4.4.5) and Eq. (1.4.4.6), p. 27]). Suppose s and $t \leq s$ are non-negative real numbers and p and $q \geq p$ are defined by the relationship when k is replaced by s in Theorem 4.3.14. Here, if

$$s - \frac{d}{p} \geq t - \frac{d}{q} \tag{4.3.26}$$

is satisfied, we get

$$W^{s,p}(\Omega;\mathbb{R}) \subset W^{t,q}(\Omega;\mathbb{R}). \tag{4.3.27}$$

Furthermore, when Ω is a Lipschitz domain and $k < k + \sigma = s - d/p < k + 1$ (k is a non-negative integer), we get

$$W^{s,p}(\Omega;\mathbb{R}) \subset C^{k,\sigma}(\Omega;\mathbb{R}). \tag{4.3.28}$$

4.4 Operators

In Sect. 4.3, various function spaces were defined and we looked at how these become a Banach space or a Hilbert space. In function optimization problems, it can be said that we had been looking at linear spaces involving design variables. In optimum design problems with functions as design variables, it can also be linear spaces containing the state variables which are the solution to the state determination problems. Next, we want to examine how the derivative of the cost function is defined when the cost function is given by the integral (functional) consisting of a design variable or a state variable. Here, as a preparation for this, a mapping from between two Banach spaces is defined as an operator. Moreover, operators with range as real numbers are defined as functionals. Furthermore, the set of functionals which domain is a function space is a Banach space and defined as a dual space with respect to that function space. This dual space is an important function space containing gradients when defining the derivative of a functional in Sect. 4.5. In this section we shall also describe important theorems (trace theorem and Riesz representation theorem) relating to operators other than dual spaces.

4.4.1 Bounded Linear Operator

Performing a calculation on a function is to define a mapping between two function spaces. The mapping in this case in particular is called an operator. An operator with linearity is called a linear operator and can be defined as follows. Let X and Y be Banach spaces on K. With respect to arbitrary $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $\alpha_1, \alpha_2 \in K$, if the mapping $f : X \to Y$ satisfies

$$f\left(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2\right) = \alpha_1 f\left(\boldsymbol{x}_1\right) + \alpha_2 f\left(\boldsymbol{x}_2\right), \tag{4.4.1}$$

then f is called a linear mapping or a linear operator. Moreover, f can also be called a linear form. Furthermore, when the mapping $f : X \to Y$ is a bijection (one-to-one and onto mapping), f is said to be an isomorphism.

For example, the derivative operator $\mathcal{D} = \left(\partial / \partial x_i\right)_{i \in \{1,\ldots,d\}} : C^1\left(\mathbb{R}^d; \mathbb{R}\right) \to C\left(\mathbb{R}^d; \mathbb{R}^d\right)$ of a function $u \in C^1\left(\mathbb{R}^d; \mathbb{R}\right)$ is a linear operator based on the fact that

$$\mathcal{D}\left(\alpha_1 u_1 + \alpha_2 u_2\right) = \alpha_1 \mathcal{D}\left(u_1\right) + \alpha_2 \mathcal{D}\left(u_2\right)$$

is satisfied.

Furthermore, if a linear operator f satisfies

$$\sup_{\boldsymbol{x} \in X \setminus \{\mathbf{0}_X\}} \frac{\left\|f\left(\boldsymbol{x}\right)\right\|_Y}{\left\|\boldsymbol{x}\right\|_X} < \infty, \tag{4.4.2}$$

then f is called a bounded linear operator. In this book, the entire set of bounded linear operators from X to Y is expressed as $\mathcal{L}\left(X; Y\right)$. Moreover, if

f satisfies Eq. (4.4.2), then $f : X \to Y$ is in fact continuous. This is because there exists some positive constant β and

$$\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_Y \le \beta \|\boldsymbol{x} - \boldsymbol{y}\|_X$$

holds with respect to an arbitrary $\boldsymbol{x}$, $\boldsymbol{y} \in X$. Moreover, if $f : X \to Y$ is continuous, then f is bounded (cf. [8, Theorem 2.8-3, p. 104]). Here, a bounded linear operator is also called a continuous linear operator. Furthermore, the following results can be obtained with respect to the entire set of bounded linear operators, $\mathcal{L}(X; Y)$ (cf. [9, Theorem 7.6, p. 150]).

Proposition 4.4.1 (Bounded linear operators) When X and Y are Banach spaces, $\mathcal{L}(X; Y)$ is a Banach space with

$$\|f\|_{\mathcal{L}(X;Y)} = \sup_{\boldsymbol{x} \in X \setminus \{\boldsymbol{0}_X\}} \frac{\|f(\boldsymbol{x})\|_Y}{\|\boldsymbol{x}\|_X}$$

as the norm $\|\cdot\|_{\mathcal{L}(X;Y)}$. □

Let us give an example of bounded linear operators. With n and m as natural numbers, the matrix $\mathbb{R}^{n \times m}$ with n rows and m columns is a bounded linear operator and the set of all $\mathbb{R}^{n \times m}$ can be expressed as $\mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. When we set $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ with respect to $\boldsymbol{x} \in \mathbb{R}^m$, the norm of the matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}$ is defined by

$$\|\boldsymbol{A}\|_{\mathbb{R}^{n \times m}} = \|\boldsymbol{y}\|_{\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)} = \sup_{\boldsymbol{x} \in \mathbb{R}^m \setminus \{\boldsymbol{0}_{\mathbb{R}^m}\}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_{\mathbb{R}^n}}{\|\boldsymbol{x}\|_{\mathbb{R}^m}}. \tag{4.4.3}$$

From this definition, when $\boldsymbol{A}$ is a positive definite real matrix ($n = m$) and the Euclidean norm $\|\boldsymbol{x}\|_{\mathbb{R}^n}$ is used, the norm $\|\boldsymbol{A}\|_{\mathbb{R}^{n \times n}}$ is given by the maximum eigenvalue.

4.4.2 Trace Theorem

In boundary value problems of partial differential equations, which will be looked at in detail in Chap. 5, operations for extracting values at the boundary from functions defined on a domain will be studied. Such an operation is conducted using trace operators. These operators are bounded linear operators. Let us express the outward unit normal at the boundary (Definition A.5.4) as $\boldsymbol{\nu}$ and write $\partial_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\nabla}$. In such a case, the following trace theorem can be obtained (cf. [5, Theorem 1.5.1.2, p. 37, and Theorem 1.5.1.3, p. 38])

Theorem 4.4.2 (Trace theorem) Let $k, l \in \mathbb{N}$, $\sigma \in (0, 1)$, $p \in (1, \infty)$, $s - 1/p = l + \sigma$ and $s \le k + 1$. A boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^d$ is a $C^{k,1}$ class boundary when $k \ge 1$ and a Lipschitz boundary when $k = 0$. In this case, a bounded linear operator $\gamma : W^{s,p}(\Omega; \mathbb{R}) \to \prod_{i \in \{0,1,\dots,l\}} W^{s-i-1/p,p}(\partial\Omega; \mathbb{R})$ that satisfies

$$\gamma f = \left\{ f|_{\partial\Omega}, \partial_\nu f|_{\partial\Omega}, \dots, \partial_\nu^l f|_{\partial\Omega} \right\}$$

with respect to $f \in C^{k,1}(\Omega;\mathbb{R})$ uniquely exists. This operator has a continuous right inverse operator (if $\gamma^{-1} g = f$, $\gamma f = g$ is satisfied) not dependent on p. □

The mapping γ in Theorem 4.4.2 is called a trace operator. In this book, the case $s = 1$ ($l = 0$) is assumed on the whole so it is used with the meaning $\gamma f = f|_{\partial\Omega}$.

When the dimension of a domain with respect to a function was changed from d dimensions to $d-1$ dimensions via trace operators, the order of the derivative changed from s to $t = s - 1/p$. The reason for such a change follows from the fact that the index of differentiability remains unchanged as

$$s - \frac{d}{p} = \left(s - \frac{1}{p}\right) - \frac{d-1}{p} = t - \frac{d-1}{p}.$$

Moreover, the following results can be obtained for a function in $W_0^{s,p}(\Omega;\mathbb{R})$ (Definition 4.3.10) (cf. [5, Theorem 1.5.1.5, p. 38 and Corollary 1.5.1.6, p. 39]).

Theorem 4.4.3 (Trace theorem with respect to $W_0^{s,p}(\partial\Omega;\mathbb{R})$) Let $k, l \in \mathbb{N}$, $\sigma \in (0,1)$, $p \in (1,\infty)$, $s - 1/p = l + \sigma$ and $s \leq k+1$. A boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^d$ is a $C^{k,1}$ class boundary when $k \geq 1$ and a Lipschitz domain when $k = 0$. Here, $f \in W_0^{s,p}(\Omega;\mathbb{R})$ is equivalent to $f \in W^{s,p}(\Omega;\mathbb{R})$ and

$$\gamma f = \gamma \partial_\nu f = \cdots = \gamma \partial_\nu^l f = 0 \quad \text{on } \partial\Omega$$

being satisfied. □

In view of Theorem 4.4.3, the space $H_0^1(\Omega;\mathbb{R}) = W_0^{1,2}(\Omega;\mathbb{R})$ is defined as

$$H_0^1(\Omega;\mathbb{R}) = \left\{ u \in H^1(\Omega;\mathbb{R}) \;\middle|\; u = 0 \text{ on } \partial\Omega \right\}.$$

This function space is used beyond Chap. 5 when considering the solution to partial differential equations using the condition for which it is zero at the boundary (homogeneous Dirichlet condition). The space $H_0^1(\Omega;\mathbb{R})$ is a linear subspace of $H^1(\Omega;\mathbb{R})$ and a real Hilbert space. In this case, the inner product and norm used are the same as for $H^1(\Omega;\mathbb{R})$.

4.4.3 Calderón Extension Theorem

In shape optimization problems with varying domain examined in Chap. 9, it is assumed that the domain in which a boundary value problem of partial differential equation is defined itself fluctuates. Therefore, the given functions used in defining the boundary value problems and the solution function need to be elements of function spaces such that they are also defined in the domain after variation. Here, the following Calderón extension theorem is used in order to extend a bounded domain Ω to $\mathbb{R}^d$. The bounded linear operator which existence is guaranteed in this theorem is called the extension operator (cf. [1, Theorem 5.28, p. 156]).

Theorem 4.4.4 (Calderón extension theorem) Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. For every $k \in \mathbb{N}$ and $p \in (1, \infty)$, there exists a bounded linear operator

$$e_\Omega : W^{k,p}(\Omega; \mathbb{R}) \to W^{k,p}\left(\mathbb{R}^d; \mathbb{R}\right)$$

and with respect to an arbitrary $u \in W^{k,p}(\Omega; \mathbb{R})$, we have

$$e_\Omega(u) = u \quad \text{in } \Omega,$$
$$\|e_\Omega(u)\|_{W^{k,p}(\mathbb{R}^d;\mathbb{R})} \le c \|u\|_{W^{k,p}(\Omega;\mathbb{R})},$$

where c is a constant dependent only on k and p. □

Note that it is $k \ge 1$ in Theorem 4.4.4.

4.4.4 Bounded Bilinear Operators

Operators with bilinearity can also be defined. Let X, Y, Z be a Banach space on K. If the mapping $f : X \times Y \to Z$ satisfies

$$f(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y}_1) = \alpha_1 f(\boldsymbol{x}_1, \boldsymbol{y}_1) + \alpha_2 f(\boldsymbol{x}_2, \boldsymbol{y}_1),$$
$$f(\boldsymbol{x}_1, \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2) = \alpha_1 f(\boldsymbol{x}_1, \boldsymbol{y}_1) + \alpha_2 f(\boldsymbol{x}_1, \boldsymbol{y}_2)$$

with respect to arbitrary $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Y$ and $\alpha_1, \alpha_2 \in K$, then f is called a bilinear operator or a bilinear form.

Furthermore, when

$$\sup_{\boldsymbol{x} \in X \setminus \{\boldsymbol{0}_X\},\ \boldsymbol{y} \in Y \setminus \{\boldsymbol{0}_Y\}} \frac{\|f(\boldsymbol{x}, \boldsymbol{y})\|_Z}{\|\boldsymbol{x}\|_X \|\boldsymbol{y}\|_Y} < \infty$$

is satisfied with respect to an arbitrary $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$, f is called a bounded bilinear operator. In this section the entire set of bounded linear operator from $X \times Y$ to Z is expressed as $\mathcal{L}(X, Y; Z)$.

Examples of bounded bilinear operators include the kinetic energy $\kappa(u, \dot{u})$ used in Eq. (4.1.1) and elastic potential energy $\pi_{\mathrm{I}}(u)$ used in Eq. (4.1.8). The expression $\kappa(u, \dot{u})$ can be written as $b(\dot{u}, \dot{u})$, where $b(u, v)$ is defined by Eq. (4.6.10), focusing on bilinearity with respect to u. Meanwhile, the functional $\pi_{\mathrm{I}}(u)$ can be written as $a(u, u)$ using $a(u, v)$ defined by Eq. (4.6.17), focusing on the bilinearity with respect to u. Although these are bounded bilinear operators, they are operators with $\mathbb{R}$ as the range. Such an operator is a functional defined more detail in the next section. Here, $b(\,\cdot\,,\,\cdot\,)$ or $a(\,\cdot\,,\,\cdot\,)$ are examples of bounded bilinear functionals.

4.4.5 Bounded Linear Functional

Operators with $\mathbb{R}$ (or $\mathbb{C}$) as range are called functionals. In function optimization problems, the cost function is given as a mapping from a function space to real numbers. Hence, to know the properties of functionals one needs to know the properties of cost functions.

The linearity and boundedness of a functional is defined by the relationships of Eq. (4.4.1) and Eq. (4.4.2) taking $Y = \mathbb{R}$. However, expressing them as $f(\,\cdot\,) = \langle \boldsymbol{\phi}, \,\cdot\, \rangle : X \to \mathbb{R}$, a bounded linear functional is defined in the following way. Let X be a Banach space on K. If the functional $\langle \boldsymbol{\phi}, \,\cdot\, \rangle : X \to \mathbb{R}$ satisfies

$$\langle \boldsymbol{\phi}, \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \rangle = \alpha_1 \langle \boldsymbol{\phi}, \boldsymbol{x}_1 \rangle + \alpha_2 \langle \boldsymbol{\phi}, \boldsymbol{x}_2 \rangle$$

with respect to arbitrary $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $\alpha_1, \alpha_2 \in K$, then $\langle \boldsymbol{\phi}, \,\cdot\, \rangle$ is called a linear functional on X. Furthermore, when

$$\sup_{\boldsymbol{x} \in X \setminus \{\boldsymbol{0}_X\}} \frac{|\langle \boldsymbol{\phi}, \boldsymbol{x} \rangle|}{\|\boldsymbol{x}\|_X} < \infty$$

is satisfied, $\langle \boldsymbol{\phi}, \,\cdot\, \rangle$ is called a bounded linear functional on X.

If X is a finite-dimensional vector space $\mathbb{R}^d$ and $\boldsymbol{\phi} \in \mathbb{R}^d$ is selected and fixed, a functional $(\boldsymbol{\phi}, \,\cdot\,)_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}$ using the inner product is a bounded linear functional on $X = \mathbb{R}^d$.

4.4.6 Dual Space

If X were a finite-dimensional vector space $\mathbb{R}^d$, choosing a bounded linear functional $(\boldsymbol{\phi}, \,\cdot\,)_{\mathbb{R}^d}$ on $X = \mathbb{R}^d$ is equivalent to choosing an element $\boldsymbol{\phi}$ on $\mathbb{R}^d$. Here the entire set $\mathbb{R}^d$ of $\boldsymbol{\phi}$ identified as a bounded linear functional, distinguishing it from X, is called the dual space of X and expressed as $X' = \mathbb{R}^d$. Generalizing this, the dual space can be defined as follows.

Definition 4.4.5 (Dual space) When X is a Banach space, the entire set $\mathcal{L}(X; \mathbb{R})$ of bounded linear functionals on X is written as X' and is called the dual space of X. Moreover, $\langle \cdot, \cdot \rangle : X' \times X \to \mathbb{R}$ is also expressed as $\langle \cdot, \cdot \rangle_{X' \times X}$ and is called a dual product. □

The dual space in Definition 4.4.5 can also be called an adjoint space.

Based on this definition, from the fact that a dual space of a Banach space is a set of all bounded linear operators $\mathcal{L}(X; \mathbb{R})$, a dual space of a Banach space is a Banach space with

$$\|\boldsymbol{\phi}\|_{X'} = \sup_{\boldsymbol{x} \in X \setminus \{\boldsymbol{0}_X\}} \frac{|\langle \boldsymbol{\phi}, \boldsymbol{x} \rangle|}{\|\boldsymbol{x}\|_X} \tag{4.4.4}$$

as the norm from Proposition 4.4.1.

Weak complete and dual weak complete

In discussions so far it became clear that not only the Banach spaces themselves but their dual spaces are also Banach spaces (complete norm spaces) with respect to norms such as Eq. (4.4.4). In other words, it indicates that Cauchy series measured by the various norms always converge. In the case of a finite-dimensional vector space, its dual space is also the same finite-dimensional vector space. Hence, convergence could be measured using the same norm. However, the definition of the norm generally differs for a Banach space and its dual space. For this reason, convergence other than that using the norm can be defined too. Here, let us define convergence using the dual product.

Definition 4.4.6 (Weak convergence) Let X be a Banach space and X' its dual space. When an infinite sequence of points $\{x_n\}_{n\in\mathbb{N}} \in X$ with respect to an arbitrary $\boldsymbol{\phi} \in X'$ satisfies

$$\lim_{n,m\to\infty} \langle \boldsymbol{\phi}, x_n - x_m \rangle = 0,$$

$\{x_n\}_{n\in\mathbb{N}}$ is called a weak Cauchy sequence. The convergence of a weak Cauchy sequence is called a weak convergence and is written as $x_n \to x$ weakly in X. When any weak Cauchy sequence of X converges within X, then X is said to be weak complete. Furthermore, when an arbitrary infinite sequence of points of a subset V of weakly complete X includes an infinite subsequence that weakly converges within V, then V is referred to as weak compact. □

By contrast, convergence relating to the norm is called a strong convergence and written as $x_n \to x$ strongly in X. Also, if the purposes of a Banach space and its dual space are reversed, a definition of another convergence is possible.

Definition 4.4.7 (Dual weak convergence) Let X be a Banach space and X' be its dual space. When the infinite sequence of points $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}} \in X'$ satisfies

$$\lim_{n,m\to\infty} \langle \boldsymbol{\phi}_n - \boldsymbol{\phi}_m, \boldsymbol{x} \rangle = 0$$

with respect to an arbitrary $\boldsymbol{x} \in X$, then $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}} \in X'$ is called a dual weak Cauchy sequence. Convergence of a dual weak Cauchy sequence is called a dual weak convergence and is expressed as $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ $*$-weakly in X'. When any of the dual weak Cauchy sequence in X' converges to a point within X', X' is said to be dual weak complete. Furthermore, if an arbitrary infinite sequence of points of the subset V' of dual weak complete X' includes an infinite subsequence that is dual weakly convergent in V', then V' is said to be dual weak compact. □

As shown later, the Fréchet derivative of the cost function in a function optimization problem is defined using the dual product. Weak completeness and dual weak completeness are qualities which are necessary when seeking the minimum point using the Fréchet derivative of the cost function.

Let us consider the conditions which guarantee that a Banach space has weak completeness. For this purpose, a reflexive Banach space is define as follows.

Definition 4.4.8 (Reflexive Banach space) Let X be a Banach space. Let X' and $X'' = (X')'$ be the dual space and second dual space of X, respectively.

If an evaluation map $\boldsymbol{\tau} : X \to X''$ ($\boldsymbol{x} \in X$ generates a scalar with respect to $\boldsymbol{f} \in X'$) such that

$$\langle \boldsymbol{f}, \boldsymbol{\tau}(\boldsymbol{x}) \rangle_{X' \times X''} = \langle \boldsymbol{f}, \boldsymbol{x} \rangle_{X' \times X}$$

holds with respect to all $(\boldsymbol{x}, \boldsymbol{f}) \in X \times X'$ is a one-to-one and onto mapping, then X is called a reflexive Banach space. □

The following results can be obtained with respect to a reflexive Banach space (cf. [9, Theorem 8.33, p. 193]).

Proposition 4.4.9 (Weak complete) A reflexive Banach space is weakly complete. □

From Proposition 4.4.9, if they are in a reflexive Banach space, weak Cauchy series or dual weak Cauchy series are guaranteed to converge to an element within that space. A Sobolev space is a Banach space (Proposition 4.3.11), but regarding reflexivity of a Sobolev space, the following result can be obtained (cf. [10, Theorem 2.25, p. 41]).

Proposition 4.4.10 (Separability and reflexivity of Sobolev spaces) Let $\Omega \subset \mathbb{R}^d$. If $p \in (1, \infty)$ (note it is not $[1, \infty]$) and $k \in \mathbb{N} \cup \{0\}$, then $W^{k,p}(\Omega; \mathbb{R})$ and $W_0^{k,p}(\Omega; \mathbb{R})$ are reflexive. □

Since $H^k(\Omega; \mathbb{R})$ is a Sobolev space when $p = 2$, it is a reflexive Banach space. Therefore, from Proposition 4.4.10, $H^k(\Omega; \mathbb{R})$ is weakly complete. On the other hand, from the fact that $L^1(\Omega; \mathbb{R})$ or $L^\infty(\Omega; \mathbb{R})$ are not reflexive, they are not weakly complete.

Moreover, the following is known about the compactness of Banach spaces. A unit sphere in infinite-dimensional space is not compact. This is because, selecting an infinite number of basic vectors such as $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $\ldots$, an infinite sequence of points which does not include a Cauchy sequence can be constructed (cf. [12, Subsection 1.2.1, p. 15]). However, the following result is known (cf. [9, Theorem 8.36, p. 194]).

Proposition 4.4.11 (Weak compact) A closed unit sphere in a reflexive Banach space is a weak compact. □

Dual space of Sobolev space

With respect to the dual space of a Sobolev space, a clear result can be obtained. Firstly, with respect to index p of $L^p(\Omega; \mathbb{R})$, a duality index q is defined as follows.

Definition 4.4.12 (Duality index) For $p \in [1, \infty)$, a constant $q \in [1, \infty]$ satisfying

$$\frac{1}{q} + \frac{1}{p} = 1$$

is called a duality index. Moreover, with respect to $L^p(\Omega;\mathbb{R})$, $L^q(\Omega;\mathbb{R})$ is called a dual space of $L^p(\Omega;\mathbb{R})$ and is expressed as $(L^p(\Omega;\mathbb{R}))'$. □

Using the duality index, let us see how the dual space of Sobolev space $W^{k,p}(\Omega;\mathbb{R})$ for $k \geq 1$ is defined. Firstly, let $\Omega = (0,1)$ and consider the dual space $\left(H^1((0,1);\mathbb{R})\right)'$ of $H^1((0,1);\mathbb{R})$. When an arbitrary $f \in \left(H^1((0,1);\mathbb{R})\right)'$ is selected, f becomes a bounded linear functional with respect to an arbitrary $v \in H^1((0,1);\mathbb{R})$. According to the Riesz representation theorem (Theorem 4.4.17) which will be shown later, there exists a unique $u \in H^1((0,1);\mathbb{R})$ which satisfies

$$\langle f, v\rangle = (u,v)_{H^1((0,1);\mathbb{R})} = \int_0^1 (uv + u'v')\,\mathrm{d}x \tag{4.4.5}$$

with respect to an arbitrary $v \in H^1((0,1);\mathbb{R})$. Hence, $f_0, f_1 \in L^2((0,1);\mathbb{R})$ exist which satisfy

$$\langle f, v\rangle = f(v) = \int_0^1 (f_0 v + f_1 v')\,\mathrm{d}x. \tag{4.4.6}$$

The norm of f is defined by

$$\|f\|_{(H^1((0,1);\mathbb{R}))'} = \sup_{v\in H^1((0,1);\mathbb{R})} \frac{|\langle f, v\rangle|}{\|v\|_{H^1((0,1);\mathbb{R})}}.$$

Here, since

$$\begin{aligned}|\langle f, v\rangle| &= \left|\int_0^1 (f_0 v + f_1 v')\,\mathrm{d}x\right| \\ &\leq \left(\|f_0\|_{L^2((0,1);\mathbb{R})} + \|f_1\|_{L^2((0,1);\mathbb{R})}\right)\|v\|_{H^1((0,1);\mathbb{R})}\end{aligned}$$

is established from the Schwarz inequality (see Theorem A.9.1), we expect that

$$\begin{aligned}&\|f\|_{(H^1((0,1);\mathbb{R}))'} \\ &\quad = \inf_{f_0, f_1\in L^2((0,1);\mathbb{R})}\left\{ \|f_0\|_{L^2((0,1);\mathbb{R})} + \|f_1\|_{L^2((0,1);\mathbb{R})} \;\middle|\; \text{Eq. (4.4.6)}\right\}\end{aligned}$$

can also be established.

Generalizing this leads to the following result (cf. [1, Theorem 3.8 and Theorem 3.9, p. 62], [10, Theorem 2.20 and Theorem 2.21, p. 38]).

Proposition 4.4.13 (Dual space of $W^{k,p}(\Omega;\mathbb{R})$) Let $\Omega \subset \mathbb{R}^2$. Let $k \in \mathbb{N}\cup\{0\}$ and $p \in [1,\infty)$. Also, let q be a dual index with respect to p and $\left(W^{k,p}(\Omega;\mathbb{R})\right)'$ be the dual space of $W^{k,p}(\Omega;\mathbb{R})$. For an arbitrary $f \in \left(W^{k,p}(\Omega;\mathbb{R})\right)'$, there exists $f_{\boldsymbol{\beta}} \in L^q(\Omega;\mathbb{R})$ which satisfies

$$f(v) = \sum_{|\boldsymbol{\beta}|\leq k} \int_\Omega \nabla^{\boldsymbol{\beta}} v f_{\boldsymbol{\beta}}\,\mathrm{d}x \tag{4.4.7}$$

for all $v \in W^{k,p}(\Omega;\mathbb{R})$. Furthermore,

$$\|f\|_{(W^{k,p}(\Omega;\mathbb{R}))'} = \inf_{f_{\boldsymbol{\beta}} \in L^q(\Omega;\mathbb{R}),\ |\boldsymbol{\beta}| \le k} \left\{ \sum_{|\boldsymbol{\beta}| \le k} \|f_{\boldsymbol{\beta}}\|_{L^q(\Omega;\mathbb{R})} \,\middle|\, \text{Eq. (4.4.7)} \right\}$$

holds. □

In Proposition 4.4.13, the case of $p = \infty$ is removed. An explanation of the dual space of $L^\infty(\Omega;\mathbb{R})$ can be found in [13, Example 5, p. 118].

The following results can be obtained with respect to the dual space of the Sobolev space $W_0^{k,p}(\Omega;\mathbb{R})$. Here, $f_{\boldsymbol{\beta}}$ is an element of $L^q(\Omega;\mathbb{R})$ shown to exist by Proposition 4.4.13. Functions g and $g_{\boldsymbol{\beta}}$ are taken to be elements of $(C_0^\infty(\Omega;\mathbb{R}))'$ satisfying

$$g(\phi) = \sum_{|\boldsymbol{\beta}| \le k} (-1)^{|\boldsymbol{\beta}|} \nabla^{\boldsymbol{\beta}} g_{\boldsymbol{\beta}}(\phi), \quad g_{\boldsymbol{\beta}}(\phi) = \int_\Omega \phi f_{\boldsymbol{\beta}} \,\mathrm{d}x \quad \text{for } |\boldsymbol{\beta}| \le k \tag{4.4.8}$$

with respect to an arbitrary $\phi \in C_0^\infty(\Omega;\mathbb{R})$. Here, from the fact that

$$\nabla^{\boldsymbol{\beta}} g_{\boldsymbol{\beta}}(\phi) = (-1)^{|\boldsymbol{\beta}|} \int_\Omega \nabla^{\boldsymbol{\beta}} \phi f_{\boldsymbol{\beta}} \,\mathrm{d}x$$

holds with respect to an arbitrary $\phi \in C_0^\infty(\Omega;\mathbb{R})$, the identity

$$g(\phi) = \sum_{|\boldsymbol{\beta}| \le k} g_{\boldsymbol{\beta}}\left(\nabla^{\boldsymbol{\beta}} \phi\right) = f(\phi)$$

can be obtained. Here, Eq. (4.4.7) was used. This result shows that $f \in \left(W^{k,p}(\Omega;\mathbb{R})\right)'$ is a function which expands $C_0^\infty(\Omega;\mathbb{R})$ on the Schwartz distribution $g \in (C_0^\infty(\Omega;\mathbb{R}))'$ to $W^{k,p}(\Omega;\mathbb{R})$. As a result, it should be noted that with respect to embedding relationships of $C_0^\infty(\Omega;\mathbb{R}) \subset W_0^{k,p}(\Omega;\mathbb{R}) \subset W^{k,p}(\Omega;\mathbb{R})$, the embedding relationships of their dual spaces become $\left(W^{k,p}(\Omega;\mathbb{R})\right)' \subset \left(W_0^{k,p}(\Omega;\mathbb{R})\right)' \subset (C_0^\infty(\Omega;\mathbb{R}))'$ (Practice **4.4**).

These relationships give the following results (cf. [1, Theorem 3.12, p. 64], [10, Theorem 2.3, p. 40]. With respect to $\left(H_0^1(\Omega;\mathbb{R})\right)'$ [3, Theorem 1, p. 283], [12, Example 3.4, p. 80]).

Proposition 4.4.14 (Dual space of $W_0^{k,p}(\Omega;\mathbb{R})$) Let $\Omega \subset \mathbb{R}^d$, $k \in \mathbb{N}$ and $p \in [0,\infty)$. Also, let $v_{\boldsymbol{\beta}}$ be an element of $L^q(\Omega;\mathbb{R})$ which has been identified by Proposition 4.4.13. Let $\left(W_0^{k,p}(\Omega;\mathbb{R})\right)'$ be the dual space of $W_0^{k,p}(\Omega;\mathbb{R})$. In this case, $g \in \left(W_0^{k,p}(\Omega;\mathbb{R})\right)'$ is uniquely determined in the sense of the Schwartz distribution $g \in (C_0^\infty(\Omega;\mathbb{R}))'$ by Eq. (4.4.8). Furthermore,

$$\|g\|_{\left(W_0^{k,p}(\Omega;\mathbb{R})\right)'} = \inf_{g_{\boldsymbol{\beta}} \in L^q(\Omega;\mathbb{R}), |\boldsymbol{\beta}| \le k} \left\{ \sum_{|\boldsymbol{\beta}| \le k} \|g_{\boldsymbol{\beta}}\|_{L^q(\Omega;\mathbb{R})} \,\middle|\, \text{Eq. (4.4.8)} \right\}$$

is established. □

From Proposition 4.4.14, the element of $\left(W_0^{k,p}(\Omega;\mathbb{R})\right)'$ is a function for which the k-th integral (note that it is not a differential) is q-th order integrable. From this fact, $\left(W_0^{k,p}(\Omega;\mathbb{R})\right)'$ can also be expressed as $W^{-k,q}(\Omega;\mathbb{R})$. Moreover, the case $k=0$ is excluded in Proposition 4.4.14 because $W_0^{0,p}(\Omega;\mathbb{R}) = L^p(\Omega;\mathbb{R})$ was defined in Definition 4.3.10.

4.4.7 Rellich–Kondrachov Compact Embedding Theorem

The result which rewrites the embedding relationship of Sobolev space given by Sobolev embedding theorem (Theorem 4.3.14) as an embedding relationship with compactness is called the Rellich–Kondrachov compact embedding theorem. Here, a Banach space X compactly embedded in a Banach space Y is defined by:

(1) $\|\boldsymbol{\phi}\|_Y \leq c\,\|\boldsymbol{\phi}\|_X$ being established with respect to an arbitrary $\boldsymbol{\phi} \in Y$ and

(2) an arbitrary bounded infinite sequence of points in X including a subsequence converging to within Y with the norm of Y (X is relative compact)).

Here, it is written as $X \Subset Y$. Therefore, if X is weakly complete, X is complete with the norm $\|\cdot\|_Y$ of Y.

The following result has been obtained (cf. [1, Theorem 6.3, p. 168], [10, Chap. 7, p. 153]).

Theorem 4.4.15 (Rellich–Kondrachov compact embedding theorem) Let $\Omega \subset \mathbb{R}^d$, $k \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. In this case:

(1) if $k - d/p < 0$, with $p^* = d/\{(d/p) - k\}$, then the embedding

$$W^{k+j,p}(\Omega;\mathbb{R}) \Subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [1,p^*)\,, \tag{4.4.9}$$

(2) if $k - d/p = 0$, then the embedding

$$W^{k+j,p}(\Omega;\mathbb{R}) \Subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [1,\infty)\,, \tag{4.4.10}$$

(3) if $k - d/p = j + \sigma > 0$ $(\sigma \in (0,1))$, or when $k = d$ and $p = 1$, then the embedding

$$W^{k+j,p}(\Omega;\mathbb{R}) \Subset W^{j,q}(\Omega;\mathbb{R}) \quad \text{for } q \in [p,\infty) \tag{4.4.11}$$

holds. Furthermore, if Ω is a Lipschitz domain, then

(4) for $k - d/p = j + \sigma > 0$ $(\sigma \in (0,1))$, or when $k = d$ and $p = 1$, we have

$$W^{k+j,p}(\Omega;\mathbb{R}) \Subset C^{j,\lambda}\left(\bar{\Omega};\mathbb{R}\right) \quad \text{for } \lambda \in (0,\sigma]\,. \tag{4.4.12}$$

□

If the Sobolev embedding theorem (Theorem 4.3.14) and Rellich–Kondrachov compact embedding theorem (Theorem 4.4.15) are compared, the condition $q \in [p, p^*]$ in Eq. (4.3.16) is different in that it is $q \in [1, p^*)$ in Eq. (4.4.9).

Based on Theorem 4.4.15, the following result can be obtained in relation to the completeness of $H^k(\Omega;\mathbb{R})$.

Proposition 4.4.16 (Completeness of $H^k(\Omega;\mathbb{R})$ in $L^2(\Omega;\mathbb{R})$) An arbitrary infinite sequence of points of $H^k(\Omega;\mathbb{R})$ ($k \in \{1, 2, \ldots\}$) includes an infinite subsequence which strongly converges in $H^{k-1}(\Omega;\mathbb{R})$ using $\|\cdot\|_{H^{k-1}(\Omega;\mathbb{R})}$. □

The result of Proposition 4.4.16 relates in the following way to the optimum design problem defined in function space shown in Chap. 7 and beyond. In shape optimization problems shown in Chaps. 8 and 9, we assume that the linear space X which contains the design variables is defined with $H^1(\Omega;\mathbb{R})$ or $H^1(\Omega;\mathbb{R}^d)$, and the admissible set $\mathcal{D}$ is defined with $H^2(\Omega;\mathbb{R})$ or $H^2(\Omega;\mathbb{R}^d)$. Here if the trial points are updated using the gradient method or the Newton method, such a sequence of points can be thought to be an infinite sequence of points on $\mathcal{D}$. When considering its convergence, Proposition 4.4.16 shows the fact that the existence of a strongly convergent subsequence using the norm of X is guaranteed.

4.4.8 Riesz Representation Theorem

Before we end this section (Sect. 4.4), let us discuss the Riesz representation theorem which is used when showing the unique existence of the solution to boundary value problem of elliptic partial differential equation (Definition A.7.1) (cf. [1, Theorem 1.12, p. 6], [12, Theorem 3.6, p. 79]).

Theorem 4.4.17 (Riesz representation theorem) Let X be a Hilbert space, X' be the dual space of X, $(\cdot\,,\cdot)_X$ the inner product on X and $\langle\cdot\,,\cdot\rangle_{X'\times X}$ be the dual product. For any $\boldsymbol{\phi} \in X'$, there exist some unique $\boldsymbol{x} \in X$ such that

$$\langle \boldsymbol{\phi}, \boldsymbol{y}\rangle_{X'\times X} = (\boldsymbol{x}, \boldsymbol{y})_X, \quad \|\boldsymbol{\phi}\|_{X'} = \|\boldsymbol{x}\|_X$$

holds for every $\boldsymbol{y} \in X$. Moreover, there exists an isomorphism $\boldsymbol{\tau} : X' \to X$ which satisfies

$$\langle \boldsymbol{\phi}, \boldsymbol{y}\rangle_{X'\times X} = (\boldsymbol{\tau}\boldsymbol{\phi}, \boldsymbol{y})_X, \quad \|\boldsymbol{\tau}\|_{\mathcal{L}(X';X)} = 1.$$

□

When X is a finite-dimensional vector space $\mathbb{R}^d$, the dual product matches the inner product and $X' = X$ with $\boldsymbol{\tau}$ becoming the identity mapping.

Using the isomorphism $\boldsymbol{\tau}$ whose existence was shown via Theorem 4.4.17, the inner product for the dual space X' of a Hilbert space X can be defined as $(\boldsymbol{\phi}, \boldsymbol{\varphi})_{X'} = (\boldsymbol{\tau}\boldsymbol{\phi}, \boldsymbol{\tau}\boldsymbol{\varphi})_X$. If this inner product is used, X' becomes a Hilbert space too.

The Riesz representation theorem will be referred to as the Lax–Milgram theorem (Theorem 5.2.4) in Chap. 5 and will be used in Chap. 5 as well as in Chap. 7 and beyond.

4.5 Generalized Derivatives

In Sect. 4.4, bounded linear operators and bounded linear functionals in Banach space were defined and we saw how the set of all bounded linear functionals becomes a dual space. In this section, this relationship is used to show the definition of a derivative with respect to operators or functionals. Here, the definition of a Gâteaux derivative, which is also called a directional derivative, is shown first, then the definition of the Fréchet derivative, which may define gradients, is provided. In particular, a gradient in the Fréchet derivative of a functional is defined as an element of dual space with respect to a Banach space of a variation vector. This relationship is an essential relationship when applying optimization theorems using derivatives of cost functions, as seen in Chap. 2, or gradient methods, looked at in Chap. 3, to function optimization problems.

4.5.1 Gâteaux Derivative

Firstly, let us start by looking at the Gâteaux derivative defined in loose conditions. In this section, X and Y are assumed to be Banach spaces and functions (operators) are seen to be given by mappings from X to Y.

Definition 4.5.1 (k-th order Gâteaux derivative) Let X and Y be Banach spaces on $\mathbb{R}$. With respect to a neighborhood (open set) $B \subset X$ of $\boldsymbol{x} \in X$, suppose $f : B \to Y$ is defined. Choose $\boldsymbol{y} \in X$ to be a variation vector and fix it. Let $k \in \mathbb{N}$. When the mapping $\epsilon \mapsto f(\boldsymbol{x} + \epsilon \boldsymbol{y})$ is an element of $C^k(\mathbb{R}; Y)$ with respect to an arbitrary $\epsilon \in \mathbb{R}$,

$$f^{(k)}(\boldsymbol{x})[\boldsymbol{y}] = \left.\frac{\mathrm{d}^k}{\mathrm{d}\epsilon^k} f(\boldsymbol{x} + \epsilon \boldsymbol{y})\right|_{\epsilon=0}$$

is called the k-th order Gâteaux derivative of f on $\boldsymbol{x}$ in the direction $\boldsymbol{y}$. □

There are times when an alternative definition of a Gâteaux derivative to Definition 4.5.1 is used. When a beneficial result can be derived from these definitions, those definitions should be used. However, in this book, we will not step out any further since we do not go on to proper discussions using Gâteaux derivatives.

Let us apply the definition of a Gâteaux derivative with respect to f of variational Problem 4.1.1.

Exercise 4.5.2 (Gâteaux derivative of an extended action integral) Show the Gâteaux derivative and second-order Gâteaux derivative of an extended action integral

$$f(u) = \int_0^{t_{\mathrm{T}}} l(u)\ \mathrm{d}t - m\beta u(t_{\mathrm{T}})$$

defined in 4.1.1. □

Answer In Problem 4.1.1, the set of functions u satisfying $u(0) = \alpha$ is expressed as U and the set of functions v satisfying $v(0) = 0$ is expressed as V. The set U is not a linear space. However, V is a linear space. Here, the Banach space X in Definition 4.5.1 is chosen to be V and $X = \left\{ v \in H^1((0, t_{\mathrm{T}}); \mathbb{R}) \mid v(0) = 0 \right\}$ (explained in Sect. 4.6.1). The problem of finding $u \in U$ is equivalent to seeking $u - u_0 \in V$ after selecting and fixing a $u_0 \in U$. Moreover, the Banach space Y is taken as the range $\mathbb{R}$.

Under these assumptions, let the fixed variation vector be $v \in X$ and seek the Gâteaux derivative in this direction. From Definition 4.5.1, we set

$$\begin{aligned} f(u + \epsilon v) = &\int_0^t \left\{ \frac{1}{2} m (\dot{u} + \epsilon \dot{v})^2 - \frac{1}{2} k (u + \epsilon v)^2 + p(u + \epsilon v) \right\} \mathrm{d}t \\ &- m\beta (u(t) + \epsilon v(t)) \end{aligned}$$

for an arbitrary $\epsilon \in \mathbb{R}$. Hence, the Gâteaux derivative is computed as follows:

$$\begin{aligned} f^{(1)}(u)[v] = f'(u)[v] &= \left. \frac{\mathrm{d}f}{\mathrm{d}\epsilon} \right|_{\epsilon=0} \\ &= \left. \left[\int_0^t \{ m(\dot{u} + \epsilon \dot{v}) \dot{v} - k(u + \epsilon v) v + pv \} \, \mathrm{d}t - m\beta v(t) \right] \right|_{\epsilon=0} \\ &= \int_0^t (m\dot{u}\dot{v} - kuv + pv) \, \mathrm{d}t - m\beta v(t) . \end{aligned}$$

This equation matches $f'(u)[v]$ in Eq. (4.1.4). However, $f'(u)[v]$ in Eq. (4.1.4) was shown with respect to an arbitrary $v \in X$. Moreover, a second-order Gâteaux derivative becomes

$$f^{(2)}(u)[v] = f''(u)[v] = \left. \frac{\mathrm{d}^2 f}{\mathrm{d}\epsilon^2} \right|_{\epsilon=0} = \int_0^t \left(m\dot{v}^2 - kv^2 \right) \, \mathrm{d}t.$$

This equation is the same as Eq. (4.1.4). Even in this case it is different from Eq. (4.1.4) in that $v \in X$ is fixed. □

Next, let us look at an example for which Gâteaux differentiation is possible but Fréchet differentiation shown in the next section is not possible.

Exercise 4.5.3 (Example which is only Gâteaux differentiable) Find the Gâteaux derivative of the following function in two-dimensional space:

$$f(\boldsymbol{x}) = \begin{cases} \dfrac{x_1^3}{x_1^2 + x_2^2} & \text{for } \boldsymbol{x} \neq \boldsymbol{0}_{\mathbb{R}^2}, \\ 0 & \text{for } \boldsymbol{x} = \boldsymbol{0}_{\mathbb{R}^2} \end{cases}$$

at $\boldsymbol{x} = \boldsymbol{0}_{\mathbb{R}^2}$. □

Answer The Gâteaux derivative of $f(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{0}_{\mathbb{R}^2}$ is given by

$$f'(\boldsymbol{0}_{\mathbb{R}^2})[\boldsymbol{y}] = \begin{cases} \dfrac{y_1^3}{y_1^2 + y_2^2}, & \text{for } \boldsymbol{y} \neq \boldsymbol{0}_{\mathbb{R}^2}, \\ 0, & \text{for } \boldsymbol{y} = \boldsymbol{0}_{\mathbb{R}^2}. \end{cases}$$

In view of the above expression, $f'(\boldsymbol{0}_{\mathbb{R}^2})[\boldsymbol{y}]$ is continuous with respect to $\boldsymbol{y}$ but is non-linear. □

4.5.2 Fréchet Derivative

The Gâteaux derivative was defined as a derivative with respect to a variation vector once its direction was specified. However, a derivative that is required in this book is a derivative which can define a gradient. In other words, it is a derivative of a functional which may be given as a dual product of an arbitrary variation vector and gradient. The Fréchet derivative shown next is a definition of a derivative generalized as a derivative of operators between Banach spaces and not only limited to functionals.

Definition 4.5.4 (k-th order Fréchet derivative) Let X and Y be Banach spaces on $\mathbb{R}$. With respect to a neighborhood $B \subset X$ of $\boldsymbol{x} \in X$, suppose $f : B \to Y$ is defined. If there exists a bounded linear operator $f'(\boldsymbol{x})[\,\cdot\,] \in \mathcal{L}(X; Y)$ satisfying

$$\lim_{\|\boldsymbol{y}_1\|_X \to 0} \frac{\|f(\boldsymbol{x} + \boldsymbol{y}_1) - f(\boldsymbol{x}) - f'(\boldsymbol{x})[\boldsymbol{y}_1]\|_Y}{\|\boldsymbol{y}_1\|_X} = 0 \tag{4.5.1}$$

with respect to an arbitrary variation vector $\boldsymbol{y}_1 \in X$, then $f'(\boldsymbol{x})[\boldsymbol{y}_1]$ is called the Fréchet derivative of f at $\boldsymbol{x}$. Moreover, when there exists $f'(\boldsymbol{x})[\boldsymbol{y}_1]$ with respect to all $\boldsymbol{x} \in B$ and those are in $C(B; \mathcal{L}(X; Y))$, it is expressed as $f \in C^1(B; Y)$.

Furthermore, if there exists a functional $f''(\boldsymbol{x})[\boldsymbol{y}_1, \,\cdot\,] \in \mathcal{L}(X; \mathcal{L}(X; Y))$ which satisfies

$$\lim_{\|\boldsymbol{y}_2\|_X \to \boldsymbol{0}_X} \frac{\|f'(\boldsymbol{x} + \boldsymbol{y}_2)[\boldsymbol{y}_1] - f'(\boldsymbol{x})[\boldsymbol{y}_1] - f''(\boldsymbol{x})[\boldsymbol{y}_1, \boldsymbol{y}_2]\|_Y}{\|\boldsymbol{y}_2\|_X} = 0$$

with respect to an arbitrary $\boldsymbol{y}_2 \in X$, then $f''(\boldsymbol{x})[\boldsymbol{y}_1, \boldsymbol{y}_2]$ is called a second-order Fréchet derivative of f at $\boldsymbol{x}$. The space $\mathcal{L}(X; \mathcal{L}(X; Y))$ is expressed as $\mathcal{L}^2(X \times X; Y)$. In addition, with respect to all $\boldsymbol{x} \in B$, if there exists a second-order Fréchet derivative and $f''(\boldsymbol{x})[\,\cdot\,, \,\cdot\,] \in C(B; \mathcal{L}(X; \mathcal{L}(X; Y)))$, then we find that $f \in C^2(B; Y)$. Similarly, a $k \in \{3, 4, \ldots\}$-th order Fréchet derivative $f^{(k)}$ can be defined and is expressed as $f \in C^k(B; Y)$. □

Equation (4.5.1) used in Definition 4.5.4 can be expressed using a Taylor expansion such as

$$f(\boldsymbol{x} + \boldsymbol{y}_1) = f(\boldsymbol{x}) + f'(\boldsymbol{x})[\boldsymbol{y}_1] + o(\|\boldsymbol{y}_1\|_X).$$

Here, $o(\|\boldsymbol{y}_1\|_X)$ is called the Bachmann–Landau small-o symbol and it is assumed that the limit

$$\lim_{\|\boldsymbol{y}_1\|_X \to 0} \frac{o(\|\boldsymbol{y}_1\|_X)}{\|\boldsymbol{y}_1\|_X} = \mathbf{0}_Y$$

holds.

Moreover, in Definition 4.5.4, if $Y = \mathbb{R}$, $f : B \to \mathbb{R}$ becomes a functional. Here, the Fréchet derivative of f can be written as

$$f'(\boldsymbol{x})[\boldsymbol{y}] = \langle \boldsymbol{g}, \boldsymbol{y} \rangle_{X' \times X} . \tag{4.5.2}$$

In this case, $\boldsymbol{g} \in X'$ is called a gradient. The second-order Fréchet derivative of f is written as

$$f''(\boldsymbol{x})[\boldsymbol{y}_1, \boldsymbol{y}_2] = h(\boldsymbol{x})[\boldsymbol{y}_1, \boldsymbol{y}_2], \tag{4.5.3}$$

where $h(\boldsymbol{x}) \in \mathcal{L}^2(X \times X; \mathbb{R})$ is called the Hesse form of f at $\boldsymbol{x}$.

Let us review the first and second variation of f in the variational Problem 4.1.1 using the definition of a Fréchet derivative in the following exercise.

Exercise 4.5.5 (Fréchet derivative of extended action integral)
Obtain the Fréchet derivative and second-order Fréchet derivative of the extended action integral

$$f(u) = \int_0^{t_\mathrm{T}} l(u)\,\mathrm{d}t - m\beta u(t_\mathrm{T})$$

defined in Problem 4.1.1, where $u(0) = \alpha$. □

Answer Let $X = \{v \in H^1((0, t_\mathrm{T}); \mathbb{R}) \mid v(0) = 0\}$, $Y = \mathbb{R}$ and $f'(u)[\,\cdot\,]$ in Eq. (4.1.5) be in $\mathcal{L}(X; Y)$ (explained in Sect. 4.6.1). Therefore, $f'(u)[v_1]$ can be viewed as a Fréchet derivative and it can be expressed as $f'(u)[v_1] = \langle g, v_1 \rangle_{X' \times X}$. In this case, the gradient is given by $g \in X'$.

Next, when v_1 is fixed in $f'(u)[v_1]$ and an arbitrary variation $v_2 \in X$ is added to u, we have

$$\begin{aligned} f'(u + v_2)[v_1] &= \int_0^{t_\mathrm{T}} \{m(\dot{u} + \dot{v}_2)\dot{v}_1 - k(u + v_2)v_1 + pv_1\}\,\mathrm{d}t - m\beta v_1(t_\mathrm{T}) \\ &= f'(u)[v_1] + \int_0^{t_\mathrm{T}} (m\dot{v}_1\dot{v}_2 - kv_1v_2)\,\mathrm{d}t \\ &= f'(u)[v_1] + f''(u)[v_1, v_2] . \end{aligned}$$

Here, it follows that $f''(u)[v_1, v_2]$ is the same as $f''(u)[v, v]$ of Eq. (4.1.4). □

The gradient $\boldsymbol{g}(\boldsymbol{x})$ defined in Eq. (4.5.2) is an element of the dual space X'. So, its norm is given by

$$\|\boldsymbol{g}(\boldsymbol{x})\|_{X'} = \sup_{\boldsymbol{y} \in X \setminus \{\mathbf{0}_X\}} \frac{\left|\langle \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{y} \rangle_{X' \times X}\right|}{\|\boldsymbol{y}\|_X}$$

$$= \sup_{\boldsymbol{y} \in X,\ \|\boldsymbol{y}\|_X = 1} \left| \langle \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{y} \rangle_{X' \times X} \right| \tag{4.5.4}$$

with respect to the definition of norm for a bounded linear functional (see Fig. 3.3.1).

4.6 Function Spaces in Variational Principles

Various function spaces were defined in Sect. 4.3 and it was shown that they satisfy the requirements of a Banach space and Hilbert space. In Sect. 4.4, an operator was defined as a mapping from between Banach spaces and in Sect. 4.5, the derivative of operators was defined. Among them, we saw that the Fréchet derivative of a functional can be defined by a dual product of the function space containing the variable of the functional and the dual space containing the gradient. In this section, the contents seen from Sect. 4.2 to Sect. 4.5 are used to review variational principles and optimization control problems shown in Sect. 4.1.

4.6.1 Hamilton's Principle

Summarizing the definitions used in the expanded Hamilton's principle (Problem 4.1.1) we obtain the following. The set of functions $u : (0, t_{\mathrm{T}}) \to \mathbb{R}$ satisfying $u(0) = \alpha$ was expressed as U. Moreover, the set of functions $v : (0, t_{\mathrm{T}}) \to \mathbb{R}$ satisfying $v(0) = 0$ was denoted as V. A functional expressing extended action integral with respect to $u \in U$ was set as

$$f(u) = \int_0^{t_{\mathrm{T}}} \left(\frac{1}{2} m \dot{u}^2 - \frac{1}{2} k u^2 + p u \right) \mathrm{d}t - m \beta u(t_{\mathrm{T}}) . \tag{4.6.1}$$

Moreover, a stationary condition of $f(u+v)$ with respect to an arbitrary $v \in V$ satisfying $v(0) = 0$ was obtained:

$$f'(u)[v] = \int_0^{t_{\mathrm{T}}} (m \dot{u} \dot{v} - k u v + p v) \, \mathrm{d}t - m \beta v(t_{\mathrm{T}}) = 0 \tag{4.6.2}$$

during the calculation of Eq. (4.1.3).

In order for integrals of Eq. (4.6.1) and Eq. (4.6.2) to have meaning, there is a need to clarify the function spaces with respect to u, v and p. We do this as follows.

Firstly, let us think about what U and V were. A element $u \in U$ has to satisfy $u(0) = \alpha$. Hence, let us set

$$U = \left\{ u \in H^1((0, t_{\mathrm{T}}); \mathbb{R}) \mid u(0) = \alpha \right\} . \tag{4.6.3}$$

On the other hand, since there was a need for $v \in V$ to satisfy the boundary condition of a homogeneous form $v(0) = 0$, we let

$$V = \left\{ v \in H^1((0, t_{\mathrm{T}}); \mathbb{R}) \mid v(0) = 0 \right\} . \tag{4.6.4}$$

Here, U is not a linear space because of the non-homogeneous boundary condition $u(0) \neq 0$. On the other hand, V is a linear space. Moreover, U is an affine subspace (Definition 4.2.7) with respect to V. In fact, if an element u_0 of $H^1((0,t_{\mathrm{T}});\mathbb{R})$ which satisfies $u(0)=\alpha$ is chosen and fixed, U is equivalent to $V(u_0)$. Let us set $\tilde{u} = u - u_0$ and let it be an element of V.

When V and $\tilde{u}$ are set in this way, and if Minkowski's inequality (Theorem A.9.2) and Hölder's inequality (Theorem A.9.1) are applied to the integral of $\dot{u}\dot{v}$, which is on the right-hand side of Eq. (4.6.2), then one obtains

$$\begin{aligned}\int_0^{t_{\mathrm{T}}} \dot{u}\dot{v}\,\mathrm{d}t &\leq \left\|\dot{\tilde{u}}\dot{v}\right\|_{L^1((0,t_{\mathrm{T}});\mathbb{R})} + \left\|\dot{u}_0\dot{v}\right\|_{L^1((0,t_{\mathrm{T}});\mathbb{R})} \\ &\leq \left\|\dot{\tilde{u}}\right\|_{L^2((0,t_{\mathrm{T}});\mathbb{R})} \left\|\dot{v}\right\|_{L^2((0,t_{\mathrm{T}});\mathbb{R})} + \left\|\dot{u}_0\right\|_{L^2((0,t_{\mathrm{T}});\mathbb{R})} \left\|\dot{v}\right\|_{L^2((0,t_{\mathrm{T}});\mathbb{R})} .\end{aligned} \tag{4.6.5}$$

Furthermore, if the Poincaré inequality system (Corollary A.9.4) is used for the right-hand side of Eq. (4.6.5), the inequality

$$\int_0^{t_{\mathrm{T}}} \dot{u}\dot{v}\,\mathrm{d}t \leq \|\tilde{u}\|_V \|v\|_V + \|u_0\|_{H^1((0,t_{\mathrm{T}});\mathbb{R})} \|v\|_V \tag{4.6.6}$$

is established. This confirms that if u_0 is an element of $H^1((0,t_{\mathrm{T}});\mathbb{R})$ and $\tilde{u}$ and v are elements of V, the right-hand side of Eq. (4.6.6) is bounded. Moreover, the integral of $\dot{u}^2$ and u^2 on the right-hand side of Eq. (4.6.1) as well as the integral of uv on the right-hand side of Eq. (4.6.2) are also bounded.

The fact that the boundary values $u(t_{\mathrm{T}})$ and $v(t_{\mathrm{T}})$ of u and v which appear in Eq. (4.6.1) and Eq. (4.6.2) are defined can be verified as follows. From the Sobolev embedding theorem (Theorem 4.3.14), the inclusion $H^1((0,t_{\mathrm{T}});\mathbb{R}) \subset C^{0,1/2}([0,t_{\mathrm{T}}];\mathbb{R})$ is established. Here, u and v are continuous functions and boundary values are set (trace is taken).

Meanwhile, for the integrals of pu and pv on the right-hand side of Eq. (4.6.1) and Eq. (4.6.2) to be bounded, it suffices to show if p is an element of $L^2((0,t_{\mathrm{T}});\mathbb{R})$. This is because the inequality

$$\int_0^{t_{\mathrm{T}}} pv\,\mathrm{d}t \leq \|p\|_{L^2((0,t_{\mathrm{T}});\mathbb{R})} \|v\|_V \tag{4.6.7}$$

in fact holds.

In the case that $u - u_0 \in V$, $v \in V$ and $p \in L^2((0,t_{\mathrm{T}});\mathbb{R})$, the integral $f'(u)[v]$ of Eq. (4.6.2) makes sense and $f'(u)[v]$ is seen as a bounded linear functional with respect to $v \in V$. Here, we can write

$$f'(u)[v] = \langle g, v\rangle_{V' \times V}, \tag{4.6.8}$$

where g is an element of the dual space V' of V and is called the gradient of f. The fact that Eq. (4.6.8) becomes zero with respect to an arbitrary $v \in V$ is equivalent to the equation of motion (Eq. (4.1.6)) and the terminal condition for velocity (Eq. (4.1.7)) being established.

In order to use the above ideas in Chap. 5 and beyond, let us rewrite Eq. (4.6.1) and Eq. (4.6.2) with emphasis on the bilinearity and linearity of these functionals. For the elastic potential energy, kinetic energy and external work, we set

$$a(u,v) = \int_0^{t_{\mathrm{T}}} kuv \,\mathrm{d}t, \tag{4.6.9}$$

$$b(u,v) = \int_0^{t_{\mathrm{T}}} muv \,\mathrm{d}t, \tag{4.6.10}$$

$$l(v) = \int_0^{t_{\mathrm{T}}} pv \,\mathrm{d}x - m\beta v(t_{\mathrm{T}}). \tag{4.6.11}$$

Then, Eq. (4.6.1) can be written as

$$f(u) = \frac{1}{2} b(\dot{u},\dot{u}) - \frac{1}{2} a(u,u) + l(u). \tag{4.6.12}$$

If these definitions are used, the problem of seeking the displacement of spring mass system based on the expanded Hamilton's principle can be rewritten as follows.

Problem 4.6.1 (Stationary problem of extended action integral) Let $V = \left\{ v \in H^1((0,t_{\mathrm{T}});\mathbb{R}) \mid v(0) = 0 \right\}$. Let a, b and l be given Eq. (4.6.9), Eq. (4.6.10) and Eq. (4.6.11), respectively. Suppose $u_0 \in H^1((0,t_{\mathrm{T}});\mathbb{R})$ satisfies $u_0(0) = \alpha$. Find $u - u_0 \in V$ at which $f(u)$ of Eq. (4.6.12) is stationary. □

When the time t_{T} is taken sufficiently smaller than half cycle of the natural vibration $\pi/\sqrt{k/m}$, the stationary condition can be changed to the minimum condition that indicates the existence of a unique minimum point ([4, Section 36.2, p. 159]).

Furthermore, the validity of the equation $f'(u)[v] = 0$ for an arbitrary $v \in V$ is equivalent to the equation

$$a(u,v) - b(\dot{u},\dot{v}) = l(v) \tag{4.6.13}$$

that holds for any $v \in V$. Hence, Problem 4.6.1 can also be expressed as follows.

Problem 4.6.2 (Variational problem of extended action integral) Let $V = \left\{ v \in H^1((0,t_{\mathrm{T}});\mathbb{R}) \mid v(0) = 0 \right\}$. Let a, b and l be given by Eq. (4.6.9), Eq. (4.6.10) and Eq. (4.6.11), respectively. Suppose $u_0 \in H^1((0,t_{\mathrm{T}});\mathbb{R})$ satisfies $u_0(0) = \alpha$. Find $u - u_0 \in V$ satisfying Eq. (4.6.13) with respect to an arbitrary $v \in V$. □

4.6.2 Minimum Principle of Potential Energy

Let us consider a function space with respect to functions used in the minimum principle of potential energy (Problem 4.1.2). Potential energy was defined in

Eq. (4.1.8) and can be rewritten as follows:

$$\pi(u) = \int_0^l \frac{1}{2} e_{\mathrm{Y}} \nabla u \nabla u \, a_{\mathrm{S}} \, \mathrm{d}x - \int_0^l b u \, a_{\mathrm{S}} \, \mathrm{d}x - p_{\mathrm{N}} u(l) \, a_{\mathrm{S}}(l) \tag{4.6.14}$$

for arbitrary $u \in U$. Moreover, as a stationary condition of $\pi(u+v)$ with respect to an arbitrary $v \in U$,

$$\pi'(u)[v] = \int_0^l (e_{\mathrm{Y}} \nabla u \nabla v - bv) \, a_{\mathrm{S}} \, \mathrm{d}x - p_{\mathrm{N}} v(l) \, a_{\mathrm{S}}(l) = 0 \tag{4.6.15}$$

was obtained during the calculation of Eq. (4.1.9). In order for these integrals to make sense, relationships such as those seen in Eq. (4.6.5), Eq. (4.6.6) and Eq. (4.6.7) should be used and the conditions

$$\begin{aligned} &u, v \in U = \left\{ u \in H^1((0,l);\mathbb{R}) \mid u(0) = 0 \right\}, \quad e_{\mathrm{Y}} \in L^\infty((0,l);\mathbb{R}), \\ &b \in L^2((0,l);\mathbb{R}), \quad a_{\mathrm{S}} \in W^{1,\infty}((0,l);\mathbb{R}) \end{aligned}$$

should be assumed. Here, a_{S} needs to be $W^{1,\infty}$ class in the neighborhood of $x = l$ in order to use its boundary value.

With these assumptions, the integrals of Eq. (4.6.14) and Eq. (4.6.15) have meanings. In this situation, $\pi'(u)[v]$ becomes a bounded linear functional with respect to $v \in U$ and can be written as

$$\pi'(u)[v] = \langle g, v \rangle_{U' \times U}, \tag{4.6.16}$$

where g is an element of U' and is called the gradient of π.

We shall again concentrate on the bilinearity or linearity of u and v which are elastic potential energy and external work and consider the functionals

$$a(u,v) = \int_0^l e_{\mathrm{Y}} \nabla u \cdot \nabla v a_{\mathrm{S}} \, \mathrm{d}x, \tag{4.6.17}$$

$$l(v) = \int_0^l b v a_{\mathrm{S}} \, \mathrm{d}x + a_{\mathrm{S}}(l) \, p_{\mathrm{N}} v(l). \tag{4.6.18}$$

Hence, Eq. (4.6.14) can be written as

$$\pi(u) = \frac{1}{2} a(u,u) - l(u). \tag{4.6.19}$$

If these definitions are used, the problem for seeking the displacement of one-dimensioal elastic bodies can be written as follows.

Problem 4.6.3 (Minimization problem of potential energy) Let $U = \{ v \in H^1((0,l);\mathbb{R}) \mid v(0) = 0 \}$. Suppose π is given by Eq. (4.6.19). In this case, find an element u which satisfies

$$\min_{u \in U} \pi(u).$$

□

The fact that $\pi'(u)[v] = 0$ holds with respect to an arbitrary $v \in U$ is equivalent to

$$a(u, v) = l(v) \tag{4.6.20}$$

that holds for any $v \in U$. In this case, Problem 4.6.3 can also be rewritten in the following way.

Problem 4.6.4 (Variational problem of potential energy)
Let $U = \{v \in H^1((0,l);\mathbb{R}) \mid v(0) = 0\}$. Let a and l be given by Eq. (4.6.17) and Eq. (4.6.18), respectively. In this case, find a $u \in U$ which satisfies Eq. (4.6.20) with respect to an arbitrary $v \in U$. □

The existence of unique solutions to Problem 4.6.3 and Problem 4.6.4 is shown in Section 5.2.

4.6.3 Pontryagin's Minimum Principle

With respect to the optimum control problem (Problem 4.1.4) of a linear system, the Lagrange function was defined as Eq. (4.1.21) and can be rewritten as follows:

$$\begin{aligned}
&\mathscr{L}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) \\
&\quad = \mathscr{L}_0(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0) + \mathscr{L}_1(\boldsymbol{\xi}, p) \\
&\quad = f_0(\boldsymbol{\xi}, \boldsymbol{u}) - \int_0^{t_{\mathrm{T}}} (\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}) \cdot \boldsymbol{z} \, \mathrm{d}t + \int_0^{t_{\mathrm{T}}} \left(\frac{\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d}}{2} - 1 \right) p \, \mathrm{d}t \\
&\quad = \int_0^{t_{\mathrm{T}}} \left\{ \frac{\|\boldsymbol{u}\|^2_{\mathbb{R}^n}}{2} + \frac{\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d}}{2} - (\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}) \cdot \boldsymbol{z} + \left(\frac{\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d}}{2} - 1 \right) p \right\} \mathrm{d}t \\
&\qquad + \frac{1}{2} \|\boldsymbol{u}(t_{\mathrm{T}})\|^2_{\mathbb{R}^n}
\end{aligned} \tag{4.6.21}$$

with respect to $(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) \in \Xi \times U \times Z \times P$. Moreover, the first variation of $\mathscr{L}$ can be summarized as

$$\begin{aligned}
&\mathscr{L}'(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\boldsymbol{\eta}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{z}}_0, \hat{p}] \\
&\quad = \mathscr{L}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\boldsymbol{\eta}] + \mathscr{L}_{\boldsymbol{u}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\hat{\boldsymbol{u}}] + \mathscr{L}_{\boldsymbol{z}_0}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\hat{\boldsymbol{z}}_0] \\
&\qquad + \mathscr{L}_p(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\hat{p}]
\end{aligned} \tag{4.6.22}$$

with respect to an arbitrary variation $(\boldsymbol{\eta}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{z}}_0, \hat{p}) \in \Xi \times V \times W \times P$ of $(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p) \in \Xi \times U \times Z \times P$, where

$$\mathscr{L}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\boldsymbol{\eta}] = \int_0^{t_{\mathrm{T}}} \left((1+p)\boldsymbol{\xi} + \boldsymbol{B}^{\top}\boldsymbol{z}_0 \right) \cdot \boldsymbol{\eta} \, \mathrm{d}t = \langle \boldsymbol{g}, \boldsymbol{\eta} \rangle, \tag{4.6.23}$$

$$\mathscr{L}_{\boldsymbol{u}}(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p)[\hat{\boldsymbol{u}}] = \int_0^{t_{\mathrm{T}}} \left(\boldsymbol{u} + \dot{\boldsymbol{z}}_0 + \boldsymbol{A}^{\top}\boldsymbol{z}_0 \right) \cdot \hat{\boldsymbol{u}} \, \mathrm{d}t$$

$$+ \left(\boldsymbol{u}\left(t_{\mathrm{T}}\right) - \boldsymbol{z}_0\left(t_{\mathrm{T}}\right)\right) \cdot \hat{\boldsymbol{u}}\left(t_{\mathrm{T}}\right), \tag{4.6.24}$$

$$\mathscr{L}_{\boldsymbol{z}_0}\left(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p\right)\left[\hat{\boldsymbol{z}}_0\right] = -\int_0^{t_{\mathrm{T}}} \left(\dot{\boldsymbol{u}} - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\xi}\right) \cdot \hat{\boldsymbol{z}}_0 \, \mathrm{d}t, \tag{4.6.25}$$

$$\mathscr{L}_p\left(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{z}_0, p\right)[q] = \int_0^{t_{\mathrm{T}}} \left(\frac{\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2}{2} - 1\right) q \, \mathrm{d}t. \tag{4.6.26}$$

In order for the integrals of Eq. (4.6.21) and Eq. (4.6.22) to make sense, the assumptions

$$\begin{aligned}
\Xi &= L^2\left((0, t_{\mathrm{T}}); \mathbb{R}^d\right), \\
U &= \left\{ \boldsymbol{u} \in H^1\left((0, t_{\mathrm{T}}); \mathbb{R}^n\right) \;\middle|\; \boldsymbol{u}(0) = \boldsymbol{\alpha} \right\}, \\
V &= \left\{ \boldsymbol{v} \in H^1\left((0, t_{\mathrm{T}}); \mathbb{R}^n\right) \;\middle|\; \boldsymbol{v}(0) = \boldsymbol{0}_{\mathbb{R}^n} \right\}, \\
Z &= \left\{ \boldsymbol{z} \in H^1\left((0, t_{\mathrm{T}}); \mathbb{R}^n\right) \;\middle|\; \boldsymbol{z}(t_{\mathrm{T}}) = \boldsymbol{u}(t_{\mathrm{T}}) \right\}, \\
W &= \left\{ \boldsymbol{w} \in H^1\left((0, t_{\mathrm{T}}); \mathbb{R}^n\right) \;\middle|\; \boldsymbol{w}(t_{\mathrm{T}}) = \boldsymbol{0}_{\mathbb{R}^n} \right\}
\end{aligned}$$

need to be made based on relationships such as those seen in Sect. 4.6.1. Here, U and Z are affine subspaces with respect to V and W respectively. If elements $\boldsymbol{u}_0$ and $\boldsymbol{z}_\top$ of $H^1((0, t_{\mathrm{T}}); \mathbb{R})$ which satisfy $\boldsymbol{u}(0) = \boldsymbol{\alpha}$ and $\boldsymbol{z}(t_{\mathrm{T}}) = \boldsymbol{u}(t_{\mathrm{T}})$, respectively, are selected and fixed, U and Z are equivalent to $V(\boldsymbol{u}_0)$ and $W(\boldsymbol{z}_\top)$, respectively.

If definitions such as those above are used, Problem 4.1.3 can be written as follows.

Problem 4.6.5 (Linear system with control) Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times d}$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ and control force $\boldsymbol{\xi} \in \Xi$ be given. Find $\boldsymbol{u} - \boldsymbol{u}_0 \in V$ such that Eq. (4.6.25) is zero with respect to an arbitrary $\hat{\boldsymbol{z}}_0 \in W$. □

The adjoint problem determining $\boldsymbol{z}_0$ (Problem 4.1.5) can be written in the following way.

Problem 4.6.6 (Adjoint problem with respect to f_0) Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be given as in Problem 4.6.5. Find $\boldsymbol{z}_0 - \boldsymbol{z}_\top \in W$ such that Eq. (4.6.24) is zero with respect to an arbitrary $\hat{\boldsymbol{u}} \in V$. □

A similar expression is possible with respect to an optimum control problem of non-linear systems (Problem 4.1.7) but it is omitted here.

4.7 Summary

In Chap. 4, we looked at what an optimization problem is when a design variable is a function defined on the domain expressing time or location (function optimization problem) by relating to the basics of functional analysis. The key points in this chapter are listed as follows:

(1) The equation of motion of a one-degree-of-freedom spring mass system can be obtained as the stationary condition (Hamilton's principle) of an action integral (Sect. 4.1.1). Moreover, the elasticity equation of a one-dimensional elastic body can be obtained as the minimum condition (minimum principle of potential energy) of potential energy (Sect. 4.1.2). Furthermore, the optimum solution for optimum control problem can be given as the minimal condition (Pontryagin's minimizing principle) of the Hamiltonian (Sect. 4.1.3).

(2) A linear space (vector space) is a set such that all the linear combinations among all elements are included. The set of all continuous functions is a linear space (Sect. 4.2.1).

(3) The linear space for which a norm is defined is called the norm space. Furthermore, a linear space which is complete (all Cauchy sequences converge) with respect to the norm is called a Banach space. The set of all continuous functions becomes a Banach space with the maximum value as the norm (Sect. 4.2.4).

(4) A linear space in which an inner product is defined is called the inner product space. Furthermore, a linear space which is complete with the norm defined by an inner product is called a Hilbert space. A finite-dimensional vector space is a Hilbert space (Sect. 4.2.5).

(5) Function spaces of the Hölder space $C^{k,\sigma}(\Omega;\mathbb{R})$, Lebesgue space $L^p(\Omega;\mathbb{R})$ and Sobolev space $W^{k,p}(\Omega;\mathbb{R})$ are Banach spaces with their corresponding norms. Moreover, $L^2(\Omega;\mathbb{R})$ and $H^k(\Omega;\mathbb{R}) = W^{k,2}(\Omega;\mathbb{R})$ are Hilbert spaces (Sect. 4.3). The embedding relationships of these function spaces are given by Sobolev embedding theorems (Theorem 4.3.14).

(6) The set of all bounded linear functionals on a Banach space is called a dual space. The Fréchet derivative of a functional is defined by a dual product of variation vector and gradient (Sect. 4.4.6).

(7) Hamilton's principle, minimum principle of potential energy and optimum control problems can be defined as function optimization problems on function spaces $H^1((0,t_{\mathrm{T}});\mathbb{R})$, $H^1((0,l);\mathbb{R})$ and $L^2\left((0,t_{\mathrm{T}});\mathbb{R}^d\right)$ respectively (Sect. 4.6).

4.8 Practice Problems

4.1 Introducing time $t \in (0,t_{\mathrm{T}})$ into Problem 4.1.2 representing the minimum principle of potential energy with respect to a one-dimensional elastic body, show the equation of motion and the terminal condition of the velocity via the expanded Hamilton's principle using the following order. In this case, let $\rho : (0,l) \to \mathbb{R}$ $(\rho > 0)$ be the density, $\alpha : (0,l) \to \mathbb{R}$ be displacement when $t = 0$, $\beta : (0,l) \to \mathbb{R}$ be the velocity at $t = t_{\mathrm{T}}$,

$b : (0,l) \times (0,t_{\mathrm{T}}) \to \mathbb{R}$ be the volume force and $p_{\mathrm{N}} : (0,t_{\mathrm{T}}) \to \mathbb{R}$ be the boundary force.

- Define U as a set of displacements $u : (0,l) \times (0,t_{\mathrm{T}}) \to \mathbb{R}$ that satisfy

$$u(0,t) = 0 \quad t \in (0,t_{\mathrm{T}}), \quad u(x,0) = \alpha(x) \quad x \in (0,l).$$

  Define V as a set of variational displacement v expressing an arbitrary variation of $u \in U$.

- Let

$$f(u) = \int_0^{t_{\mathrm{T}}} \left\{ \int_0^l \left(\frac{1}{2}\rho \dot{u}^2 - \frac{1}{2} e_{\mathrm{Y}} (\nabla u)^2 + bu \right) a_{\mathrm{S}} \,\mathrm{d}x + p_{\mathrm{N}} u(l,t)\, a_{\mathrm{S}}(l) \right\} \mathrm{d}t - \int_0^l \rho \beta u(x,t_{\mathrm{T}})\, a_{\mathrm{S}} \,\mathrm{d}x$$

  be an extended action integral and seek the stationary condition of f for an arbitrary $v \in V$.

- Determine the appropriate function spaces for ρ, α, β, b and p_{N}.

4.2 Function spaces relating to a generalized displacement $\boldsymbol{u}$ of an $n \in \mathbb{N}$-degree-of-freedom system and its variation $\boldsymbol{v}$ are set to be

$$U = \left\{ \boldsymbol{u} \in H^1((0,t_{\mathrm{T}});\mathbb{R}^n) \mid \boldsymbol{u}(0) = \boldsymbol{\alpha},\ \boldsymbol{u}(t_{\mathrm{T}}) = \boldsymbol{\beta} \right\},$$
$$V = \left\{ \boldsymbol{v} \in H^1((0,t_{\mathrm{T}});\mathbb{R}^n) \mid \boldsymbol{v}(0) = \boldsymbol{0}_{\mathbb{R}^n},\ \boldsymbol{v}(t_{\mathrm{T}}) = \boldsymbol{0}_{\mathbb{R}^n} \right\},$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are elements of $\mathbb{R}^n$. Suppose the kinetic energy $\kappa(\boldsymbol{u}, \dot{\boldsymbol{u}})$ and potential energy $\pi(\boldsymbol{u}, \dot{\boldsymbol{u}})$ with respect to $\boldsymbol{u} \in U$ are given. Moreover, suppose that the Lagrange function in mechanics be defined by $l(\boldsymbol{u}, \dot{\boldsymbol{u}}) = \kappa(\boldsymbol{u}, \dot{\boldsymbol{u}}) - \pi(\boldsymbol{u}, \dot{\boldsymbol{u}})$. Furthermore, assume that the action integral is defined by

$$f(\boldsymbol{u}, \dot{\boldsymbol{u}}) = \int_0^{t_{\mathrm{T}}} l(\boldsymbol{u}, \dot{\boldsymbol{u}}) \,\mathrm{d}t.$$

Show that the Lagrange equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial l}{\partial \dot{\boldsymbol{u}}} - \frac{\partial l}{\partial \boldsymbol{u}} = \boldsymbol{0}_{\mathbb{R}^n}$$

can be obtained from the stationary condition (Hamilton's principle) of $f(\boldsymbol{u} + \boldsymbol{v}, \dot{\boldsymbol{u}} + \dot{\boldsymbol{v}})$ with respect to an arbitrary $\boldsymbol{v} \in V$.

4.3 Introducing a generalized momentum $\boldsymbol{q} \in Q = H^1((0,t_{\mathrm{T}});\mathbb{R}^n)$ to Practice **4.2** and calling $\mathscr{H}(\boldsymbol{u}, \boldsymbol{q}) = -l(\boldsymbol{u}, \boldsymbol{q}) + \boldsymbol{q} \cdot \dot{\boldsymbol{u}}$ the Hamiltonian, let

$$f(\boldsymbol{u}, \boldsymbol{q}) = \int_0^{t_{\mathrm{T}}} \left(-\dot{\boldsymbol{q}} \cdot \boldsymbol{u} - \mathscr{H}(\boldsymbol{u}, \boldsymbol{q}) \right) \mathrm{d}t$$

be the action integral. In this case, show that the stationary condition of $f(\boldsymbol{u}, \boldsymbol{q})$ becomes Hamilton's equation of motion

$$\dot{\boldsymbol{q}} = -\frac{\partial \mathscr{H}}{\partial \boldsymbol{u}}, \quad \dot{\boldsymbol{u}} = \frac{\partial \mathscr{H}}{\partial \boldsymbol{q}}.$$

Moreover, when Hamilton's equation of motion holds, show that $\dot{\mathscr{H}}(\boldsymbol{u}, \boldsymbol{q}) = 0$ (the Hamiltonian is conserved). Furthermore, find $\mathscr{H}(\boldsymbol{u}, \boldsymbol{q})$ with respect to the spring mass system of Fig. 4.1 assuming an external force $p = 0$.

4.4 Let Y and Z be Banach spaces and Y be compactly embedded within Z ($Y \Subset Z$) (Theorem 4.4.15). Show that $Z' \Subset Y'$ holds with respect to the dual spaces Y' and Z' of Y and Z.

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