Notation

Use of Letters

The use of variables will adhere to the following basic rules.

a, α, $\cdots$	Lower-case italic Latin and Greek letters (Table 1) represent scalars, vectors and functions.
$\boldsymbol{a}$, $\boldsymbol{\alpha}$, $\cdots$	Bold lower italic Latin and Greek letters represent finite-dimensional vectors and functions with such ranges.
A, $\mathcal{A}$, Γ, $\cdots$	Capital italic Latin and Greek characters, as well as their cursive representations, represent sets.
$\boldsymbol{A}$, $\boldsymbol{\Gamma}$, $\cdots$	Bold capital italic Latin and Greek letters represent finite dimension matrices and functions with such ranges ranges.
$\mathscr{L}$, $\mathscr{H}$	Represent Lagrange and Hamilton functions, respectively.
a_{A}, a_{div}	Subscripts on upright Latin letters represent the initial of a terminology or an abbreviation.

Below, m, n and d denote natural numbers.

Sets

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	Represent the set of natural numbers (positive integers), the integers, the rational numbers, the real numbers, and the complex numbers, respectively.
$\mathbb{R}^d$	Represents the d-dimensional real linear space (real vector space).
$A = \{a_1, \cdots, a_m\}$	Represents a set A consisting of a finite number of elements, $a_1, \cdots, a_m$.
$\|A\|$	Represents the number of elements in a finite set A.

Table 1: Greek letters

Capital		Lower-case		Pronunciation	Capital		Lower-case		Pronunciation
A	$\boldsymbol{A}$	α	$\boldsymbol{\alpha}$	alpha	N	$\boldsymbol{N}$	ν	$\boldsymbol{\nu}$	nu
B	$\boldsymbol{B}$	β	$\boldsymbol{\beta}$	beta	Ξ	$\boldsymbol{\Xi}$	ξ	$\boldsymbol{\xi}$	xi
Γ	$\boldsymbol{\Gamma}$	γ	$\boldsymbol{\gamma}$	gamma	O	$\boldsymbol{O}$	o	$\boldsymbol{o}$	omicron
Δ	$\boldsymbol{\Delta}$	δ	$\boldsymbol{\delta}$	delta	Π	$\boldsymbol{\Pi}$	π ϖ	$\boldsymbol{\pi}$ $\boldsymbol{\varpi}$	pi
E	$\boldsymbol{E}$	ϵ ε	$\boldsymbol{\epsilon}$ $\boldsymbol{\varepsilon}$	epsilon	P	$\boldsymbol{P}$	ρ ϱ	$\boldsymbol{\rho}$ $\boldsymbol{\varrho}$	rho
Z	$\boldsymbol{Z}$	ζ	$\boldsymbol{\zeta}$	zeta	Σ	$\boldsymbol{\Sigma}$	σ ς	$\boldsymbol{\sigma}$ $\boldsymbol{\varsigma}$	sigma
H	$\boldsymbol{H}$	η	$\boldsymbol{\eta}$	eta	T	$\boldsymbol{T}$	τ	$\boldsymbol{\tau}$	tau
Θ	$\boldsymbol{\Theta}$	θ ϑ	$\boldsymbol{\theta}$ $\boldsymbol{\vartheta}$	theta	Υ	$\boldsymbol{\Upsilon}$	υ	$\boldsymbol{\upsilon}$	upsilon
I	$\boldsymbol{I}$	ι	$\boldsymbol{\iota}$	iota	Φ	$\boldsymbol{\Phi}$	ϕ φ	$\boldsymbol{\phi}$ $\boldsymbol{\varphi}$	phi
K	$\boldsymbol{K}$	κ	$\boldsymbol{\kappa}$	kappa	X	$\boldsymbol{X}$	χ	$\boldsymbol{\chi}$	chi
Λ	$\boldsymbol{\Lambda}$	λ	$\boldsymbol{\lambda}$	lambda	Ψ	$\boldsymbol{\Psi}$	ψ	$\boldsymbol{\psi}$	psi
M	$\boldsymbol{M}$	μ	$\boldsymbol{\mu}$	mu	Ω	$\boldsymbol{\Omega}$	ω	$\boldsymbol{\omega}$	omega

$a \in A$ Indicates that a is an element of the set A.

$\{0\}$ Represents a set consisting of the single element 0.

$\{a_k\}_{k\in\mathbb{N}}$ Denotes an infinite sequence, $\{a_1, a_2, \cdots\}$.

$A \subset B$ Indicates that the set A is a subset of the set B.

$A \cup B$, $A \cap B$, $A \setminus C$ Represent the union, intersection and subtraction operations of sets, respectively.

$(0,1)$, $[0,1]$, $(0,1]$ Denote the intervals $\{x \in \mathbb{R} \mid 0 < x < 1\}$, $\{x \in \mathbb{R} \mid 0 \le x \le 1\}$, $\{x \in \mathbb{R} \mid 0 < x \le 1\}$, respectively.

Vectors and Matrices

The following notation concerns vectors and matrices of $\mathbb{R}^d$, $\mathbb{R}^m$, and $\mathbb{R}^n$.

$\boldsymbol{x} = (x_1, \cdots, x_d)^\top \in \mathbb{R}^d$ Represents a d-dimensional vertical real vector. x_i represents the i-th element of $\boldsymbol{x}$. $\boldsymbol{x}^\top$ represents the transposition of $\boldsymbol{x}$.

$\boldsymbol{A} = (a_{ij})_{ij} \in \mathbb{R}^{m\times n}$ Represents the real matrix with m rows and n columns. Can also be written as $\boldsymbol{A} = (a_{ij})_{(i,j)\in\{1,\cdots,m\}\times\{1,\cdots,n\}}$.

$\boldsymbol{0}_{\mathbb{R}^d}$, $\boldsymbol{0}_{\mathbb{R}^{m\times n}}$ Represents the zero elements of $\mathbb{R}^d$ and $\mathbb{R}^{m\times n}$.

$\boldsymbol{x} \ge \boldsymbol{0}_{\mathbb{R}^d}$ Represents $x_i \ge 0$ for $i \in \{1, \cdots, d\}$.

$\|\boldsymbol{x}\|_{\mathbb{R}^d,p}$ When $p \in [1, \infty)$, represents the p-powered norm $\sqrt[p]{|x_1|^p + \cdots + |x_d|^p}$ of $\boldsymbol{x} \in \mathbb{R}^d$. When $p = \infty$, represents the maximum norm $\max\{|x_1|^p, \cdots, |x_d|^p\}$. If there is no confusion, it is written as $\|\boldsymbol{x}\|_p$.

$\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{A} \cdot \boldsymbol{B}$ Represents the inner product (scalar product) $\sum_{i \in \{1, \cdots, m\}} a_i b_i$, $\sum_{(i,j) \in \{1, \cdots, m\}^2} a_{ij} b_{ij}$ with respect to $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{A} = (a_{ij})_{ij}$, $\boldsymbol{B} = (b_{ij})_{ij} \in \mathbb{R}^{m \times m}$.

$\|\boldsymbol{x}\|_{\mathbb{R}^d}$ Represents the Euclidean norm $\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ of $\boldsymbol{x} \in \mathbb{R}^d$. If there is no confusion, it is written as $\|\boldsymbol{x}\|$.

δ_{ij} Represents the Kronecker delta $\delta_{ij} = 1$ $(i = j)$, $\delta_{ij} = 0$ $(i \neq j)$.

$\boldsymbol{I}_{\mathbb{R}^{m \times m}}$ Represents the unit matrix $(\delta_{ij})_{ij} \in \mathbb{R}^{m \times m}$. If there is no confusion, it is written as $\boldsymbol{I}$.

$(\,\cdot\,)^{\mathrm{s}}$ Represents $\left((\,\cdot\,)^\top + (\,\cdot\,)\right)/2$.

Domains and Functions

Here below, functions on domains in $\mathbb{R}^d$ are considered.

$\Omega \subset \mathbb{R}^d$ Represents a domain (simply-connected open set) of $\mathbb{R}^d$.

$\bar{\Omega}$ Represents the closure of Ω.

$\partial\Omega$ Represents the boundary of Ω, i.e. $\bar{\Omega} \setminus \Omega$.

$|\Omega|$ Represents $\displaystyle\int_\Omega \mathrm{d}x$.

S° Represents the interior of a closed set S.

$\boldsymbol{\nu}$ Represents an outward unit normal defined at the boundary $\partial\Omega$.

$\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_{d-1}$ Represents the tangent defined at the boundary $\partial\Omega$.

κ Represents $\boldsymbol{\nabla} \cdot \boldsymbol{\nu}$ ($d-1$ times the mean curvature) defined at the boundary $\partial\Omega$.

$\boldsymbol{\nabla} u$ Represents the gradient $\partial u / \partial \boldsymbol{x} \in \mathbb{R}^d$ of a function $u : \mathbb{R}^d \to \mathbb{R}$.

Δu Represents the Laplace operator $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ of a function $u : \mathbb{R}^d \to \mathbb{R}$.

$\partial_\nu u$ Represents $(\boldsymbol{\nu} \cdot \boldsymbol{\nabla})\, u$ defined for a function $u : \Omega \to \mathbb{R}$ at the boundary $\partial\Omega$.

$\partial_\nu \boldsymbol{u}$ Represents $(\boldsymbol{\nu} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} = \left(\boldsymbol{\nabla} \boldsymbol{u}^\top\right)^\top \boldsymbol{\nu}$ for function $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ defined at the boundary $\partial\Omega$.

$\mathrm{d}x$, $\mathrm{d}\gamma$, $\mathrm{d}\varsigma$	Represents the measures used in the integration in the domain $\Omega \subset \mathbb{R}^d$, integration in the boundary $\Gamma \subset \partial\Omega$ and integration in the boundary of boundary $\partial\Gamma$.
$\operatorname{ess\,sup}_{\text{a.e. } \boldsymbol{x} \in \Omega} \lvert u(\boldsymbol{x}) \rvert$	Represents the essential bound of $u : \Omega \to \mathbb{R}$. The letters a.e. mean that it is almost everywhere on the measurable sets.
χ_Ω	Represents the characteristic function $\chi_\Omega : \mathbb{R}^d \to \mathbb{R}$ ($\chi_\Omega(\Omega) = 1$, $\chi_\Omega\left(\mathbb{R}^d \setminus \bar{\Omega}\right) = 0$) with respect to a domain $\Omega \subset \mathbb{R}^d$.

Banach Spaces

Here below, V will be a normed space and X and Y are Banach spaces.

$\lVert \boldsymbol{x} \rVert_V$	Represents the norm of $\boldsymbol{x} \in V$. If there is no confusion, it is written as $\lVert \boldsymbol{x} \rVert$.
$f : X \to Y$	Represents a mapping (operator) from X to Y.
$f(\boldsymbol{x}) : X \ni \boldsymbol{x} \mapsto f \in Y$	Represents a mapping expressing elements.
$f \circ g$	Represents the composite mapping $f(g)$.
$\mathcal{L}(X; Y)$	Represents the universal set of bounded linear operators from X to Y.
$\mathcal{L}^2(X \times X; Y)$	Represents $\mathcal{L}(X; \mathcal{L}(X; Y))$.
X'	Represents the dual space of X, in other words the universal set $\mathcal{L}(X; \mathbb{R})$ of bounded linear functions on X.
$\langle \boldsymbol{y}, \boldsymbol{x} \rangle_{X' \times X}$	Represents the dual product of $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in X'$. If there is no confusion, it is written as $\langle \boldsymbol{y}, \boldsymbol{x} \rangle$.
$f'(\boldsymbol{x})[\boldsymbol{y}]$	Represents the Fréchet derivative $\langle f'(\boldsymbol{x}), \boldsymbol{y} \rangle_{X' \times X}$ of $f : X \to \mathbb{R}$ at $\boldsymbol{x} \in X$ with respect to an arbitrary variation $\boldsymbol{y} \in X$.
$f_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})[\boldsymbol{z}]$, $\partial_X f(\boldsymbol{x}, \boldsymbol{y})[\boldsymbol{z}]$	Represents the Fréchet partial derivative $\langle \partial f(\boldsymbol{x}, \boldsymbol{y}) / \partial \boldsymbol{x}, \boldsymbol{z} \rangle_{X' \times X}$ of $f : X \times Y \to \mathbb{R}$ at $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$ with respect to an arbitrary variation $\boldsymbol{z} \in X$. $f(\boldsymbol{x}, \boldsymbol{y}) / \partial \boldsymbol{x} \in X'$ is written as $f_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y})$.
$C^k(X; Y)$	Represents the universal set of the $k \in \{0, 1, \cdots\}$-th Fréchet differentiable mappings.
$C^k_{\mathrm{S}'}(X; Y)$	Represents the universal set of the $k \in \{0, 1, \cdots\}$-th shape differentiable mappings.

$C^k_{\mathrm{S}'}(X;Y)$ Represents the universal set of the $k \in \{0,1,\cdots\}$-th shape differentiable mappings.

$C^k_{\mathrm{S}^*}(X;Y)$ Represents the universal set of the $k \in \{0,1,\cdots\}$-th partial shape differentiable mappings.

Function Spaces

Here below, Ω is a domain in $\mathbb{R}^d$.

$C(\Omega;\mathbb{R}^n)$, $C^0(\Omega;\mathbb{R}^n)$ Represents the universal set of the continuous function $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ defined on Ω.

$C_{\mathrm{B}}(\Omega;\mathbb{R}^n)$, $C^0_{\mathrm{B}}(\Omega;\mathbb{R}^n)$ Represents the universal set of bounded functions in $C(\Omega;\mathbb{R}^n)$.

$C_0(\Omega;\mathbb{R}^n)$ Represents the universal set of $\boldsymbol{f} \in C(\Omega;\mathbb{R}^n)$ such that the support of $\boldsymbol{f}$ becomes a compact set of Ω.

$C^k(\Omega;\mathbb{R}^n)$ Represents the universal set of $\boldsymbol{f} \in C(\Omega;\mathbb{R}^n)$ for which up to the $k \in \{0,1,\cdots\}$-th derivative of $\boldsymbol{f}$ belongs to $C(\Omega;\mathbb{R}^n)$.

$C^k_{\mathrm{B}}(\Omega;\mathbb{R}^n)$ Represents the universal set of $\boldsymbol{f} \in C^k(\Omega;\mathbb{R}^n)$ for which up to the k-th derivative of $\boldsymbol{f}$ belongs to $C_{\mathrm{B}}(\Omega;\mathbb{R}^n)$.

$C^k_0(\Omega;\mathbb{R}^n)$ Represents $C^k(\Omega;\mathbb{R}^n) \cap C_0(\Omega;\mathbb{R}^n)$.

$C^{k,\sigma}(\Omega;\mathbb{R}^n)$ Represents the universal set of $\boldsymbol{f} \in C^k(\Omega;\mathbb{R}^n)$ for which up to the k-th derivative of $\boldsymbol{f}$ is a Hölder continuous function with the Hölder index $\sigma \in (0,1]$. When $k = 0$ and $\sigma = 1$, the function is said to be Lipschitz continuous.

$L^p(\Omega;\mathbb{R}^n)$ Represents the universal set of p-th powered Lebesgue integrable functions $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ for $p \in [1,\infty)$, and the universal set of functions which are essentially bounded for $p = \infty$.

$W^{k,p}(\Omega;\mathbb{R}^n)$ Represents the universal set of functions for which up to the $k \in \{0,1,\cdots\}$-th derivative belongs to $L^p(\Omega;\mathbb{R}^n)$.

$W^{k,p}_0(\Omega;\mathbb{R}^n)$ Represents the closure of $C^\infty_0(\Omega;\mathbb{R})$ in $W^{k,p}(\Omega;\mathbb{R})$.

$H^k(\Omega;\mathbb{R}^n)$ Represents $W^{k,2}(\Omega;\mathbb{R}^n)$.

$H^k_0(\Omega;\mathbb{R}^n)$ Represents $W^{k,2}_0(\Omega;\mathbb{R}^n)$.