Special Mathematics Lecture
Differential geometry

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Handwritten notes taken by L. Zhang

Differential Geometry

Extrinsic / Intrinsic ways to study DG (they're not so different)

from outside of the manifold — on the manifold

Extrinsic : to look at curves or surfaces from outside in a bigger space (in Calculus II) simple for visualization

Intrinsic : no more any ambiant space, like a 2D animal in a flatland ~~witho~~ without a 3^{rd} dimension, useful in general relativity & universe (mostly used in this course)

However, a manifold can always embedded in a higher dimensional space (not always one more) (Nash embedding thm)

I) Differentiable manifolds

I.1 Topological manifolds (+ topology)

Def. a TOPOLOGICAL MANIFOLD of dimension n is a $\boxed{\text{topological space}}$ $\mathcal{M}$ s.t. :

1) $\mathcal{M}$ is $\boxed{\text{Hausdorff}}$
2) Any $p \in \mathcal{M}$ has a $\boxed{\text{neighborhood}}$ V $\boxed{\text{homeorphic}}$ to an $\boxed{\text{open set}}$ $U \subset \mathbb{R}^n$.
3) $\mathcal{M}$ is $\boxed{\text{second countable}}$.

$\boxed{\text{Def}}$ a TOPOLOGICAL SPACE $\mathcal{M} \equiv (\mathcal{M}, \mathcal{T})$ (curly T)

is a set $\mathcal{M}$ together with a collection $\mathcal{T}$ of subsets satisfying :

1) $\phi, \mathcal{M} \in \mathcal{T}$ (empty set 空集)
2) If $V_\alpha \in \mathcal{T}$, then $\bigcup_\alpha V_\alpha \in \mathcal{T}$ (union 並集) ($\mathcal{T}$ is STABLE FOR ARBITRARY UNION)
3) If $V_1, \cdots, V_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} V_i \in \mathcal{T}$ (intersection 交集) (↳ UNDER FINITE INTERSECTION)

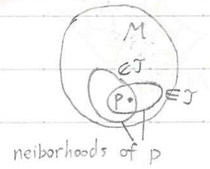
The elements of $\mathcal{T}$ are called the OPEN SETS.

Their complement ($\mathcal{M} \setminus V$, $V \in \mathcal{T}$) is called a CLOSED SET.

$\boxed{\text{Def}}$ Let $(\mathcal{M}, \mathcal{T})$ be a topological space (t.s.), and let $p \in \mathcal{M}$.

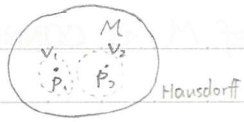
a NEIBORHOOD of p is any open set containing p.

We write $\mathcal{V}_p$ (Greek nu) for the set of all neighborhoods of p.



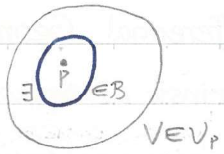
$\boxed{\text{Def}}$ $(\mathcal{M}, \mathcal{T})$ is HAUSDORFF if

$$\forall p_1, p_2 \in \mathcal{M},\ p_1 \neq p_2 : \exists V_1 \in \mathcal{V}_{p_1},\ V_2 \in \mathcal{V}_{p_2} : V_1 \cap V_2 = \phi$$



It is often difficult to describe all open sets in $(\mathcal{M}, \mathcal{T})$

⇒ Introduce the notion of a basis. (related to Second Countable)



Def. A subset $\mathcal{B} := \{V_\alpha\} \subset \mathcal{T}$ is a BASIS of $(M, \mathcal{T})$ if

$$\forall p \in M \; \forall V \in \mathcal{V}_p : \exists U \in \mathcal{B} : p \in U \subset V$$

Example: $M = \mathbb{R}^n$ with $\mathcal{T} = \{$all open sets in $\mathbb{R}^n\}$ is a topological manifold.

An OPEN SET in $\mathbb{R}^n$ is a set V s.t. $\forall p \in V$:

there is a small ball centered at p and contained in V.

We set $B(p, r)$ = a ball centered at p and of radius r.

$$B(p,r) := \{x \in \mathbb{R}^n \mid \underbrace{\|x-p\|}_{\text{norm in } \mathbb{R}^n} < r\}$$

Then: (all balls centered at any point)

$\mathcal{B} := \{B(x,r) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis for $\mathbb{R}^n$.

$\boxed{\text{Def.}}$ $(M, \mathcal{T})$ is SECOND COUNTABLE if it has a countable basis. (countable: in a 1 to 1 (= bijective) relation with $\mathbb{N}$.)

For $\mathbb{R}^n$, we can set

$\mathcal{B} := \{B(x, \frac{1}{n}) \mid x \in \mathbb{Q}^n, n \in \mathbb{N}\}$ and it is a countable basis for $\mathbb{R}^n$.

$\Rightarrow \mathbb{R}^n$ is second countable.

$\boxed{\text{Def.}}$ Let $(M, \mathcal{T}), (\mathcal{N}, \mathcal{S})$ be 2 t.s., and let $f: M \mapsto \mathcal{N}$.

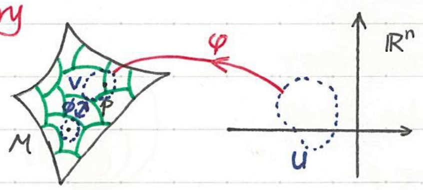
f is CONTINUOUS if $f^{-1}(U) \in \mathcal{T} \; \forall U \in \mathcal{S}$

with the PRE-IMAGE $f^{-1}(U) := \{p \in M \mid f(p) \in U\}$

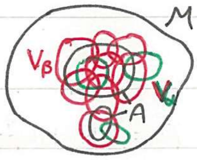
Exersise: When $M = \mathcal{N} = \mathbb{R}$ and $\mathcal{T} = \mathcal{S} = \{$open sets in $\mathbb{R}\}$, check if this def corresponds to the ε-δ def of coutinuousity.

If f is bijective and f, f^{-1} are continuous, we say that f is HOMEOMORPHIC.

Summary



Def. M is CONNECTED if it is not the disjoint union of 2 non-empty open sets.



Def. Let A be a subset of M.

1) An OPEN COVER for A is a subfamily $\{V_\alpha\} \subset \mathcal{T}$ s.t. $A \in \bigcup_\alpha V_\alpha$ finite or infinite

2) a SUBCOVER of an open cover for A (in which the green subsets are unnecessary) is a subfamily $\{V_\beta\} \subset \{V_\alpha\}$ which still covers A.

3) A is COMPACT (small in this setting) if any open cover of A admits a finite subcover
(If $A = \mathbb{R}^n$, A is compact iff A is closed and bounded)

$(\mathbb{N}, \mathcal{T})$ topo. space

$\mathcal{T}_0 := \{[a,b] \cap \mathbb{N} \mid a$ is not odd and b is not even; $a<b$; $a, b \in \mathbb{N} \cup \{\infty\}\}$

$\mathcal{T} := \{I \mid I = \bigcup_\alpha I_\alpha,\ \forall \alpha : I_\alpha \in \mathcal{T}_0\} \cup \{\emptyset\}$

$\mathcal{T} := \{(\bigcup_\alpha [A_\alpha, B_\alpha]) \cap \mathbb{N} \mid \forall \alpha : A_\alpha$ is not odd and B_α is not even; $A_\alpha < B_\alpha$; $A_\alpha, B_\alpha \in \mathbb{N} \cup \{\infty\}\}$
$\cup \{\emptyset\}$,

In the example on P_2, $B = \{B(x, \frac{1}{m}) \mid x \in \mathbb{Q}^n, m \in \mathbb{N}\}$

Let us define a half-space:

$$H^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

$$\partial H^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$$ for the boundary.

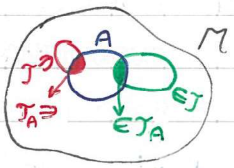
Def. a TOPOLOGICAL MANIFOLD of dimension n with a boundary is a Hausdoff second-countable topological space M with each point $p \in M$ having a neighborhood V eigther homomorphic to an open subset of $H^n \setminus \partial H^n$ or to an open subset of H^n with the image of p inside ∂H^n.

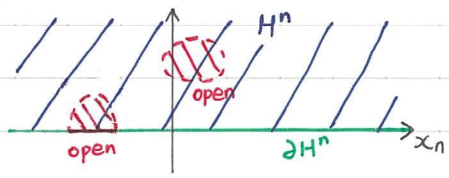
Remark: If $(M, \mathcal{T})$ is a topo. space and if $A \subset M$.

Then the topology on A is given by $\mathcal{T}_A := \{V \cap A \mid V \in \mathcal{T}\}$

(called RELATIVE or SUBSPACE TOPOLOGY)

⚠ An open set for A (in $\mathcal{T}_A$) is not always an open set for M (in $\mathcal{T}$).





I.2 Smooth manifolds & Smooth maps

$\swarrow C^\infty$

Def. a SMOOTH (or C^∞) MANIFOLD M is a topo. manifold together with a family of homeomorphisms

$\varphi_\alpha : \mathbb{R}^n \supset \underset{\text{open}}{U_\alpha} \longmapsto M$ s.t.

1) $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$

2) If $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) =: V_{\alpha\beta} \neq \phi$ then

$$\begin{cases} \varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(V_{\alpha\beta}) \mapsto \varphi_\beta^{-1}(V_{\alpha\beta}) \\ \varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(V_{\alpha\beta}) \mapsto \varphi_\alpha^{-1}(V_{\alpha\beta}) \end{cases} \quad \text{(TRANSITION FUNCTIONS)}$$

are of C^∞ (from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$).

3) The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ is maximal.

$\mathcal{A}$ is called a C^∞ $\underbrace{\text{MAXIMAL}}_{3)}$ $\underbrace{\text{ATLAS}}_{1)\,\&\,2)}$.

MAXIMAL: If $\varphi : U \overset{\subset \mathbb{R}^n,\ \text{open}}{\mapsto} M$ satisfies $\varphi^{-1} \circ \varphi_\alpha$ and $\varphi_\alpha^{-1} \circ \varphi$ (whenever defined) is smooth then $(U, \varphi) \in \mathcal{A}$.

Remark: ◦ it is often easy to describe an atlas, but not the maximal one.

◦ A topological manifold can be endowed with different inequivalent maximal atlases. (see the P_1 on today's handout) (very deep)

INEQUIVALENT: take 2 max atlases, if the union is not an atlas (some transision functions are not C^∞) then the 2 atlases are not equivalent.

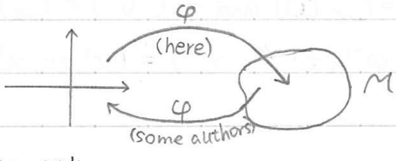
Exercises

1) Provide an example of smooth manifolds with an atlas.
   (n-sphere, group of matrices, Lie groups, real projective space $P(\mathbb{R}^n)$, etc)

2) Show the uniqueness (starting from a certain atlas) of the maximal atlas.

3) Look at inequivalent atlases on n-sphere.
   $\overset{\shortparallel}{\text{differential structure}}$

Remark: 1)



2) For $(U, \varphi) \in \mathcal{A}$ and $p \in \varphi(U) \subset M$, we set

$$\varphi^{-1}(p) =: (x^1(p), x^2(p), \cdots, x^n(p))$$

and call it a LOCAL COORDINATE of p. It means

$\varphi^{-1}(\cdot) =: (x^1(\cdot), x^2(\cdot), \cdots, x^n(\cdot))$ (a CHART or a LOCAL COORDINATE FUNCTION)

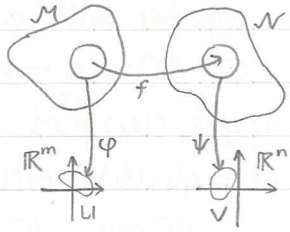
is an homeomorphism from an open subset of M to an open subset of $\mathbb{R}^n$.

Def: Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds of dim m and n respectively.

A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a SMOOTH MAP if

$\forall$ charts (U, φ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$:

$\psi \circ f \circ \varphi^{-1}$ is smooth wherever defined.



The function $\psi \circ f \circ \varphi^{-1}$ is called a LOCAL REPRESENTATION of f.

We set $C^\infty(\mathcal{M}, \mathcal{N}) :=$ the set of such smooth functions.

and $C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$.

Def. If $f \in C^\infty(\mathcal{M}, \mathcal{N})$ is bijective and if $f^{-1} \in C^\infty(\mathcal{N}, \mathcal{M})$, we call f a DIFFEOMORPHISM.

Remark: • a diffeomorphism is also a homeomorphism.

• A map $f: \mathcal{M} \mapsto \mathcal{N}$ is a LOCAL DIFFEOMORPHISM at $p \in \mathcal{M}$ if

$\exists V \in \mathcal{V}_p$ and $W \in \mathcal{V}_{f(p)}$: $f|_V : V \mapsto W$ is a diffeomorphism.

Def. Let $f: \mathcal{M} \mapsto \mathcal{N}$ be a smooth function and let (U, φ) (V, ψ) be charts of $\mathcal{M}$ & $\mathcal{N}$ respectively.

For $p \in \mathcal{M}$, the RANK of f at p $(=: \operatorname{rank}(f)_p)$ corresponds to

the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix} (\varphi(p)) \quad \text{with } F := \psi \circ f \circ \varphi^{-1}$$

This rank is independent of the charts.

Thm. (not so easy) Framework as before. (Constant rank thm)

Suppose that $\operatorname{rank}(f)_p = k \ \forall p \in \mathcal{M}$, with $k \in \mathbb{N}$. Then

$\forall p \in \mathcal{M} \ \exists (U, \varphi), (V, \psi)$ charts of $\mathcal{M}, \mathcal{N}$ respectively s.t.

- $\varphi(p) = \mathbf{0} \in \mathbb{R}^m$ and $\psi(f(p)) = \mathbf{0} \in \mathbb{R}^n$; Cube in $\mathbb{R}^n$ centered at $\mathbf{0}$ and with $x^i \in (-\varepsilon, \varepsilon)$
- $\varphi(U) = C^m_\varepsilon(\mathbf{0})$ and $\psi(V) = C^n_\varepsilon(\mathbf{0})$ $\exists \varepsilon > 0$;
- $\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, \underbrace{0, \dots, 0}_{n-k})$

I.3 Tangent Space

Recall that a PARAMETRIC SURFACE in $\mathbb{R}^3$ is a map $m : \mathbb{R}^2 \supset \Omega \mapsto \mathbb{R}^3$

Set $M := m(\Omega)$. For $p \in M$ and $c : (-\varepsilon, \varepsilon) \mapsto M \subset \mathbb{R}^3$ with $c(0) = p$ and if c is smooth,

$v := c'(0)$ is TANGENT to M at p.

The set of all such vectors generate the TANGENT PLANE.

Intrinsively, if M is a smooth manifold and if (U, φ) a chart at $p \in M$, then we could set $v := \left[\frac{d}{dt}(\varphi \circ c)\right](0) \in \mathbb{R}^n$ and call it a tangent vector. (well-defined)

$\mathbb{R}^n \leftarrow M \leftarrow (-\varepsilon, \varepsilon)$

But it depends too much on the choice of a chart.

Def. For $p \in M$ (a s.m.) we denote by $C^\infty(p)$ the EQUIVALENCE CLASS of smooth real functions defined on a neighborhood of p.

Two functions are identified if they coincide (are identically same) on a neighborhood of p.

The elements of $C^\infty(p)$ are called GERMS of C^∞-function at p.

Observations: $C^\infty(p)$ is a vector space with the multiplication of functions

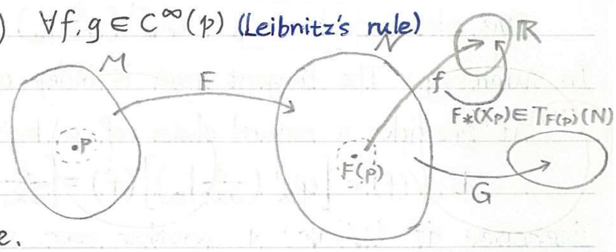
$\Leftrightarrow$ $C^\infty(p)$ is an algebra.

Def. The TANGENT SPACE $T_p(M)$ of M at p is the set of all maps

$X_p : C^\infty(p) \mapsto \mathbb{R}$ satisfying

1) $X_p(\alpha f + g) = \alpha X_p(f) + X_p(g) \quad \forall f, g \in C^\infty(p), \forall \alpha \in \mathbb{R}$

2) $X_p(fg) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g) \quad \forall f, g \in C^\infty(p)$ (Leibnitz's rule)



$T_p(M)$ is endowed with

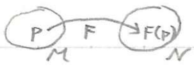
1) $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$

2) $(\alpha X_p)(f) = \alpha X_p(f)$

which makes $T_p(M)$ a real vector space.

⚠ A tangent vector at p is any $X_p : C^\infty(p) \mapsto \mathbb{R}$.

Observe that this def is indep of any chart, and is intrusic.

Thm. 

(proof as exercise) (simple)

Let $F : M \mapsto N$ be a smooth map between smooth manifolds. For any $p \in M$:

$F^* : C^\infty(F(p)) \mapsto C^\infty(p)$, $F^*(f) := f \circ F$

$F_* : T_p(M) \mapsto T_{F(p)}(N)$, $[F_*(X_p)](f) := X_p(F^*(f)) = X_p(f \circ F) \quad \forall f \in C^\infty(F(p))$

Then F^* is a homomorphism of algebra $(F^*(f + \alpha g) = F^*(f) + \alpha F^*(g),\ F^*(fg) = F^*(f) F^*(g))$

and F_* is a homomorphism (means preserving structures) of vector space. $(F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p))$

If $H = G \circ F$, $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

F_* is called the DIFFRENTIAL of F and also denoted by $dF \equiv DF \equiv F'$

Now consider a local version of this result, with $\mathcal{N} = \mathbb{R}^n$.

Let $p \in M$ and (U, φ) a coordinate system ($\equiv$ a chart) at p

Then $\varphi_* : T_p(M) \mapsto T_{\varphi(p)}(\mathbb{R}^n)$ is a homomorphism $\forall p \in U$

If $a := \varphi(p) \in \varphi(U)$, then $\varphi^{-1}{}_* : T_a(\mathbb{R}^n) \mapsto T_p(M)$ is a homomorphism

It implies that φ_* and $\varphi^{-1}{}_*$ are isomorphisms.

$\rightsquigarrow$ We can borrow information from $T_a(\mathbb{R}^n)$

$:C^\infty(a) \mapsto \mathbb{R}$ — There exists one unique

Lemma: $\forall X_a \in T_a(\mathbb{R}^n)\ \exists!\ v \in \mathbb{R}^n$ s.t.

$$X_a(f) = \sum_{j=1}^n v_j \left(\frac{\partial F}{\partial x_j}\right)(a) = v \cdot [\nabla f](a) = [D_v f](a) \text{ (directional derivative)}$$

and any $v \in \mathbb{R}^n$ defines an element of $T_a(\mathbb{R}^n)$ by $X_a = D_v$.

In other words $T_a(\mathbb{R}^n) \ni X_a \xleftrightarrow{\text{bijective}} v \in \mathbb{R}^n$

$\xrightarrow[\text{simple}]{\text{less simple}}$ (to prove)

We conclude that $T_a(\mathbb{R}^n)$ is of dim n.

A basis of $T_a(\mathbb{R}^n)$ is given by $\left\{ \frac{\partial}{\partial x_1}\Big|_a, \frac{\partial}{\partial x_2}\Big|_a, \cdots, \frac{\partial}{\partial x_n}\Big|_a \right\}$

which can be written by $E_{i,a} = \frac{\partial}{\partial x_i}\Big|_a$ with $\{E_{i,a}\}_{i=1}^n$ a basis of $T_a(\mathbb{R}^n)$

$\Rightarrow$ For any coordinate system (U, φ) on M, the image

$\left\{ \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right) \right\}$ is a basis of $T_{\varphi^{-1}(a)}(M)$.

We also write $E_{i,p} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)$ and call these bases the COORDINATE FRAMES.

In summary: The tangent space is indep of any coordinate systems, but once one is given it provides a natural choice of a basis, namely if $f \in C^\infty(p)$, then

$$E_{i,p}(f) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\Big|_a\right)\right](f) = \left[\frac{\partial}{\partial x_i}(f \circ \varphi^{-1})\right](\varphi(p))$$

Exercise: if (V, ψ) is another coor. system, what are the relations between these bases?

Corollary: If $F: M \mapsto \mathcal{N}$ is smooth and if $p \in M$,

the rank of F at p is equal to the dim of $F_*(T_p(M))$ in $T_{F(p)}(\mathcal{N})$

(another def of rank indep of the coor. systems)

Back to curves:

Consider $c : (-\varepsilon, \varepsilon) \mapsto \mathcal{M}$ a smooth map. ↗are smooth manifolds

On $(-\varepsilon,\varepsilon)$ all tangent vector at $t_0 \in (-\varepsilon,\varepsilon)$ are given by $v \underbrace{\frac{d}{dt}\Big|_{t_0}}_{\text{unit vector}}$ for $v \in \mathbb{R}$

Then $c_*\left(\frac{d}{dt}\Big|_{t_0}\right) f = \left[\frac{d}{dt}(f \circ c)\right](t_0) =: \circledast$ $(f \in C^\infty(c(t_0)))$

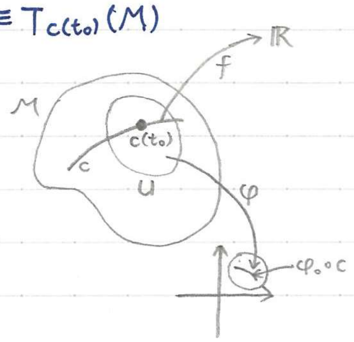
If (U, φ) is a coor. system at $c(t_0)$

and if we set $c^i := (\varphi \circ c)^i \quad \forall i = 1, \cdots, n$

$$\circledast = \left[\frac{d}{dt}\left(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n} \circ \underbrace{\varphi \circ c}_{\mathbb{R}^n \leftarrow (-\varepsilon,\varepsilon)}\right)\right](t_0) = \left[\frac{d}{dt}\left(f \circ \varphi^{-1}(c^1, c^2, \cdots, c^n)\right)\right](t_0) \quad \leftarrow \text{Calculus II}$$

$$= \sum_{j=1}^{n} \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(\varphi \circ c(t_0))\, c^{j\prime}(t_0) = \sum_{j=1}^{n} c^{j\prime}(t_0)\, E_{j, c(t_0)}(f) \in T_{c(t_0)}(\mathcal{M})$$

$\Rightarrow$ A curve defines an element of $T_{c(t_0)}(\mathcal{M})$



The converse:

Lemma For any $p \in \mathcal{M}$ and any $X_p \in T_p(\mathcal{M})$

$\exists c : (-\varepsilon,\varepsilon) \mapsto \mathcal{M}$, smooth and with $c(0) = p$, s.t.

$c_*\left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$

I.4 Vector fields

We consider a map $X : M \mapsto \bigcup_{p \in M} T_p(M)$,
$$p \mapsto X_p \in T_p(M)$$

How can one impose some smoothness on X?

1^{st} solution: (best) (too abstract)

consider $T(M) = \bigcup_{p \in M} T_p(M)$ with a certain topology making a smooth manifold.

$T(M)$ is called TANGENT BUNDLE. → describe this: exercise for mathematiciens.

Then consider X as a smooth map between smooth manifolds.

2^{nd} solution:

for a coordinate system (U, φ) on M and for $p \in U$,

we consider the basis $\{E_{j,p}\}_{j=1}^{n}$

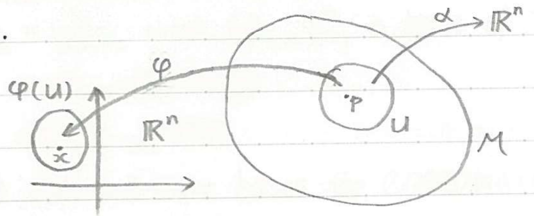
Then $X_p \in T_p(M)$ and $X_p = \sum_{j=1}^{n} \underbrace{\alpha_j(p)}_{\in \mathbb{R}} E_{j,p}$ (a decomposition of X_p on this basis)

By moving p in U, the coefficients $\alpha_j(p)$ is also varying.

So we can impose that

$$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n \quad \text{is smooth.}$$

This requirement $\Leftrightarrow$ first solution.



Def. a C^∞-VECTOR FIELD on M is a map $X : M \mapsto T(M)$ whose components α_j in the coordinate frame $\{E_{i,p}\}$ of any coordinate system satisfy

$$\mathbb{R}^n \supset \varphi(U) \ni x \longmapsto (\alpha \circ \varphi^{-1})(x) \in \mathbb{R}^n \quad \text{is smooth.}$$

The set of all C^∞-vector fields is denoted by $\mathfrak{X}(M)$.

Lemma: $X : M \mapsto T(M)$ is a C^∞-vector field iff

$$\forall f \in C^\infty(M, \mathbb{R}) : Xf : M \mapsto \mathbb{R},\ [Xf](p) \equiv [Xf]_p := X_p f \quad \text{is smooth.}$$

(another equivalent def) (could be an exercise)

Observe that in this lemma, X can be considered as a map

$$C^\infty(M) \ni f \xmapsto{X} Xf \in C^\infty(M)$$

Remark: $\mathfrak{X}(M)$ is a vector space and has ~~on~~ additional structures:

1) $\mathfrak{X}(M)$ is a $C^\infty(M)$-MODULE

$:\Leftrightarrow \forall f \in C^\infty(M) \, \forall X \in \mathfrak{X}(M) : \exists f X \in \mathfrak{X}(M)$ defined by $[fX]_p := f(p) X_p$

2) $\mathfrak{X}(M)$ is a Lie-algebra (very important)

$:\Leftrightarrow$ We can endow $\mathfrak{X}(M)$ with a Lie bracket:

$$\begin{array}{ccccc} \mathfrak{X}(M) & \times & \mathfrak{X}(M) & \longmapsto & \mathfrak{X}(M) \\ \Cup & & \Cup & & \Cup \\ (X & , & Y) & \longmapsto & [X,Y] := XY - YX \end{array}$$

given by … satisfying

i) linearity in each element

ii) antisymmetry: $[X,Y] = -[Y,X]$

iii) Jacobi identity: $[X,[Y,Z]] = [Y,[Z,X]] = [Z,[X,Y]]$

Exercise: show that 1) and 2) hold

In particular check that $[X,Y]_p$ satisfies Leibniz's rule.

Recall that for any $X_p \in T_p(M)$ $\exists c : (-\varepsilon, \varepsilon) \mapsto M$ with $c(0) = p$ and

$$\dot{c}(0) := c_*\left(\frac{d}{dt}\Big|_{t=0}\right) = X_p$$

Thm. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$.

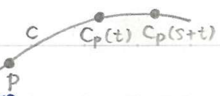
$$\forall p \in M \underset{\text{there exists a unique}}{\exists!} c_p : (-\varepsilon, \varepsilon) \mapsto M \text{ with } c_p(0) = p \text{ and } \dot{c}_p(t) = X_{c_p(t)}$$

Remarks: The curve c_p is called the INTEGRAL CURVE of X at p,

and we call $c_p((-\varepsilon,\varepsilon))$ the ORBITAL of p.

⚠ The value ε depends on p.

Whenever it is well-defined, the following relation holds:



$$C_p(s+t) = C_{C_p(t)}(s)$$

in Appendix for 2nd lecture

Thm. The orbit of p is either the single point p or an immersion of $(-\varepsilon,\varepsilon)$ in M.

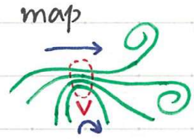
$\qquad$ if $X_p = 0$ $\qquad$ if $X_p \neq 0$

Thm. For any $x \in \mathfrak{X}(M)$ and any $p \in M$ $\exists V \in \mathcal{V}_p$, $\varepsilon > 0$ and a smooth map

$\mathcal{V}_p := \{\text{open sets} \ni p\}$

$$F : \underbrace{(-\varepsilon,\varepsilon) \times V}_{\text{s.m.}} \longmapsto M \text{ satisfying}$$

$$F(0,q) = q \in V \text{ and } \dot{F}(t,q) = X_{F(t,q)} \quad \forall \begin{smallmatrix} t \in (-\varepsilon,\varepsilon) \\ q \in V \end{smallmatrix}$$



※ The map F is called the LOCAL FLOW of X at p. Note that $F(t,p) = c_p(t)$

Def. Let $X \in \mathfrak{X}(M)$ and $p \in M$. If $X_p = 0$ then p is called a SINGULAR POINT of the vector field. Since $c_p(t) = p \, \forall t$ if p is sigular, these points are very special and we can study the integral curves around them.

The possible behaviors depend on the topology. Nice subject but we can't go further.

Def. A C^∞-vector field is COMPLETE if
at any $p \in M$, c_p is defined on all $\mathbb{R}$.
A complete vector field can contain some singular points.

Thm. Any C^∞-vector field on a compact manifold is complete.

Remark: Let $X \in \mathfrak{X}(M)$, $p \in M$ and c_p the corresponding integral curve.
Then for any $f \in C^\infty(p)$:

$$X_p f = \frac{d}{dt} f(c_p(t))\Big|_{t=0} = \lim_{t \to 0} \frac{f(c_p(t)) - f(p)}{t}$$

If $f \in C^\infty(M, \mathbb{R})$ recall that
$Xf \equiv L_X f$ is defined by $[Xf]_p = X_p f$
↳ called the LIE DERIVATIVE of f
interpreted as the derivative of f in the direction given by X.

If $Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y \in \mathfrak{X}(M)$ of Y is defined by

$$[L_X Y]_p := \lim_{t \to 0} \frac{1}{t}\Big(\underbrace{\overset{F(-t,\cdot):\ \mapsto V_p}{F(-t, \underbrace{V_{c_p(t)}}_{\in V_{c_p(t)}})_*}}_{T_p(M) \leftarrow T_{c_p(t)}(M)} \underbrace{Y_{c_p(t)}}_{\in T_{c_p(t)}(M)} - \underbrace{Y_p}_{\in T_p(M)} \Big)$$

Lemma: $L_X Y = [X, Y]$

II: Tensors, tensor fields and differential forms

II.1 Tensors [Bo 199-214][GN 62-69]

: Let V be a finite dimensional and real vector space ($\mathbb{R}^n$)
and let V^* be its DUAL.
(= the set of all linear maps $V \mapsto \mathbb{R}$, such a map is called a LINEAR FUNCTIONAL on V)

Prop. If $\dim V = n$, then $\dim V^* \overset{\text{exercise}}{=} n$

Def. a TENSOR ϕ on V is a multilinear map

$$\phi : \underbrace{V \times V \times V \times \cdots \times V}_{r \text{ terms}} \times \underbrace{V^* \times V^* \times \cdots \times V^*}_{s \text{ terms}} \longmapsto \mathbb{R}$$

e.g.

$$\phi(v_1, \underbrace{\alpha v_2 + \beta v_2'}_{\in V \times V}, \underbrace{\omega_1}_{\in V^*}) = \alpha \phi(v_1, v_2, \omega_1) + \beta \phi(v_1, v_2', \omega_1)$$

We say that ϕ is r-times COVARIANT and s-times CONTRAVARIANT.

We write $\quad \phi \in \mathcal{T}^r_s(V) \equiv \mathcal{T}^{r,s}(V, V^*)$

Examples

1) $r=1, s=0$: $\phi : V \mapsto \mathbb{R}$ is an element of $V^* = \mathcal{T}^1_0(V)$

2) $r=1, s=1$: $\phi(v, \omega) \equiv \omega(v) \equiv \langle \omega, v \rangle$ related to scalar product

Lemma: $\mathcal{T}^r_s(V)$ is a vector space of dim n^{r+s}. (exercise)

Remark: If $\phi_j \in \mathcal{T}^{r_j}_0(V)$, $j = 1, 2$, we set

$\phi_1 \otimes \phi_2 \in \mathcal{T}^{r_1+r_2}_0(V)$ with

$$\phi_1 \otimes \phi_2 (v_1, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_{r_1+r_2}) := \underbrace{\phi_1(v_1, \cdots, v_{r_1})}_{\in \mathbb{R}} \underbrace{\phi_2(v_{r_1+1}, \cdots, v_{r_1+r_2})}_{\in \mathbb{R}}$$

Similar def for $\phi_j \in \mathcal{T}^0_{s_j}(V)$, $j = 1, 2$

If $\phi_1 \in \mathcal{T}^r_0(V) =: \mathcal{T}^r(V)$, $\phi_2 \in \mathcal{T}^0_s(V) =: \mathcal{T}_s(V)$, then

$\phi_1 \otimes \phi_2 \in \mathcal{T}^r_s(V)$ with

$$\phi_1 \otimes \phi_2 (v_1, \cdots, v_r, \omega_1, \cdots, \omega_s) := \phi_1(v_1, \cdots, v_2)\, \phi_2(\omega_1, \cdots, \omega_2)$$

⚠ This product is not commutative!

Def. A tensor $\phi \in \mathcal{T}^r(V)$ is SYMMETRIC if invariant under the permutation of 2 argumen
(for example if $\phi(v_1, v_2) = \phi(v_2, v_1)$)
and ALTERNATING if it changes the sign under the permutation of 2 arguments.
(for example if $\phi(v_1, v_2) = -\phi(v_2, v_1)$)

Same def for $\phi \in \mathcal{T}_s(V)$.

We write $\Sigma^r(V)$ for the set of symmetric tensors in $\mathcal{T}^r(V)$

and $\Lambda^r(V)$ for " alternating " $\mathcal{T}^r(V)$.

Note that $\Sigma^r(V)$ and $\Lambda^r(V)$ are vector spaces.

Let S_k denote the group of all permutation of $\{1,\cdots,k\}$

$\sigma\in S_k$ if σ is a bijective map from $\{1,\cdots,k\}$ to itself

with $(1,\cdots,k)\longmapsto(\sigma(1),\cdots,\sigma(k))$

We set $\mathrm{sgn}(\sigma)=1$ if σ corresponds to an even number of transposition, ↳ permutation of 2 element

and $\mathrm{sgn}(\sigma)=-1$ if " odd "

Def. On $\mathcal{T}^n(V)$ one set

$\mathcal{S}:\mathcal{T}^n(V)\mapsto\mathcal{T}^n(V)$ and $\mathcal{A}:\mathcal{T}^n(V)\mapsto\mathcal{T}^n(V)$ by

$$\left.\begin{aligned}[\mathcal{S}\phi](v_1,\cdots,v_n)&:=\frac{1}{n!}\sum_{\sigma\in S_n}\phi(v_{\sigma(1)},\cdots,v_{\sigma(n)}) \quad \text{(SYMMETRIZE)}\\ [\mathcal{A}\phi](v_1,\cdots,v_n)&:=\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\phi(v_{\sigma(1)},\cdots,v_{\sigma(n)}) \quad \text{(ANTI-SYMMETRIZE)}\end{aligned}\right\}\text{linear}$$

Lemma: 1) $\mathcal{S}^2=\mathcal{S}$, $\mathcal{A}^2=\mathcal{A}$

2) $\mathcal{S}\mathcal{T}^n(V)=\Sigma^n(V)$, $\mathcal{A}\mathcal{T}^n(V)=\Lambda^n(V)$ if and only if

3) $\phi\in\Sigma^n(V)$ iff $\mathcal{S}\phi=\phi$, $\phi\in\Lambda^n(V)$ iff $\mathcal{A}\phi=\phi$.

Remark: If $F:V\mapsto W$ is a linear map between 2 vector spaces

then it induces a linear map

$F^*:\mathcal{T}^n(W)\mapsto\mathcal{T}^n(V)$ by

$$\underbrace{[F^*\phi]}_{\in\mathcal{T}^n(V)}\underbrace{(v_1,\cdots,v_n)}_{\in V^n}=\underbrace{\phi}_{\in\mathcal{T}^n(W)}\underbrace{(F(v_1),\cdots,F(v_n))}_{\in W^n}\ \forall\phi\in\mathcal{T}^n(W)$$

Now let us set

$$\Lambda(V):=\overset{\mathbb{R}}{\overset{\shortparallel}{\Lambda^0(V)}}\oplus\Lambda^1(V)\oplus\Lambda^2(V)\oplus\cdots\oplus\Lambda^j(V)\oplus\cdots$$
$$\subset\mathcal{T}^0(V)\oplus\mathcal{T}^1(V)\oplus\mathcal{T}^2(V)\oplus\cdots\oplus\mathcal{T}^j(V)\oplus\cdots=:\mathcal{T}(V) \leftarrow \text{lemoi algebra over } V$$

The elements of $\Lambda(V)$ or $\mathcal{T}(V)$ cousint in finite "sums" ← only a notation

$$\phi^0+\phi^1+\phi^2+\cdots+\phi^j+\cdots\equiv(\phi^0,\phi^1,\phi^2,\cdots,\phi^j,\cdots)=:\Phi$$

for $\phi^j\in\Lambda^j(V)$ or $\mathcal{T}^j(V)$; $\exists k\in\mathbb{N}\ \forall j\geq k:\phi^j=0$ (k different for each Φ)

Lemma: $\mathcal{T}(V)$ is a vector space and an associative algebra with $\otimes$ ← extended by linearity

↳ $(\phi\otimes\psi)\otimes\varphi=\phi\otimes(\psi\otimes\varphi)$

E.g. $(\phi_0+\phi_1)\otimes(\psi_0+\psi_1+\psi_2)$

$$=\underset{\in\mathbb{R}}{\phi_0\otimes\psi_0}+\underbrace{\phi_0\otimes\psi_1+\phi_1\otimes\psi_0}_{\in\mathcal{T}^1(V)}+\underbrace{\phi_0\otimes\psi_2+\phi_1\otimes\psi_1}_{\in\mathcal{T}^2(V)}+\underset{\in\mathcal{T}^3(V)}{\phi_1\otimes\psi_2}\ \in\mathcal{T}(V)$$

What about $\Lambda(V)$? The product $\otimes$ does not generate altenating tensor

Def. For $\phi \in \mathcal{J}^r(V)$ and $\psi \in \mathcal{J}^s(V)$ we set

$\phi \wedge \psi \in \mathcal{J}^{r+s}(V)$ with

$\phi \wedge \psi := \frac{(r+s)!}{r!\,s!} \mathcal{A}(\phi \otimes \psi)$ called EXTERIOR PRODUCT or WEDGE PRODUCT

Lemma. the Wedge product is bilinear and associative.

Corollory: $\Lambda(V)$ with the wedge product is an associative algebra

called EXTERIOR or GRASSMAN ALGEBRA over V

ts Lemma. If $\phi \in \Lambda^r(V)$ and $\psi \in \Lambda^s(V)$ then $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Thm. If $\dim V = n$

1) If $r > n$, then $\Lambda^r(V) = 0$

2) If $0 \leq r \leq n$, then $\dim \Lambda^r(V) = \binom{n}{n} := \frac{n!}{r!(n-r)!}$

In particular if $r=n$, $\dim \Lambda^n(V) = 1 \Rightarrow$ unicity of det

3) $\dim \Lambda(V) = 2^n$

(Next time: $M \mapsto \bigcup_{p \in M} \Lambda(T_p M)$)

II.2 About bases

Recall that if $\{E_1, \cdots, E_n\}$ is a basis of V, then $\exists!$ basis $\{\varphi_1, \cdots, \varphi_n\}$ of V^* s.t.

$\varphi_j(E_k) = \delta_{jk} := \begin{cases} 1, & \text{if } j=k \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow \forall v \in V: v = \sum_{j=1}^{n} \underbrace{\varphi_j(v)}_{\text{component of } v \text{ on } E_j} E_j$

We call $\{\varphi_1, \cdots, \varphi_n\}$ the DUAL BASIS.

Consider M a smooth manifold, and (U, φ) a local chart.

For any $p \in U$ a basis of $T_p(M)$ is given by the coordinate frame $\{E_{1,p}, \cdots, E_{n,p}\}$ with

$E_{j,p} := \varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)$

Thus if we consider the dual space $T_p(M)^* \equiv T_p^*(M)$

there exists a dual basis for $\{E_{1,p}, \cdots, E_{n,p}\}$, usually denoted by $\{(dx^j)_p\}_{j=1}^n$

"Justification" for the notation (change of point of view)

Let $f \in C^\infty(p)$ and $X_p \in T_p(M)$. We set $(df)_p(X_p) := X_p f \in \mathbb{R}$ and in particular

$(df)_p(E_{j,p}) = \left[\varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)\right](f) = \left[\frac{\partial}{\partial x_j}(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n})\right](\varphi(p))$

If we choose $f = \varphi^i : V_p \ni V \mapsto \mathbb{R} \Rightarrow$ $= \frac{\partial}{\partial x_j}(x^i)(\varphi(p)) = \delta_{ij}$

Observe that $(df)_p : T_p(M) \mapsto \mathbb{R}$ is linear, and thus an element of $T_p^*(M)$

$\Rightarrow (d\varphi^i)_p$ is an element of the dual basis.

If $\mathcal{M} = \mathbb{R}^n$

then φ = identity, and if $f \in C^\infty(p)$ then $(df)_p = \sum_{j=1}^{n} \lambda_i (dx^i)_p$ with (coef on a basis)

$\lambda_i = E_{p,i}(f) = \frac{\partial f}{\partial x^i}(p)$

$\Rightarrow (df)_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(dx^i)_p$

Corresponds to $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n$, seen in Calculus II.

II.2 Tensor field

Recall that a vector field is a map

$X : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} T_p(\mathcal{M}) \equiv T(\mathcal{M})$.

Def. a (r,s)-TENSOR FIELD on $\mathcal{M}$ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \mathcal{T}^r_s(T_p(\mathcal{M}))$

$p \mapsto \phi(p) \in \mathcal{T}^r_s(T_p(\mathcal{M}))$ of dimension n if $\dim \mathcal{M} = n$

Examples

1) A vector field $X : \mathcal{M} \mapsto T(\mathcal{M})$ is a $(0,1)$-tensor field. Indeed:

a $(0,1)$-tensor field ϕ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \underbrace{\mathcal{T}^0_1(T_p(\mathcal{M}))}$

linear map from $T^*_p(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T^{**}_p(\mathcal{M}) = T_p(\mathcal{M})$ (an exercise)

2) Reciprocally, a $(1,0)$-tensor field ϕ is a map

$\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \underbrace{\mathcal{T}^1_0(T_p(\mathcal{M}))} = \bigcup_{p\in\mathcal{M}} T^*_p(\mathcal{M})$

linear map from $T_p(\mathcal{M})$ to $\mathbb{R} \Rightarrow$ element of $T^*_p(\mathcal{M})$

$\bigcup_{p\in\mathcal{M}} T^*_p(\mathcal{M})$ is called a COTANGENT BUNDLE. (exercise: it's a smooth manifold)

In this case ϕ is a COVECTOR FIELD.

3) A map $\phi : \mathcal{M} \mapsto \bigcup_{p\in\mathcal{M}} \mathcal{T}^2_0(T_p(\mathcal{M}))$ is called FIELD of BILINEAR FORMS.

$\forall p \in \mathcal{M} : \phi_p : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \xrightarrow{\text{bilinear}} \mathbb{R}$.

Observation: A bilinear map can be identified with a $n\times n$ matrix:

$\alpha_{ij,p} := \phi_p(E_{i,p}, E_{j,p}) \quad (i,j \in \{1, \cdots, n\})$

About smoothness

There are several equivalent defs for the smoothness for a tensor field.

For example, if $X_1, \cdots, X_r \in \mathfrak{X}(\mathcal{M}) = \{$smooth vector fields$\}$

and if $Y_1, \cdots, Y_s$ are smooth covector fields,

then one imposes that. the map

$$\mathcal{M} \ni p \mapsto \phi_p(X_{1,p}, \cdots, X_{r,p}, Y_{1,p}, \cdots, Y_{s,p}) \in \mathbb{R} \text{ is smooth.}$$

Or, if (U, φ) is a chart, if $p \in U$ and if we consider $\{E_{j,p}\}_{j=1}^n$ and $\{(dx^i)_p\}_{i=1}^n$

the coordinate frames and coframes. Then we can write

$$\phi_p = \sum_{\substack{i_k=1 \\ j_t=1}}^{n} \underbrace{a_{i_1, \cdots, i_k}^{j_1, \cdots, j_s}(p)}_{\in \mathbb{R}} (dx^{i_1})_p \otimes \cdots \otimes (dx^{i_k})_p \otimes E_{j_1,p} \otimes \cdots \otimes E_{j_s,p}$$

(coefficient in a local basis)

and impose that the coefficients are C^∞ on U.

We call such smooth tensors C^∞-TENSOR FIELDS.

Def. The set of all smooth (r,s)-tensor fields on $\mathcal{M}$ is denoted by $\mathcal{T}_s^r(\mathcal{M})$.

Lemma: 1) $\mathcal{T}_s^r(\mathcal{M})$ is a vector field

2) $\mathcal{T}_0^r(\mathcal{M})$ is a $C^\infty(\mathcal{M})$-module $:\Leftrightarrow \phi(x_1, \cdots, f x_j, \cdots, x_n) = f\phi(x_1, \cdots, x_j, \cdots, x_n)$

3) If $\phi \in \mathcal{T}_0^r(\mathcal{M})$ and $\psi \in \mathcal{T}_0^s(\mathcal{M})$ then $\phi \otimes \psi \in \mathcal{T}_0^{r+s}(\mathcal{M})$

with $(\phi \otimes \psi)_p := \phi_p \otimes \psi_p$

Remarks

1) If $f \in C^\infty(\mathcal{M}) \equiv C^\infty(\mathcal{M}, \mathbb{R})$ then we define a covector field by the formula

$df : \mathcal{M} \mapsto \mathcal{T}^*(\mathcal{M}) = \cup_{p \in \mathcal{M}} T_p^*(\mathcal{M})$, $(\Leftrightarrow df \in \mathcal{T}_0^1(\mathcal{M}))$

$(df)_p(X_p) := X_p(f)$

called the DIFFERENTIAL of f

2) If $F : \mathcal{M} \mapsto \mathcal{N}$ a smooth map and if ϕ is a $(r,0)$-tensor field on $\mathcal{N}$

then we set $F^*\phi$ a $(r,0)$-tensor field on $\mathcal{M}$ by

$$(F^*\phi)_p(X_{1,p}, \cdots, X_{r,p}) := \phi_{F(p)}(\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(\mathcal{N})}, \cdots, F_*(X_{r,p}))$$

It means

$$F^* : \mathcal{T}_0^r(\mathcal{N}) \longmapsto \mathcal{T}_0^r(\mathcal{M})$$

Def. A tensor field $\phi \in \mathcal{T}^r_0(M)$ is SYMMETRIC if $\forall p \in M: \phi_p \in \overset{\nearrow \{\text{sym. tensors}\}}{\Sigma^r}(T_p(M))$

ALTERNATING if $\underset{\searrow \{\text{alt. tensors}\}}{\Lambda}$

Remark: (Very important)

A symmetric tensor field $\underbrace{\phi \in \mathcal{T}^2_0(M)}^{\text{bilinear forms on } M}$ is POSITIVE DEFINITE if

$\forall p \in M \, \forall X_p \in T_p(M): \phi_p(X_p, X_p) \geq 0$; equality $\Leftrightarrow X_p = 0$

A manifold with a symmetric positive definite bilinear form is called a RIEMANN MANIFOLD; ϕ is called a RIEMANN METRIC. ($\Rightarrow$ Integration)

(Good for geometry)

II.3 Differential forms and exterior derivative

Def. A tensor field $\phi \in \mathcal{T}^r(M)$ which is alternating is called an EXTERIOR DIFFERENTIAL FORM of degree r; or a r-FORM.

We write $\Lambda^r(M)$ for the set of all r-forms, and

$\Lambda(M) := \bigoplus_{r=0}^{n} \Lambda^r(M)$, with $\Lambda^0(M) := C^\infty(M)$.

Properties

1) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$ then $\underset{\underset{(-1)^{rs}\psi \wedge \phi}{\parallel}}{\phi \wedge \psi} \in \Lambda^{r+s}(M)$

2) $\Lambda(M)$ is an algebra with the Wedge product $\wedge$.

3) If (U, φ) is a local chart, and if $p \in U$, then the set

$\{(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p\}$ with $1 \leq i_1 < \cdots < i_r \leq n$

is a basis for $\Lambda^r(T_p(M))$, and accordingly

$\{(dx^{i_1}), \cdots, (dx^{i_r})\}$ is a basis for $\Lambda^r(U) \subset \Lambda^r(M)$.

$\Rightarrow \Lambda(M)$ is the algebra of differential forms or exterior algebra.

Main result of this chapter (for def of grad, div, curl, etc)

Thm. Let M be a smooth manifold, and $\Lambda(M)$ the exterior algebra,

There is a unique linear map

$d : \Lambda(M) \mapsto \Lambda(M)$ satisfying $\underset{\text{this map}}{d}(\underset{\text{differential of } f}{f})$

1) If $f \in \Lambda^0(M) = C^\infty(M)$, then $df = df \in \mathcal{T}^1_0(M), (df)_p(X_p) = X_p(f)$

2) If $\phi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, then

$$d(\phi \wedge \psi) = (d\phi) \wedge \psi + (-1)^r \phi \wedge (d\psi)$$

3) $d^2 = d \circ d = 0$

In local coordinates, we have an explicit formula for d:

Recall that if (U, φ) is a chart, $p \in U$, then

$\{E_{j,p}\}_{j=1}^n$ is a basis for $T_p(M)$ and $\{(dx^j)_p\}_{j=1}^n$ is a basis for $T_p^*(M)$.

Then $\phi \in \Lambda^r(M)$ can be represented by

$$\phi_p = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1, \cdots, i_r}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_r})_p \quad \text{(a special case of } \mathcal{T}^0_s(M)\text{)}$$

$$=: \sum_I a_I(p)(dx^I)_p \quad \text{with } a_I : U \mapsto \mathbb{R} \text{ smooth.}$$

Then (def)

$$(d\phi)_p := \sum_I \overbrace{(da_I)_p}^{\in \mathcal{T}^1_0(M)} \wedge \overbrace{(dx^I)_p}^{\in \Lambda^r(M)} \in \Lambda^{r+1}(M)$$

Exercise: check that this def satisfies the 3 conditions [GN p74]

Remarks

1) d is a local operator: If $U \subset M$ and $\phi \in \Lambda(U) \subset \Lambda(M)$ then $d_U \phi = d_M \phi$

2) d maps $\Lambda^r(M)$ to $\Lambda^{r+1}(M)$

3) d is called the EXTERIOR DERIVATIVE

Exercise (Thm?)

If $\omega \in \Lambda^1(M)$ and $X, Y \in \mathfrak{X}(M) := \{C^\infty\text{-vector fields}\}$, then

$$d\omega(X,Y) = X\underbrace{\omega(Y)}_{\in C^\infty(M)} - Y\underbrace{\omega(X)}_{\in C^\infty(M)} - \underbrace{\omega([X,Y])}_{\in C^\infty(M)} \in C^\infty(M)$$

$$(\omega X)_p = \omega_p(X_p) \in \mathbb{R}$$

Proof: In a chart (U, φ), $\omega_p = \sum_{j=1}^n a_j(p)(dx^j)_p$

For shortness, we write $\omega_p = f\,dg$ for $f, g \in C^\infty(M)$

Then $d\omega(X,Y) = d(f\,dg)(X,Y) \overset{\text{by def}}{=\!=} (df \wedge dg)(X,Y) \overset{\text{by def}}{=\!=} df(X)dg(Y) - df(Y)dg(X)$

$$= (Xf)(Yg) - (Yf)(Xg) \in C^\infty(M)$$

$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(f\,dg(Y)) - Y(f\,dg(X)) - f\,dg([X,Y])$

$= X(fYg) - Y(fXg) - f(XY - YX)g \overset{\text{Leibniz}}{=\!=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$

$= (Xf)(Yg) - (Yf)(Xg)$ □

$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(fdg(Y)) - Y(fdg(X)) - fdg([X,Y])$

$= X(fYg) - Y(fXg) - f(XY-YX)g \overset{\text{Leibniz}}{=\!=} XfYg + fXYg - YfXg - fYXg - fXYg + fYXg$

$= (Xf)(Yg) - (Yf)(Xg)$ □

Generalization

Prop. ["GN" 3.8.2 p.75~76] (independent of any coordinate systems)

Let $\phi \in \Lambda^r(M)$ and $X_1, \cdots, X_{n+1} \in \mathfrak{X}(M)$, then

$$[d\phi](X_1, \cdots, X_{r+1}) = \sum_{i=1}^{n+1} (-1)^{r+1} X_i \phi(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}) + \sum_{i<j} (-1)^{i+j} \underbrace{\phi([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1})}_{\in C^\infty(M)}$$

omit

Exercise: This satisfies conditions 1~3 of Thm.

Recall that if $F: M \mapsto N$ a smooth map between manifolds, then

$F^*: \mathcal{T}^r(N) \mapsto \mathcal{T}^r(M)$ by

$$(F^*\phi)_{\underset{\uparrow \in M}{p}}(X_{1,p}, \cdots, \underbrace{X_{r,p}}_{\in T_p(M)}) := \phi_{F(p)}(\underbrace{F_*(X_{1,p})}_{\in T_{F(p)}(N)}, \cdots, F_*(X_{r,p}))$$

which is also $F^*: \Lambda^r(N) \mapsto \Lambda^r(M)$ (alternating property is preserved)

Lemma: In this framework

$F^* \circ d_N = d_M \circ F^*$ [Bo Thm 8.2 p.223]

Exercise for mathematicians: about de Rham cohomology

[GN, p.76 ex 5]

II.4 Orientation on a manifold (easier)

Let V be a real vector space of dim n, and $\{E_j\}_{j=1}^n$ and $\{F_j\}_{j=1}^n$ 2 bases

Set $A \in M_{n\times n}(\mathbb{R})$ by $F_i = \sum_{j=1}^n a_{ij} E_j$ coeff. of the change of basis

Def. The two bases has the SAME ORIENTATION if $\det(A) > 0$

and of OPPOSITE ORIENTATION if $\det(A) < 0$

$\Rightarrow$ There exist 2 classes of equivalence of bases.

We say either they are either POSITIVELY ORIENTED

or NEGATIVELY ORIENTED.

Def. Let M be a smooth manifold of dim $n \geq 1$, ⚠ Convention changed

M is ORIENTABLE if there exists a covering ($\equiv$ atlas) $\{(U_j, \varphi_j)\}_j$ s.t. all transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \mapsto \varphi_j(U_i \cap U_j)$ is ORIENTATION PRESERVING

$:\Leftrightarrow$ if $\det \mathrm{Jac}(\varphi_j \circ \varphi_i^{-1}) > 0$

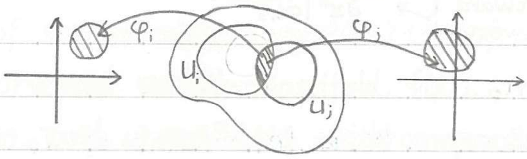
Lemma: A connected orientable manifold of dim ≥ 1 has only 2 possible orientations.

Remark: If $M = \{p\}$ (of dim 0)

an orientation is a map from p to ± 1. We need this because

$\int_a^b f'(x)dx = f(b) - f(a)$

M of dim 0, $b \mapsto +1$, $a \mapsto -1$ are what we need from orientations.



Thm. [Bo. p.218] (very deep but intruisive) ϕ is called a VOLUME FORM

A manifold is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$ ($\phi_p \in \Lambda^n(T_p(M))$)

Recall that $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and

M is a smooth manifold with boundary if every chart

$(U_\alpha, \varphi_\alpha)$ with $\varphi_\alpha : U_\alpha \mapsto H^n$ is a homeomorphism. (+ atlas conditions)

The BOUNDARY of M is denoted by ∂M and is given by

$\partial M := \bigcup_\alpha \varphi_\alpha^{-1}(\partial H_n \cap \varphi_\alpha(U_\alpha))$

which is a smooth manifold with dim $(n-1)$

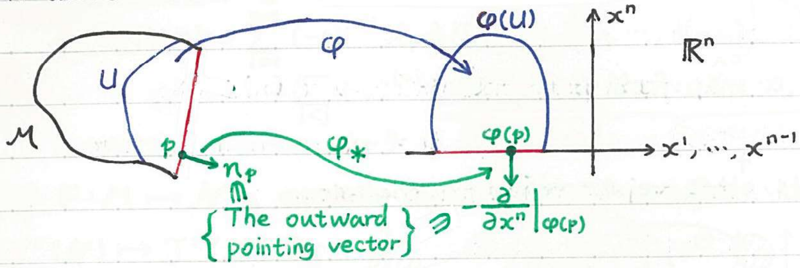
Next time: If M is oriented then it induces also an orientation on ∂M

(needed in Stroke's Thm)

Propositions

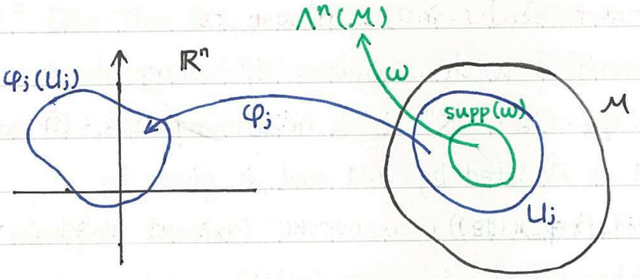
1) The boundary of a smooth manifold M of dim n is a smooth manifold ∂M of dim $(n-1)$.

2) If M is orientable then ∂M is also orientable. More precisely, if an orientation is chosen on M, then there exists an INDUCED ORIENTATION on ∂M.



We set $\varphi_*^{-1}\left(-\frac{\partial}{\partial x^n}\Big|_{\varphi(p)}\right) =: n_p$

For a basis on ∂M, we choose a basis $\{e_1, \cdots, e_{n-1}\}$ of $T_p(\partial M)$ such that $\{n_p, e_1, \cdots, e_{n-1}\}$ generates a basis of $T_p(M)$ of the same orientation as on M.



III. Integration on manifolds

III.1 Integration of n-forms

Let $\mathcal{M}$ be an oriented $\Lambda^n(\mathcal{M})$ manifold and let $\{(U_j, \varphi_j)\}_j$ be an $\overbrace{\text{oriented}}^{\text{preserving}}$ atlas.

Let $\omega \in \Lambda^n(\mathcal{M})$ with $\mathrm{supp}(\omega) \subset U_j$ and with $\mathrm{supp}(\omega)$ compact.

$$\Rightarrow \overset{(\text{or } \omega_p)}{\omega(p)} = a(p)\,(dx^1)_p \wedge \cdots \wedge (dx^n)_p \quad \text{with } a \in C^\infty(\mathcal{M})$$

Recall that $\varphi_j^{-1\,*}$ maps $\Lambda^n(\mathcal{M})$ to $\Lambda^n(\mathbb{R}^n)$

$$\Rightarrow \varphi_j^{-1\,*}(\omega) = \underbrace{a \circ \varphi_j^{-1}}_{\text{function on } \varphi_j(U_j) \subset \mathbb{R}^n}\, dx^1 \wedge \cdots \wedge dx^n.$$

Then we set

usual Riemann integral in $\mathbb{R}^n$

$$\int_{\mathcal{M}} \omega = \int_{U_j} \omega := \int_{\varphi_j(U_j)} a \circ \varphi_j^{-1}\, dx_1 \cdots dx_n \equiv \int_{\varphi_j(U_j)} a(x)\, dV \qquad (*)$$

Lemma: If $\mathrm{supp}(\omega) \subset U_k$ for an other localization map (U_k, φ_k), then

$$\int_{\varphi_k(U_k)} a \circ \varphi_k^{-1}\, dx_1 \cdots dx_n = \int_{\varphi_j(U_j)} a \circ \varphi_j^{-1}\, dx_1 \cdots dx_n$$

(independence of the coordinate system) (proof as Exercise)

Def. Let M be an oriented smooth manifold, $\{(U_j, \varphi_j)\}_j$ a covering preserving the orientation, and $\omega \in \Lambda^n(\mathcal{M})$ with compact support.

Let $\{f_j\}$ be a partition of unity of M subordinated to U_j. Then ($:\Leftrightarrow \forall j: \mathrm{supp}(f_j) = U_j$)

$$\int_{M} \omega = \int_M \sum_j f_j\, \omega = \sum_j \int_M \underbrace{f_j\, \omega}_{\in \Lambda^n(\mathcal{M})} = \sum_j \int_{U_j} f_j\, \omega \quad \text{as defind in } (*).$$

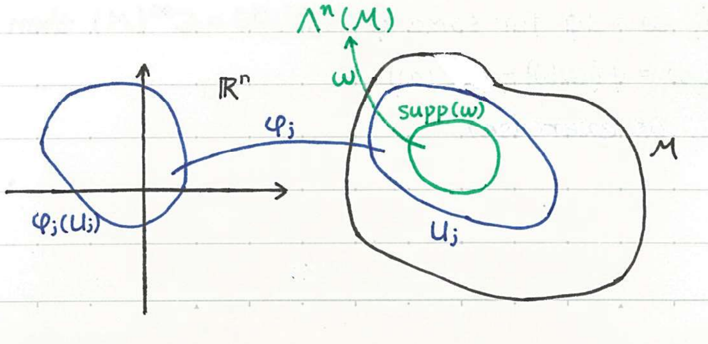
the sum is finite because supp(ω) is compact

Remarks

- $\int_M \omega$ is independent of the choice of a partition of unity. (Exercise)
- The map $\Lambda^n(\mathcal{M}) \ni \omega \mapsto \int_M \omega \in \mathbb{R}$ is a linear map.
- We can avoid the "comapactedly supported" but be careful about the convergence.
- If $F: M \mapsto \mathcal{N}$ is a diffeomorphism and if $\omega \in \Lambda^n(\mathcal{N})$, compactedly supported,

$$\int_M \underbrace{F^*\omega}_{\in \Lambda^n(\mathcal{M})} = \pm \int_{\mathcal{N}} \omega$$

($\pm$ depends on if F preserves the orientation or not)



Thm. (Stokes' Theorem) (The main thm of this chapter)

Let M be an oriented smooth manifold of dim n, with boundary ∂M. (with induced orientation).

Let $i : \partial M \mapsto M$ be the inclusion map. (identity) $\Rightarrow i^* : \Lambda^{n-1}(M) \mapsto \Lambda^{n-1}(\partial M)$

Let $\omega \in \Lambda^{n-1}(M)$ with compact support. Then

$$\int_{\partial M} \underbrace{i^*\omega}_{\in \Lambda^{n-1}(\partial M)} = \int_M \underbrace{d\omega}_{\in \Lambda^n(M)}$$

Reference for the proof: [GN p. 82-84] [Bo p. 260-261]

Remark: 1) If $\partial M = \emptyset$ then $\int_M d\omega = 0$

2) The proof is similar to the one of Calculus II on $\mathbb{R}^2$ or $\mathbb{R}^3$, and the main ingredient is $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Exercise: Show that the Green Thm, Stokes Thm in $\mathbb{R}^3$ or Divergence Thm are special cases of this theorem. See Bo p. 262-263.

Recall that M is orientable iff $\exists \phi \in \Lambda^n(M)\ \forall p \in M : \phi_p \neq 0$.

Def. Let us fix one of them, and for any $f \in C^\infty(M)$ with compact support we set

$$\int_M f := \int_M f\phi$$

⚠ This def depends on the choice of ϕ.

In particular if M is compact we set the volume of M as

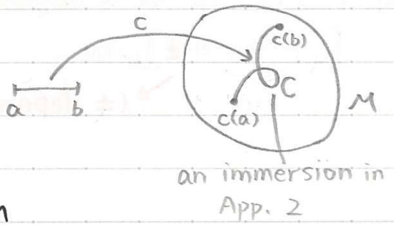
$$\mathrm{Vol}(M) := \int_M 1\phi = \int_M \phi$$

III.2 Line integrals

Let $c : [a,b] \mapsto M$ be a diffeomorphism and set $C = c([a,b])$ ↗ Capital ; it is a manifold

If $\omega \in \Lambda^1(M)$ we set

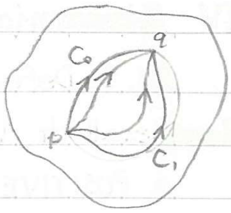
$$\int_C \omega = \int_{[a,b]} \underbrace{c^*\omega}_{\substack{\in \Lambda^1([a,b]) \\ \Downarrow \\ t \mapsto f(t)\,dt}} = \int_a^b f(t)\,dt$$



Lemma: If $\omega = d\phi$ for some $\phi \in \Lambda^0(M) = C^\infty(M)$ then

$$\int_C \omega = \phi(c(b)) - \phi(c(a))$$

(Proof as exercise)



Consider a smooth map

$$H: [0,1] \times [a,b] \mapsto M$$

$\uparrow$ parameter

with $H(s,a) = p \in M$ and $H(s,b) = q \in M \quad \forall s \in [0,1]$

We set $\left.\begin{array}{l} C_0 : [a,b] \mapsto M,\ C_0(t) = H(0,t) \\ C_1 : [a,b] \mapsto M,\ C_1(t) = H(1,t) \end{array}\right\}$ We say that C_0 and C_1 are HOMOTOPIC paths between p and q.

Thm. Let $\omega \in \Lambda^1(M)$ s.t. $d\omega = 0$ everywhere. Then

$$\int_{C_0} \omega = \int_{C_1} \omega$$

Remark: if $\omega = d\phi$ with $\phi \in C^\infty(M) = \Lambda^0(M)$, then $d\omega = d^2\phi = 0$ and the statement follows from the previous lemma.

- If M is of dim 2, the statement is "almost" a consequence of Stoke's Thm, but we don't have the smoothness of the boundary at p and q.
- More generously, see [Bo p. 271]

Remark: Smoothness can be relaxed in most of the statements.

IV Riemannian Manifolds

IV. 1 Definition and basic properties

Recall that if V is a real vector space of dimension n,
a POSITIVE DEFINATE BILINEAR FORM is a map $\phi : V \times V \mapsto \mathbb{R}$
which is linear in each argument
and s.t. $\phi(v,v) \geq 0 \ \forall v \in V$ and $\phi(v,v) = 0 \Leftrightarrow v = 0$.
ϕ is SYMMETRIC if $\phi(v_1, v_2) = \phi(v_2, v_1)$.

Def. A smooth manifold with a positive definite symmetric bilinear tensor field is called a RIEMANNIAN MANIFOLD.

$\Leftrightarrow \exists \phi \in \mathcal{J}^2(M)$:

$$\phi_p \in \Sigma^2(T_p(M)) \wedge [\forall X_p \in T_p(M) : \phi_p(X_p, X_p) \geq 0 \text{ with } =0 \Leftrightarrow X_p = 0]$$

We call ϕ a RIEMANNIAN METRIC.

Lemma: If $F : M \mapsto N$ is an IMMERTION ($:\Leftrightarrow \dim F(M) = \dim M$; see App. 2)
and if ϕ is ~~an~~ a Riemannian metric on N,
Then $F^*(\phi) \in \mathcal{J}^2(M)$ is a Riemannian metric on M.

Proof as exercise ; recall that

$$(F^*\phi)(X_p, Y_p) := \phi(\underbrace{F_*(X_p)}_{\in T_p(N)}, F_*(Y_p)) \qquad = 0 \text{ iff } Y_p = 0$$

Thm. Any smooth manifold can be endowed with a Riemannian metric.

"2 proofs" : ① Use a covering + local coordinate system + Lemma above
② Use Whitney Imbedding Thm + Lemma above

Remark: For a Riemannian manifold, $T_p(M)$ has an inner product provided by ϕ
$\Rightarrow$ We can now define orthonormal bases on $T_p(M)$ at every $p \in M$.

Thm. Let (M, ϕ) be a Riemannian manifold which is oriented.
fix ϕ from now on
Then $\exists !$ volume form Ω s.t. $\forall p \in M : \Omega_p(F_{1,p}, \cdots, F_{n,p}) = 1$ (*)
whenever $\{F_{1,p}, \cdots, F_{n,p}\}$ is an oriented orthonormal basis of $T_p(M)$.

Proof: Since $\dim(\Lambda^n(T_p(M))) = 1$, then Ω is uniquely defined by (*).
We have to show that it does not vanish.
Let (U, φ) be a local chart with $p \in U$;
Let $\{E_{1,p}, \cdots, E_{n,p}\}$ be the corresponding basis for $T_p(M)$. (Coordinate frame at p)

Set $g_{ij}(p) := \phi_p(E_{i,p}, E_{j,p})$.

Since $E_{i,p} = \sum_{k=1}^n \alpha_i^k F_{k,p}$ and since $\phi_p(F_{i,p}, F_{j,p}) = \delta_{ij}$

$\Rightarrow$ $g_{i,j}(p) = \phi_p(E_{i,p}, E_{j,p}) = \phi_p\left(\sum_{k=1}^n \alpha_i^k F_{k,p}, \sum_{l=1}^n \alpha_j^l F_{k,p}\right)$

$= \sum_{k=1}^n \alpha_i^k \alpha_j^k = ({}^T A A)_{ij}$ with $A_{ij} = \alpha_i^j$

$\Rightarrow$ $\det(g_{ij}(p))_{ij} = \det({}^T A A) = (\det(A))^2 > 0$

$\Rightarrow$ $\sqrt{\det(g_{ij}(p))_{ij}} > 0$

$\Rightarrow$ $\Omega_p(E_{1,p}, \cdots, E_{n,p}) \overset{\text{exercise}}{=} \det(A) \underbrace{\Omega_p(F_{1,p}, \cdots, F_{n,p})}_{=1 \text{ by def}} = \underbrace{\det(A)}_{>0 \text{ by choice of orientation of } (F_{1,p}, \cdots, F_{n,p})} = \sqrt{\det(g_{ij})} > 0$

Since p, (U, φ) are arbitrary, then Ω is a volume form.

Smoothness is automatic. □

Ω is called the NATURAL VOLUME ELEMENT on the oriented Riemannian manifold (M, ϕ).

We often see $\underbrace{\varphi^* \Omega}_{\in \Lambda^n(\mathbb{R}^n)} = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n$

$\quad\quad \sqrt{g} := \det(g_{ij} \circ \varphi^{-1})$

Remark: We can use Ω to define

$\int_M f := \int_M f \Omega \quad \forall f \in C^\infty(M)$

Let $c : [a,b] \mapsto M$ be a smooth curve on a Riemannian manifold (M, ϕ).

The tangent vector is

$c_*\left(\frac{d}{dt}\Big|_t\right) =: \dot{c}(t) \in T_{c(t)}(M)$

Def. The LENGTH of the curve is defined by

$L := \int_a^b [\phi_{c(t)}(\dot{c}(t), \dot{c}(t))]^{\frac{1}{2}} dt$

Exercise: This is indep. of the parametrization.

The ARC LENGTH is defined by $s : [a,b] \mapsto [0, L]$,

$s(t) := \int_a^t [\phi_{c(\tau)}(\dot{c}(\tau), \dot{c}(\tau))]^{\frac{1}{2}} d\tau$

We often write $\left[\left(\frac{ds}{dt}\right)^2 = \phi(\dot{c}, \dot{c})\right]$

Thm. [Bo. p. 189~191] A connected manifold is a metric space with the metric defined by $d(p,q) = \inf$ on the length of all paths (:= curves of C^1 or C^∞) between p and q.

The metric topology and the manifold topology coincide.

Reminder: a METRIC SPACE is a pair (M, d) with $d : M \times M \mapsto \mathbb{R}_+$ s.t.

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \Leftrightarrow x = y$

3) $d(x,y) = d(y,x)$

4) $d(x,z) \leq d(x,y) + d(y,z)$ ($\triangle$ inequality)

Def. Two R_0 (→ Riemannian) manifolds $(\mathcal{M}_1, \phi_1)$ and $(\mathcal{M}_2, \phi_2)$ are ISOMETRIC if

$\exists F : \mathcal{M}_1 \mapsto \mathcal{M}_2$ a diffeomorphism such that $F^* \phi_2 = \phi_1$

$\Rightarrow d_1(p,q) = d_2(F(p), F(q))$ (→ distance ←)

Remark: (Nash embedding thm) asserts that

any R_0 manifold can be isometrically embedded in $\mathbb{R}^d$, for $d \geq \frac{n(3n+11)}{2}$.

IV.2 Differentiation

Differentiation is important for the description of an evolution or a transport.

Example: In $\mathbb{R}^3$ for a fixed relavence system, $\dot{x}(t) = v$

One can also consider a moving reference system. (moving frame)

Example: We attach a reference system to a point moving in $\mathbb{R}^3$.

Let $s \mapsto c(s)$ be a curve in $\mathbb{R}^3$, with the arc length parameter.

Set $T(s) := c'(s)$, with the property $\|T(s)\| = 1$.

Then $\dot{T}(s) \equiv T'(s) \perp T(s)$ and set $T(s) = K(s) N(s)$ with $K(s) \geq 0$ and $\|N(s)\| =$

Consider $\{T(s), N(s), B(s)\}$ (↑ the curvature) (↑ suppose $K(s) \neq 0$)

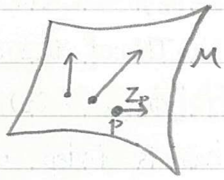
as a basis at $c(s)$ orthonormal

The equation of motion of this frame is given by the Sevret-Frenet formula

There are 2 parameters:

$K(s)$ = the curvature

$J(s)$ = the torsion



Example: Let $\mathcal{M}$ be a manifold of dim n in $\mathbb{R}^d$.

Let $Z \in \mathfrak{X}(\mathbb{R}^d)$ and let $p \in \mathcal{M}$. $\Rightarrow Z_p \in T_p(\mathbb{R}^d)$ but not always $Z_p \in T_p(\mathcal{M})$.

If $Z_p \in T_p(\mathcal{M})$ (tangent to $\mathcal{M}$ at p) for any $p \in \mathcal{M}$,

we say that Z is a tangent vector field.

Since $\mathbb{R}^d$ has a scalar product, it endows $\mathcal{M}$ with a scalar project

$\Rightarrow T_p(\mathbb{R}^d)$ has a scalar product, as well as $T_p(\mathcal{M})$.

$\Rightarrow T_p(\mathbb{R}^d) = T_p(\mathcal{M}) \oplus T_p(\mathcal{M})^\perp$

$\Rightarrow \exists \Pi_p$ and $\Pi_p^\perp$ two orthogonal projections on $T_p(\mathcal{M})$ and $T_p(\mathcal{M})^\perp$.

Def. Let $Y \in \mathfrak{X}(M) \subset \mathfrak{X}(\mathbb{R}^d)$ and consider $t \mapsto c(t) \in M \subset \mathbb{R}^d$ a curve on M.

Set $Y(t) := Y_{c(t)} \in T_{c(t)}(M)$ and consider

$$\frac{DY}{dt}(t) := \Pi_{c(t)}\left(\frac{d}{dt}Y(t)\right) \in T_{c(t)}(M)$$

↖ computation in $\mathbb{R}^d$

called the COVARIANT DERIVATIVE of Y along c.

Thus, Y and $\frac{DY}{dt}$ belong to $\mathfrak{X}(M)$ but the defination of $\frac{DY}{dt}$ uses $\mathbb{R}^d$.

Prop. 1) $\frac{D}{dt}(Y_1 + Y_2) = \frac{DY_1}{dt} + \frac{DY_2}{dt}$

2) $\frac{D}{dt}(fY) = f'Y + f\frac{DY}{dt}$ with any $f \in C^\infty(M)$

$\frac{d}{dt}\langle Y_1, Y_2 \rangle = \langle \frac{DY_1}{dt}, Y_2 \rangle + \langle Y_1, \frac{DY_2}{dt} \rangle$ with $\boxed{Y_1 = Y_1 \circ c,\ Y_2 = Y_2 \circ c}$

⚠ $\frac{DY}{dt} = 0 \not\Leftrightarrow \frac{dY}{dt} = 0$

— End of example 3

Remark

If we consider $X_p \in T_p(M)$ and

if we choose a curve $t \mapsto c(t) \in M$ with $c(t_0) = p$ and $\dot{c}(t_0) = X_p$

then $\frac{DY}{dt}(t_0)$ does not depend on $c(t)$ but only on X_p.

(proof as exercise)

It means we can define a map

$$\begin{array}{ccccc} T_p(M) & \times & \mathfrak{X}(M) & \longrightarrow & T_p(M) \\ \Psi & & \Psi & & \Psi \\ X_p & & Y & & \frac{DY}{dt}(t_0) =: \nabla_{X_p} Y \end{array}$$

↖ new notation

or more generally

$$\begin{array}{ccccc} \mathfrak{X}(M) & \times & \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ \Psi & & \Psi & & \Psi \\ X & & Y & & \nabla_X Y \end{array}$$

with $(\nabla_X Y)_p = \nabla_{X_p} Y$.

Def. An AFFINE CONNECTION on a smooth manifold M is a bilinear map

$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$

$X \quad Y \quad \mapsto \nabla(X,Y) \equiv \nabla_X Y$ satisfying

1) $\nabla_{fX} Y = f \nabla_X Y$ $C^\infty(M)$-linearity in the first variable

2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

$\Big\} \forall f \in C^\infty(M)$

Def. For any $X, Y \in \mathfrak{X}(M)$ we set

$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M)$

called the TORSION of the connection; and set

$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$

$[\tilde{R}(X,Y,Z) := R(X,Y)Z]$ $\quad \hookrightarrow : \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$; endomorphism 自同態

called the CURVATURE of the connection.

Lemma:

$T(X,Y)$ is $C^\infty(M)$-linear in both arguments;

$\tilde{R}(X,Y,Z)$ is " " in the 3 arguments.

[Exercise; see Tu (geometry) p. 44]

Def. On a R. manifold, a torsion free ($:\Leftrightarrow T(X,Y) = 0 \ \forall X,Y$) connection satisfying

$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \in C^\infty(M) \quad \forall X,Y,Z \in \mathfrak{X}(M)$

is called a RIEMANNIAN CONNECTION or LEVI CIVITA CONNECTION.

(compability condition between the Riemannian metric ϕ and the connection ∇)

$\langle X,Y\rangle : M \mapsto \mathbb{R}$; $\langle X,Y\rangle_p := \phi_p(X_p, Y_p) \in \mathbb{R}$

Thm. On a Riemannian manifold $\exists!$ Riemannian connection.

This connection satisfies

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle$$

(Koszul formula)

Lemma: Let M be a smooth manifold of dim n (Riemannian not assumed) and let ∇ be an affine connection on M.

Let (U,φ) be a chart and consider a coordinate frame on the tangent spaces.

Then ∇ is defined by n^3 functions

$\Gamma_{i,j}^k : U \longmapsto \mathbb{R}$ for $i,j,k \in \{1,\cdots,n\}$ called the CHRISTOFFEL SYMBOLS.

Proof: Let $X, Y \in \mathfrak{X}(M)$, and $\forall p \in U$:

$$X_p = \sum_{i=1}^n \underbrace{b^i(p)}_{\in\mathbb{R}} E_{i,p} \; ; \; Y_p = \sum_{i=1}^n \underbrace{a^i(p)}_{\in\mathbb{R}} E_{i,p} .$$

$\hookrightarrow$ bases of $T_p(M)$

Set

$$\overbrace{\nabla_{E_{i,p}} E_{j,p}}^{\in T_p(M)} =: \sum_{k=1}^n \Gamma_{i,j}^k(p) E_{k,p}$$

Then

$$\nabla_X Y = \nabla_{\sum_i b^i E_i} \sum_j a^j E_j \overset{\text{Linearity and 1)}}{=} \sum_{i,j} b^i \nabla_{E_i}(a^j E_j) \overset{2)}{=} \sum_{i,j} b^i \left\{ (E_i a^j) E_j + a^j \sum_k \Gamma_{i,j}^k E_k \right\}$$

$$= \sum_k \left(X a^k + \sum_{i,j} a^j b^i \Gamma_{i,j}^k \right) E_k \qquad (*)$$

$\Rightarrow \nabla$ can be expressed by $\Gamma_{i,j}^k$.

Conversely, if we start with $(*)$, it defines an affine connection. (5-min exercise) $\square$

Remark: 1) with these notations

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \sum_{ijk} (\Gamma_{i,j}^k - \Gamma_{j,i}^k) a^i b^j E_k .$$

Thus $\forall X, Y \in \mathfrak{X}(M)$: $T(X,Y) = 0 \Leftrightarrow \forall i,j,k : \Gamma_{i,j}^k = \Gamma_{j,i}^k$

2) If (M,ϕ) is Riemannian, recall that

$g_{ij}(p) = \phi_p(E_{i,p}, E_{j,p}) \; \forall i,j \in \{1,\cdots,n\}$ and then

$$\Gamma_{i,j}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$\llcorner$ inverse matrix of (g_{ij})

(proof as exercise)

A new look at the covariant derivative:

Let $c: I \ni t \mapsto c(t) \in M$ be a smooth curve on M, and let $Y \in \mathfrak{X}(M)$.

Let (U, φ) be a local chart, and for $p \in U$

$$Y_p = \sum_{k=1}^{n} b^k(p) E_{k,p}.$$

Then we set

$$\frac{DY}{dt}(t) := \left[\nabla_{\dot{c}(t)} Y\right]_{c(t)} = \sum_{k=1}^{n}\left(\dot{c}(t)\, b^k(c(t)) + \sum_{i,j} \Gamma_{i,j}^{k}(c(t))\, b^i(c(t))\, \dot{c}^j(t)\right) E_{k,c(t)}$$

$\dot{c}(t) = \sum_k \dot{c}^k(t)\, E_{k,c(t)}$

Observe that

$$\dot{c}(t)\, b^k = c_*\left(\frac{d}{dt}\right)\Big|_t b^k = \frac{d}{dt}(b^k \circ c)\Big|_t = \frac{d}{dt} b^k(c(t)).$$

$$= \sum_{k=1}^{n}\left(\frac{db^k(c(t))}{dt} + \sum_{i,j}\Gamma_{i,j}^{k}(c(t))\, b^i(c(t))\, \dot{c}^j(t)\right) E_{k,c(t)} \qquad \triangle$$

Remark: only the values of Y on the curve are taken into account.

Def. Let $c: I \mapsto M$ be a curve on M, and ∇ an affine connection on M.

A vector field $Y: I \ni t \mapsto Y(t) \in T_{c(t)}(M)$ is PARALLEL along c if

$$\frac{DY}{dt}(t) = 0 \quad \forall t \in I.$$

Since $\triangle$ is a group of first-order differential equations we have:

Prop. 1) Given a smooth curve $c: (-\varepsilon, \varepsilon) \ni t \mapsto c(t) \in M$ and given $Y_{c(0)} \in T_{c(0)}(M)$ then

$\exists !\, Y: (-\varepsilon, \varepsilon) \ni t \mapsto Y(t) \in T_{c(t)}(M)$ parallel to c.

2) If (M, ϕ) is a Riemannian manifold and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_{c(0)}(M)$ then $\exists !$ orthonormal frame at $c(t)$ which is parallel to c.

More generally on Riemannian manifolds,

parallel transport preserves the length and the inner product.

IV.3 Geodesics

Let $c : I \mapsto M$ be a curve on M and ∇ be an affine connection.

($\xleftrightarrow{\text{locally}}$ a set of Christoffel's symbols)

Def. c is GEODESIC (with respect to ∇) if $\dot{c}$ is parallel along c, which means

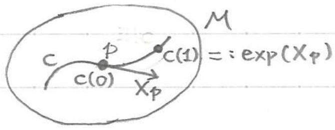
$\frac{D\dot{c}}{dt}(t) = 0 \quad \forall t \in I$

$\Leftrightarrow \ddot{c}^k + \sum_{i,j} \Gamma_{i,j}^k \dot{c}^i \dot{c}^j = 0 \quad \forall k = 1, \cdots, n$ (geodesic equation)

Remark: since the geodesic equation is a second-order differential equation,

given $p \in M$ and $X_p \in T_p(M)$,

$\exists ! \, c : (-\varepsilon, \varepsilon) \mapsto M$ geodesic s.t. $c(0) = p$ and $\dot{c}(0) = X_p$.



Note that $\forall a > 0$, if we set $c_a : (-\frac{\varepsilon}{a}, \frac{\varepsilon}{a}) \longmapsto M$ then

$c_a(0) = p$, $\dot{c}_a(0) = aX_p$ and c_a is again geodesic. Then

Def. $\exp(X_p) := c(1)$ whenever defined.

$:\Leftrightarrow \forall u \in U \; \forall a \in [0,1] : au \in U$

Prop. $\forall p \in M \; \exists$ open set $U \subset T_p(M)$ star-shaped with $0 \in U$ s.t.

$\exp : U \longmapsto M$ is a diffeomorphism onto $V \subset M$ with $p \in V$.

The proof involves some uniformity.

$\operatorname{Exp}(U)$ is called a NORMAL NEIGHBORHOOD of p on M,

and exp is called the EXPONENTIAL MAP.

Remark: If (M, ϕ) is a Riemannian manifold,

and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_p(M)$, then

$X_p = \sum_{j=1}^n \underbrace{x^j}_{\in \mathbb{R}} F_j$ (unique decomposition)

Then

$\varphi : \exp(U) \ni \exp(\underset{\in U}{X_p}) = \exp(\sum_{j=1}^n x^j F_j) \longmapsto (x^1, \cdots, x^n) \in \mathbb{R}^n$

and $(\exp(U), \varphi)$ is a coordinate system around p, called

the NORMAL COORDINATE SYSTEM around p.

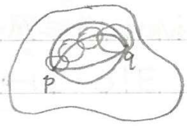
(with special properties)

In summary, for a given $p \in M$ $\exists\, v \in \mathcal{V}_p$ (neighborhood) s.t.
any $q \in v$ can be joined to p by a unique geodesic.

With more work one gets

Thm. If c is a piecewise differential path between p and q with
$\underset{\text{length of } c}{L(c)} = \underset{\text{distance between } p \text{ and } q \text{ on the Riemannian manifold } M}{d(p,q)}$ (for defs of L and d, see p.27 in IV.1)
Then c is a geodesic when parametrized by its arc length.

Idea of proof: do it locally.



⚠ The distance is not always realized by a path.
Example: $\mathbb{R}^2 \setminus \{0\}$, $p = (0,1)$, $q = (0,-1)$

Thm. (Hopf and Rinow)
Let (M, ϕ) with Levi-Civita connection ∇.
Are equivalent:
1) exp is defined every on $T_p(M)$ $\forall p \in M$;
2) (M, d) is a COMPLETE metric space (:$\Leftrightarrow$ with "no holes")
↳ every Cauchy sequence $\subset M$ has a limit $\in M$.
3) Every geodesic $c: I \mapsto M$ can be extended on $\mathbb{R}$.

Def. (M, ϕ) is GEODESICALLY COMPLETE
if one ($\Rightarrow$ all) of these conditions is satisfied.

Lemma. If (M, ϕ) is COMPACT then it is geodesically complete.

Proof: Based on the fact that any compact metric space is complete. □

V Curvature

V.1 Several curvatures

Framework: M a smooth manifold with ∇ a connection.

If (M,ϕ) is Riemannian, then ∇ is the Levi Civita connection.

Recall that the curvature R is defined on $X, Y \in \mathfrak{X}(M)$ by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$$

$$R(X,Y) : \mathfrak{X}(M) \ni Z \longmapsto R(X,Y)Z \in \mathfrak{X}(M)$$

Lemma: If ∇ is torsion free then

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

[Bianchi identity ; GN p. 125]

True also for Levi Civita connection.

In local coordinates [= with a chart (U,ϕ) and the coordinate frame $\{E_{j,p}\}_j$]

$$R(E_i, E_j)E_k = \sum_l R_{ijk}{}^l E_l$$

with $R_{ijk}{}^l = \frac{\partial}{\partial x^i}\Gamma^l_{jk} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} + \sum_m \Gamma^m_{jk}\Gamma^l_{im} - \sum_m \Gamma^m_{ik}\Gamma^l_{jm}$

↑ components of R in a basis

⚠ It can be slightly different depending on the authors

For (M,ϕ), let us also set

$$\underbrace{\phi(\underbrace{\underbrace{R(X,Y)}_{\in \mathrm{End}} Z}_{\in \mathfrak{X}(M)}, W)}_{\in C^\infty(M)} =: R(X,Y,Z,W) \in C^\infty(M)$$

$\in \mathcal{T}^4(M)$; called the RIEMANNIAN CURVATURE TENSOR

and in local coordinates

$$R_{ijkl} := \phi(R(E_i,E_j)E_k, E_l) = \sum_m R_{ijk}{}^m g_{ml}$$

Lemma: For (M,ϕ)

1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

3) $R(X,Y,Z,W) = R(Z,W,X,Y)$

[Exercise ; see Boo p. 383 and GN p. 126]

For any $p \in M$, let us denote by Π a PLANE SECTION in $T_p(M)$,
it means Π is a 2D subspace of $T_p(M)$.

Let X_p, Y_p be 2 elements in $T_p(M)$ generating a basis of Π s.t. (X_p, Y_p) is an orthonormal basis of Π.

Def. The SECTIONAL CURVATURE $K(\Pi)_p$ of the section Π with basis (X_p, Y_p) is

$$K(\Pi)_p := -R(X_p, Y_p, X_p, Y_p) = -\phi_p(R(X_p, Y_p)X_p, Y_p)$$

Exercise: $K(\Pi)_p$ depends only on the plane Π and not on the choice of a basis.

Thm. For (M, ϕ) with $\dim(M) \geq 3$:
the Riemannian curvature tensor at p is uniquely determined by the values of all sectional curvatures at p.
[Exercise; see Boo p.385 and GN p.127]

Def. 1) (M, ϕ) is ISOTROPIC at p if

$$K(\Pi)_p = K_p = \text{constant} \quad \forall \Pi;$$

2) (M, ϕ) is ISOTROPIC if it is isotropic at any $p \in M$;

3) If K_p is constant on any $p \in M$, we say that M has CONSTANT CURVATURE.

Report: manifolds with constant curvature are classified.

Remark: If $\dim(M) = 2$ then M is isotropic, and $K_p \equiv K(p)$ is called the GAUSS CURVATURE.

Report: on Gauss curvature or on Gauss-Bonnet Thm.

Lemma: If M is isotropic then locally

$$R_{ijkl}(p) = -K_p(g_{ik}g_{jl} - g_{il}g_{jk})(p)$$

> The Riemannian curvature tensor, the sectional curvature, the Ricci curvature and the scalar curvature give some information on the local structure of the manifold.

Def. The RICCI CURVATURE tensor field $Ric \equiv R \equiv S \in \mathcal{T}^2(M)$ is defined on $X, Y \in \mathfrak{X}(M)$ by

$$S_p(X_p, Y_p) := \sum_j R(F_{j,p}, X, Y, F_{j,p}) \text{ with } \{F_{j,p}\}_j \text{ an orthonormal basis of } T_p(M)$$

Remark: 1) It is independent of the choice of a basis of $T_p(M)$.
Locally, $S_{ij} = S(E_i, E_j) = \sum_k R_{kij}{}^k$

2) The above operation is called a CONTRACTION of a tensor.
If we contract the Ricci curvature we get the SCALAR CURVATURE given

$$S(p) = \sum_j S(F_{j,p}, F_{j,p}) = \sum_{i,j} S_{i,j}\, g^{ij}(p)$$

V.2 Equation of structure

Recall that a connection ∇ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$$
$$X \quad Y \longmapsto \nabla_X Y$$

which is bilinear and satisfies

1) $\nabla_{fX} Y = f \nabla_X Y$
2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

∇ is torsion free if $\nabla_X Y - \nabla_Y X - [X,Y] \, (=: T(X,Y)) = 0$ and
∇ is compatible with the metric for (M,ϕ) if
$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ making the parallel transport of 2 orthogonal vectors still orthogonal
} ⊛

Let U be an open subset of M and
let $\{F_j\}_{j=1}^n$ be a C^∞-field of frames on U $\{F_{j,p}\}$ is a basis of $T_p(M)$ $\forall p \in U$; not necessarily orthonormal nor generated by a chart
e.g. the coordinate frames given by a chart (U, φ)
Let $\{\theta^j\}_{j=1}^n$ be a dual coframe, it means $\{\theta^j\}$ is a C^∞-field of frames on $T(M)^*$
and $\{\theta_p^j\}$ is a basis of $T_p(M)^*$ with $\theta_p^j(F_{k,p}) = \delta_{jk}$ Cronecker delta

Recall that ∇ is uniquely determined by $\{\Gamma_{ij}{}^k\}$ defined by

$$\nabla_{F_i} F_j = \sum_k \Gamma_{ij}{}^k F_k$$

Def. $\theta_j{}^k := \sum_l \Gamma_{lj}{}^k \theta^l \in \mathcal{T}^1(M)$ one form

$\{\theta_j{}^k\}$ are called CONNECTION FORMS. Clearly

$\Rightarrow$ $\theta_j{}^k(F_i) = \Gamma_{ij}{}^k$, and

if $T(M) \ni X = \sum_l b^l F_l$ then

$$\nabla_X F_j = \nabla_{\sum_l b^l F_l} F_j \xlongequal[\text{and } 1)]{\text{linear}} \sum_l \underbrace{b^l}_{\in C^\infty(U)} \nabla_{F_l} F_j \overset{\Gamma}{=} \sum_l b^l \sum_k \Gamma_{lj}{}^k F_k$$
$$\xlongequal[\text{of } \theta]{\text{Def}} \sum_k \sum_l b^l \theta_j{}^k (F_l) F_k \xlongequal[\text{of 1 forms}]{\text{linearity}} \sum_k \theta_j{}^k(X) F_k$$

Thus, $\theta_j{}^k(X)$ are the components of $\nabla_X F_j$ with respect to $\{F_k\}$.

For a R_0 manifold (M, ϕ) and for the Levi Civita connection ∇,
the n^2 connection form are not indep because of the relations ⊛.

Thm. (Structure Thm of Cartan) [GN p. 133]

Let (R, ϕ) be a R_0 manifold, ∇ the Levi Civita connection, $U, \{E_i\}, \{\theta^j\}$ above.

Then the connection forms $\{\theta_j{}^k\}$ are the unique solution of the equations:

1) $d\theta^i = \sum_j \theta^j \wedge \theta^i$ wedge product; equality between 2-forms

2) $dg_{ij} = \sum_k (g_{kj}\theta_i{}^k + g_{ki}\theta_j{}^k)$ equality between 1-forms

$\hookrightarrow \in C^\infty(M)$ $\hookrightarrow$ one-forms

Remark: If $\{F_j\}$ is an orthonormal basis,

$g_{ij} := \phi(F_i, F_j) = \delta_{ij}$ and 2) becomes

2) $0 = \theta_i{}^j + \theta_j{}^i$

Similarly, one can introduce the CURVATURE FORM for $X, Y \in \mathfrak{X}(M)$:

$$\Omega_k^l(X,Y) := \underset{\in \mathfrak{X}(M)^*}{\theta^l}(\underbrace{R(X,Y)F_k}_{\in \mathfrak{X}(M)}) \in C^\infty(M) \Rightarrow \Omega_k^l \in \mathcal{J}^2(U) \subset \mathcal{J}^2(M)$$

which gives

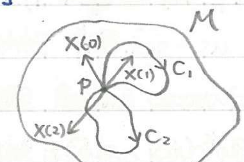
$$R(X,Y)F_k = \sum_j \Omega_k^j(X,Y) F_j$$

Thus $\Omega_k^l(X,Y)$ are components of $R(X,Y)F_k$ on the basis $\{F_j\}$

Remark: $\Omega_k^l \in \mathcal{J}^2(U) \subset \mathcal{J}^2(M)$ and one has

Thm. (Structure Thm of Cartan) [GN p. 135 ; Bo p. 391]

$$\Omega_i{}^j = d\theta_i{}^j - \sum_k \theta_i{}^k \wedge \theta_j{}^k$$ equality between 2-forms



V.3 Holonomy for a connected Riemannian manifold

⚠ Exists in a more general context of vector bundles or principal bundles.

Let $c : [0,1] \ni t \longmapsto c(t) \in M$ a smooth curve on (M, ϕ)

with $c(0) = c(1) = p$.

Let $X_p \in T_p(M)$ and let $X(t)$ be the parallel transport of X_p along c

with $X(0) = X_p$. Let

$P_c : T_p(M) \ni X_p = X_0 \longmapsto X_1 \in T_p(M)$, and clearly

$P_{c_2 \circ c_1} = P_{c_2} P_{c_1}$; $P_{c^{-1}} = P_c^{-1}$ leading to the fact that it composes a group.

composition of paths; backward

In addition $P_c \in GL(T_p(M))$ (invertible matrices) because the parallel transport is a solution

of a homogenous equation. $\Rightarrow$ linear in the initial condition

In fact $P_c \in O(T_p(M))$ orthogonal matrices on $T_p(M)$

because the parallel transport preserves norms and scalar products.

Remark: Instead of smooth curve, we can consider C^1-piecewise curves.

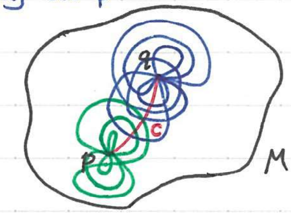
We have obtained that

$\{P_c\}_* \subset O(T_p(M))$ is a group e is the zero path

called the HOLONOMY GROUP at p and denoted $Hol(p)$.

$* :=$ "c any C^1-piecewise curve starting and ending at p"

If p and q are 2 points on M then



$Hol(p)$ is isomorphic to $Hol(q)$ since

$Hol(p) = P_c^{-1}\, Hol(q)\, P_c$

for some fixed path c between q and p.

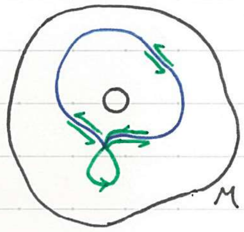
Def. We set $Hol(M) = Hol(p) \subset O(\ )$ for a fixed $p \in M$, and call it the HOLONOMY GROUP of M.

We also set $Hol^\circ(M)$ constructed only with C^1-piecewise path which can be deformed to the zero path.

Remarks:

1) These groups are representations of the group of paths on M.

2) $Hol^\circ(M)$ is a normal subgroup of $Hol(M)$.



Lemma

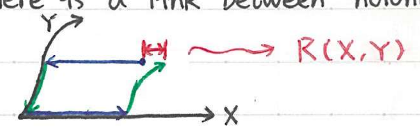
M is orientable iff $Hol(M) \subset SO(M)$ determinant 1 and never -1

Thm (deep notation)

$Hol^\circ(M)$ is compact (it is a closed set in $O(n)$)

Remark [see App. 12]

There is a link between holonomy and the curvature tensor $R(X,Y)$



They are not so many holonomy groups!

Thm. Let (M, ϕ) and suppose that $Hol^{\circ}(M) \subset O(n)$ is irreducible (no invariant subspaces of $\mathbb{R}^n$).

(For a manifold made by product of two manifolds, this is not satisfied)

Suppose that M is not LOCALLY SYMMETRIC.

Then $Hol^{\circ}(M)$ is one of the following groups:

1) $SO(n)$ generic case
2) if $n=2m$, $Hol(M) = U(m)$
3) if $n=2m$, $Hol(M) = SU(m)$
4) $n=4m$, $Hol(M) = Sp(1)Sp(m)$
5) if $n=4m$, $Hol(M) = Sp(m)$
6) $n=16$, " $= Spin(9)$
7) $n=8$, " $= Spin(7)$
8) $n=7$, " $= G_2$

Later it is found that (6) does not actually appear in any manifolds.

4) ~ 8) are in qualernions (extension of $\mathbb{C}$, 2D → 4D)

Def. M is LOCALLY SYMMETRIC if for any $p \in M$:

the geodesic symmetry S_p is an isometry (preserves the distance)

namely, we have $S_p(c(t)) = c(-t)$ for any geodesic c with $c(0) = p$

Example: $\mathbb{R}^n$ is locally symmetric (easy to show). And

$Hol(\mathbb{R}^n) = Hol^{\circ}(\mathbb{R}^n) = \{e\}$ which is not one of the 8 kinds of groups above.

VI General relativity

Def. a PSEUDO-RIEMANNIAN MANIFOLD is a pair (M, ϕ) with

M a smooth manifold and $\phi \in \mathcal{T}^2(M)$, symmetric and non-degenerate.

$\phi(X,Y) = \phi(Y,X)$; $\phi(X,Y) = 0 \ \forall Y \in \mathfrak{X}(M) \Rightarrow X = 0$

⚠ No 「positive definite」 required !

a LORENTZIAN MANIFOLD is a pseudo-Riemannian manifold

with $(g_{ij}) = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in suitable coordinates (locally). → signature (trace) $= n-2$

Facts for pseudo-Riemannian manifolds

1) unicity of Levi Civita connection when the 2 conditions are imposed.

2) Koszul formula still holds.

3) Hopf-Rinow thm + geodesically complete are no more valid.

⇒ We don't have a metric space anymore.

4) Cartan structure thm are still valid.

Recall that the length of a vector is not changed under parallel transport along a curve.

Geodesics c satisfy that $\dot{c}$ is parallel transported along c.

$\Rightarrow \phi(\dot{c}, \dot{c}) = cst$

Def. A geodesic c on a pseudo-Riemannian manifold (M, ϕ) is

TIMELIKE, NULL, or SPACELIKE if

$\phi(\dot{c},\dot{c}) < 0$, $\phi(\dot{c},\dot{c}) = 0$ or $\phi(\dot{c},\dot{c}) > 0$

<0 and $=0$ are allowed by PSEUDO METRIC ϕ

Remark: these expressions come from special relativity with $M = \mathbb{R}^4$ and

$(g_{\nu\mu}) = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}$ a special case of a Lorentzian manifold.

$\mu, \nu = 0,1,2,3$

For a Lorentzian manifold (M, ϕ) of dim 4, with the Levi Civita connection,

the Einstein field equation reads

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{for } \mu, \nu = 0,1,2,3 \qquad (*)$$

$R_{\mu\nu}$: Ricci curvature; R: scalar curvature; Λ: cosmological constant >0; $T_{\mu\nu}$: stress-energy or energy-momentum tensor

(about geometry) $G_{\mu\nu}$ Einstein tensor

$T_{\mu\nu}$ contains the physics (energy + matter)

G = gravitation constant

c = speed of light

⚠ Not so much freedom for writing a meaningful equations.

This is a system of 10 equations because of symmetry between μ and ν.

In addition the thms of structure reduces the number of indep eq.

Remark: These equations define the pseudo metric tensor $g_{\mu\nu}$.

Indeed, $R_{\mu\nu\gamma}{}^{\delta}$ and $R_{\mu\nu}$ can be expressed in terms of $\Gamma_{\mu\nu}{}^{\delta}$ and its derivativ

And $\Gamma_{\mu\nu}{}^{\delta}$ can be expressed in terms of $g_{\mu\nu}$ and its derivatives.

$\Rightarrow$ ⊛ is a system of non linear partial differential equations for $g_{\mu\nu}$.

Schwarzschild solution

Assumptions: ∘ $T_{\mu\nu} = 0$

∘ $g_{\mu\nu}$ is time independent (static solution)

∘ spherically symmetric in space ($\equiv$ in the indices 1, 2, 3)

∘ $M = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^2$ $\mathbb{R}$ for t and $\mathbb{R}_+ \times \mathbb{S}^2$ is $\mathbb{R}^3$ in spherical coordinates

Suppose that

$$g = -A^2(r)\, dt \otimes dt + B^2(r)\, dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta)\, d\varphi \otimes d\varphi \in \mathcal{T}^2(M)$$

with $A, B : \mathbb{R}_+ \mapsto \mathbb{R}$ unknown,

$(dt, dr, d\theta, d\varphi) \in \mathcal{T}^1(M)$ generate a basis of $T^*(M)$.

$\{(\frac{\partial}{\partial x^i})_p\}_{i=1}^{n}$ is a basis of $T_p(M)$, and $\{dx_p^i\}_{i=1}^{n}$ is a basis of $T_p^*(M)$.

$\Rightarrow \{dx^i \otimes dx^j\}_{i,j}$ is a basis of $\mathcal{T}^2(M)$.

Question: can we find A, B such that ⊛ is satisfied (with $T_{\mu\nu} = 0$)?

Two approaches:

1) Express $\Gamma_{\mu\nu}{}^{\delta} \rightsquigarrow R_{\mu\nu\delta}{}^{\gamma}$ and then $R_{\mu\nu}$ and R in terms of $g_{\mu\nu}$, and solve ⊛

2) Set $\left.\begin{array}{ll} \theta^0 := A(r)\, dt & \theta^2 := r\, d\theta \\ \theta^1 := B(r)\, dr & \theta^3 := r \sin(\theta)\, d\varphi \end{array}\right\} \in \mathcal{T}^1(M)$ and observe that

$g = \sum_{\mu,\nu} \eta_{\mu,\nu}\, \theta^{\mu} \otimes \theta^{\nu}$ and $\eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ and that

$\{\theta^0, \theta^1, \theta^2, \theta^3\}$ is an orthonormal coframe a basis of $T^*(M)$

- Define $\theta_{\mu}{}^{\nu}$ and $\Omega_{\mu}{}^{\nu}$ (connection and curvature tensors) and write the structure relations of Cartan.
- One obtains some differential equations for A and B, which can be solved.
- $A(r) = (1 - \frac{2m}{r})^{\frac{1}{2}}$ and $B(r) = (1 - \frac{2m}{r})^{-\frac{1}{2}}$ with $m \in \mathbb{R}$ an integration const

Conclusion

Textbooks on general relativity are now accessible (but still the theory is complicated).