Chapter 5

Traces of pseudo-differential operators

In this chapter we look at applications of the Dixmier traces in the context of pseudo-differential operators. Again, our main source of inspirations will be [SU] and [LSZ] but also the book [RT].

5.1 Pseudo-differential operators on $\mathbb{R}^d$

In this first section we recall a few classical definitions and results related to pseudo-differential operators. Our setting is clearly not the most general one and many extensions are possible. We shall use the usual multi-index notation, namely $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j \in \{0, 1, 2, \dots\}$. For shortness, we shall write $\alpha \in \mathbb{N}_0^d$ with $\mathbb{N}_0 = \{0, 1, 2 \dots\}$ (recall that the convention of Chapter 2 is that $\mathbb{N} = \{1, 2, 3 \dots\}$). We shall also use $|\alpha| := \sum_{j=1}^d \alpha_j$ and $\alpha! = \alpha_1! \dots \alpha_d!$. The other standard notations which are going to be used are $\nabla := (\partial_1, \dots, \partial_d)$ with $\partial_j := \partial_{x_j}$, $-\Delta := -\sum_{j=1}^d \partial_j^2$ which is a positive operator, and $\langle x \rangle := \big(1 + \sum_{j=1}^d x_j^2\big)^{1/2}$ for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. In the Hilbert space $L^2(\mathbb{R}^d)$, we shall also use the notation $X = (X_1, \dots, X_d)$ with X_j the self-adjoint operator of multiplication by the variable x_j, and $D = (D_1, \dots, D_d)$ with D_j the self-adjoint operator corresponding to the operator $-i\partial_j$.

Definition 5.1.1. *1) For any $m \in \mathbb{R}$, $\rho \in [0, 1]$, and $\delta \in [0, 1)$, a function $a \in C^\infty(\mathbb{R}^{3d})$ is called* an amplitude of order m *if it satisfies*

$$\big|[\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha a](x, y, \xi)\big| \le C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta(|\beta|+|\gamma|)} \tag{5.1}$$

for any $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ and all $x, y, \xi \in \mathbb{R}^d$. The set of all amplitudes satisfying (5.1) is denoted by $\mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$. Note that the constants $C_{\alpha,\beta,\gamma}$ depend also on the function a but not on x, y and ξ.

2) For any amplitude $a \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$ the corresponding amplitude operator of order *m is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by*

$$[a(X,Y,D)f](x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i(x-y)\cdot\xi}\, a(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{5.2}$$

The corresponding set of operators is denoted by $\mathfrak{Op}\big(\mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)\big)$.

Remark 5.1.2. *It can be shown that the operator defined in* (5.2) *is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$, when this space is endowed with its usual Fréchet topology. Note also that the presence of 2π is irrelevant and mainly depends on several conventions about the Fourier transform. In fact, the above operator should be denoted by $a\big(X,Y,\frac{1}{2\pi}D\big)$ according to the convention taken in [RT]. In these notes, we mainly follow the convention of [LSZ] but warn the reader(s) that some constants have not been double-checked. It is possible that sometimes the equality $2\pi = 1$ holds !*

The main interest for dealing with amplitudes is that the expression for the adjoint operator is simple. Indeed, by using the usual scalar product $\langle\cdot,\cdot\rangle$ of $L^2(\mathbb{R}^d)$ one defines the adjoint of $a(X,Y,D)$ by the relation

$$\big\langle a(X,Y,D)^* f, g\big\rangle = \big\langle f, a(X,Y,D)g\big\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{5.3}$$

It then follows that $a(X,Y,D)^*$ is also an amplitude operator of order m with symbol a^* given by

$$a^*(x,y,\xi) = \overline{a(y,x,\xi)}. \tag{5.4}$$

The adjoint operator plays an important role for the extension by duality to operators acting on tempered distributions. Indeed, if $\mathcal{S}'(\mathbb{R}^d)$ denotes the set of tempered distributions on $\mathbb{R}^d$ and if $\Psi \in \mathcal{S}'(\mathbb{R}^d)$, then we can define $a(X,Y,D) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$[a(X,Y,D)\Psi](f) := \Psi\big(a(X,Y,D)^* f\big) \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Let us now explain the link between amplitudes and more usual symbols of the class $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$. For that purpose, we define the Fourier transform for any $f \in L^1(\mathbb{R}^d)$ by

$$[\mathscr{F} f](\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x. \tag{5.5}$$

The inverse Fourier transform is then provided by $[\mathscr{F}^{-1} f](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} f(\xi)\,\mathrm{d}\xi$

Definition 5.1.3. *1) For any $m \in \mathbb{R}$, $\rho \in [0,1]$, and $\delta \in [0,1)$, a function $a \in C^\infty(\mathbb{R}^{2d})$ is called* a symbol of order m *if it satisfies*

$$\big|[\partial_x^\beta \partial_\xi^\alpha a](x,\xi)\big| \le C_{\alpha,\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{5.6}$$

for any $\alpha,\beta \in \mathbb{N}_0^d$ and all $x,\xi \in \mathbb{R}^d$. The set of all symbols satisfying (5.6) *is denoted by $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$. Note that the constants $C_{\alpha,\beta}$ depend also on the function a but not on x and ξ.*

2) For any symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ the corresponding pseudo-differential operator of order *m is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by*

$$[a(X,D)f](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, a(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi \tag{5.7}$$

and $a(X,D)f \in \mathcal{S}(\mathbb{R}^d)$. The corresponding set of operators is denoted by $\mathfrak{Op}\big(\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)\big)$.

3) We set $\mathcal{S}^{-\infty}(\mathbb{R}^d) := \bigcap_{m\in\mathbb{R}} \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ (which is independent of ρ and δ) and call a smoothing operator *a pseudo-differential operator $a(X,D)$ with $a \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$.*

Before going on, let us look at the Fourier transform of a symbol. More precisely, observe that

$$\begin{aligned}
[a(X,D)f](x) &= \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i (x-y)\cdot\xi}\, a(x,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} k(x,y) f(y)\,\mathrm{d}y
\end{aligned}$$

with

$$k(x,y) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i (x-y)\cdot\xi}\, a(x,\xi)\,\mathrm{d}\xi. \tag{5.8}$$

Remark 5.1.4. *The integral in* (5.8) *does not converge absolutely in general. This integral is usually understood as an oscillatory integral. We shall not develop this any further in these notes. However, if the function $(x,\xi) \mapsto a(x,\xi)$ decreases fast enough in ξ, then the integral can be understood in the usual sense.*

The map $(x,y) \mapsto k(x,y)$ is sometimes called the kernel (or Schwartz kernel[1]) of the operator $a(X,D)$. One of its important property is given in the following statement, see [RT, Thm. 2.3.1].

Theorem 5.1.5. *For any $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$, the corresponding kernel $k(x,y)$ defined by* (5.8) *satisfies*

$$\big|\big[\partial^\beta_{x,y} K\big](x,y)\big| \leq C_{N,\beta}|x-y|^{-N}$$

for any $N > m+n+|\beta|$ and $x \neq y$. In other words, for $x \neq y$ the map $(x,y) \mapsto k(x,y)$ is a smooth function which decays at infinity, together with all its derivatives, faster that any power of $|x-y|^{-1}$.

Additional results for pseudo-differential operators are summarized in the following statements, see [RT, Thm. 2.4.2 and Thm. 2.5.1].

[1]In the context of operators K defined by $[Kf](x) = \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y$ for $f \in C_c(\mathbb{R}^d)$ with kernel $k \in L^1_{loc}(\mathbb{R}^{2d})$ one can not prevent from recalling the important Schur's lemma which says that if the two conditions $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|k(x,y)|\mathrm{d}y < \infty$ and $\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|k(x,y)|\mathrm{d}x < \infty$ hold, then K defines a bounded operator on $L^2(\mathbb{R}^d)$.

Theorem 5.1.6 (L^2-boundedness)**.** *Let $a \in \mathcal{S}^0_{1,0}(\mathbb{R}^d)$, then the operator $a(X,D)$ extends continuously to an element of $\mathscr{B}\big(L^2(\mathbb{R}^d)\big)$.*

Theorem 5.1.7 (Composition formula)**.** *Let $a \in \mathcal{S}^{m_a}_{1,0}(\mathbb{R}^d)$ and $b \in \mathcal{S}^{m_b}_{1,0}(\mathbb{R}^d)$, then there exists $c \in \mathcal{S}^{m_a+m_b}_{1,0}(\mathbb{R}^d)$ such that the equality*

$$a(X,D)\, b(X,D) = c(X,D)$$

holds, where the product of operators in considered on the l.h.s. Moreover, one has the asymptotic formula

$$c \sim \sum_{\alpha\in\mathbb{N}_0^d} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha a)(\partial_x^\alpha b) \tag{5.9}$$

where the meaning of (5.9) *is*

$$c - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha a)(\partial_x^\alpha b) \ \in \ \mathcal{S}^{m_a+m_b-N}(\mathbb{R}^d) \tag{5.10}$$

for any $N > 0$.

Exercise 5.1.8. *Provide a proof of the previous statements, and check what happens for symbols in $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta < \rho \leq 1$. In particular for Theorem 5.1.7 show that if $a \in \mathcal{S}^{m_a}_{\rho,\delta}(\mathbb{R}^d)$ and $b \in \mathcal{S}^{m_b}_{\rho,\delta}(\mathbb{R}^d)$ then $c \in \mathcal{S}^{m_a+m_b}_{\rho,\delta}(\mathbb{R}^d)$.*

The link between amplitudes and symbols can now be established. Clearly, any symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ defines the amplitude $a \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$. Conversely one has:

Theorem 5.1.9. *For any amplitude $c \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta < \rho \leq 1$, there exits a symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ such that $a(X,D) = c(X,Y,D)$. Moreover, the symbol a admits the asymptotic expansion given by*

$$(x,\xi) \mapsto a(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} [\partial_\xi^\alpha \partial_y^\alpha c](x,x,\xi) \ \in \ \mathcal{S}^{m-(\rho-\delta)N}(\mathbb{R}^d)$$

for any $N > 0$.

The proof of the previous statement for $(\rho,\delta) = (1,0)$ can be found in [RT, Thm. 2.5.8]. Its extension to amplitudes with $(\rho,\delta) \neq (1,0)$ can be performed as an exercise.

For any operator $a(X,D)$, we define its L^2-adjoint by the formula

$$\big\langle a(X,D)^* f, g\big\rangle = \big\langle f, a(X,D) g\big\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d)$$

which corresponds to the relation (5.3) for amplitudes. Then, by the content of Theorem 5.1.9 together with the formula (5.4) for the amplitude of an adjoint one directly infers:

Corollary 5.1.10. *For any $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta < \rho \leq 1$ there exists a symbol $a^* \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ such that $a(X,D)^* = a^*(X,D)$. Moreover a^* admits the asymptotic expansion*

$$(x,\xi) \mapsto a^*(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}[\partial^\alpha_\xi \partial^\alpha_y \bar{a}](x,\xi) \;\in\; \mathcal{S}^{m-(\rho-\delta)N}(\mathbb{R}^d)$$

for any $N \geq 0$. Here $\bar{a}$ means the complex conjugate function.

The previous result implies that any pseudo-differential operator $a(X,D)$ extends to a continuous linear map from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Let us then note that a rather simple criterion allows us to know if a continuous linear operator from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is of the previous form. More precisely, if for any $\xi \in \mathbb{R}^d$ one sets $\mathrm{e}_\xi : \mathbb{R}^d \to \mathbb{C}$ by $\mathrm{e}_\xi(x) := \mathrm{e}^{2\pi i x\cdot\xi}$ then one has:

Theorem 5.1.11. *A continuous linear operator T from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is a pseudo-differential (with symbol a) if and only if the symbol a defined by*

$$a(x,\xi) := \mathrm{e}_{-\xi}(x)[T\,\mathrm{e}_\xi](x)$$

belong to $\mathcal{S}^\infty(\mathbb{R}^d) := \bigcup_{m\in\mathbb{R}} \mathcal{S}^m_{1,0}(\mathbb{R}^d)$.

Among the set of pseudo-differential operators let us still introduce those which have a *classical symbol*. For that purpose, we say that a function $a \in C^\infty(\mathbb{R}^{2d})$ is *homogeneous of order k* for some $k \in \mathbb{R}$ if for all $x \in \mathbb{R}^d$

$$a(x,\lambda\xi) = \lambda^k a(x,\xi), \quad \forall\lambda > 1,\ \forall\xi \in \mathbb{R}^d \text{ with } |\xi| \geq 1. \tag{5.11}$$

Definition 5.1.12. *1) A symbol $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ is called* classical *if there exists an asymptotic expansion $a \sim \sum_{k=0}^\infty a_{m-k}$ where each function a_{m-k} is homogeneous of order $m-k$, and if $a - \sum_{k=0}^N a_{m-k} \in \mathcal{S}^{m-N-1}_{1,0}(\mathbb{R}^d)$ for all $N \geq 0$. The set of all classical symbols of order m is denoted by $\mathcal{S}^m_{cl}(\mathbb{R}^d)$.*

2) For a classical symbol $a \in \mathcal{S}^m_{cl}(\mathbb{R}^d)$, its principal symbol *corresponds to the term a_m in the mentioned expansion.*

Note that the notion of principal symbol can be defined for more general pseudo-differential operators. For a symbol in $\mathcal{S}^m_{1,0}(\mathbb{R}^d)$ its principal symbol corresponds to the equivalent class of this symbol modulo the subclass $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$. More precisely, we set:

Definition 5.1.13. *For $a,b \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ we write $a \sim b$ if the difference $a-b$ belongs to $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$. For $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ we denote by $[a]$ the equivalent class defined by the previous equivalence relation and call it* the principal symbol *of $a(X,D)$.*

Examples 5.1.14. *1) The simplest and main example of a pseudo-differential operator is provided by the relation*

$$\big[(1-\Delta)^{m/2} f\big](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} \langle 2\pi\xi\rangle^m \hat{f}(\xi)\,\mathrm{d}\xi \qquad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{5.12}$$

In other terms the symbol corresponding to the operator $(1-\Delta)^{m/2}$ *is the map* $\xi \mapsto \langle 2\pi\xi\rangle^m$. *In addition, since the equality* $\langle x\rangle^m = |x|^m(1+|x|^{-2})^{m/2}$ *holds, by using a binomial expansion for* $|x|>1$ *one observes that any symbol which agrees with* $(2\pi)^m|x|^m$ *for* $|x|\geq 1$, *up to a symbol in* $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$, *share the same principal symbol as the one of* $(1-\Delta)^{m/2}$.

2) Let ϕ *be an element of* $C^\infty_b(\mathbb{R}^d)$, *which corresponds to the set of smooth functions with all derivatives bounded. Then the multiplication operator* $\phi(X)$ *defined by* $[\phi(X)f](x) = \phi(x)f(x)$ *is a pseudo-differential operator belonging to* $\mathcal{S}^0_{1,0}(\mathbb{R}^d)$. *In addition, the operator* $\phi(X)(1-\Delta)^{m/2}$ *belongs to* $\mathfrak{Op}\big(\mathcal{S}^m_{1,0}(\mathbb{R}^d)\big)$ *and the corresponding symbol is the map* $(x,\xi)\mapsto \phi(x)\langle 2\pi\xi\rangle^m$.

3) For any $\phi\in C^\infty_b(\mathbb{R}^d)$, *one infers from Theorem 5.1.7 that*

$$[\phi(X),(1-\Delta)^{m/2}] \in \mathfrak{Op}\big(\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)\big).$$

More generally, if $A\in\mathfrak{Op}\big(\mathcal{S}^{m_a}_{1,0}(\mathbb{R}^d)\big)$ *and* $B\in\mathfrak{Op}\big(\mathcal{S}^{m_b}_{1,0}(\mathbb{R}^d)\big)$ *then one has* $[A,B]\in\mathfrak{Op}\big(\mathcal{S}^{m_a+m_b-1}_{1,0}(\mathbb{R}^d)\big)$. *This information means also that* AB *and* BA *share the same principal symbol.*

Let us briefly mention the link between pseudo-differential operators and Sobolev spaces. First of all recall that for any $s\geq 0$ the Sobolev space $\mathcal{H}^s(\mathbb{R}^d)$ is defined by

$$\mathcal{H}^s(\mathbb{R}^d) := \big\{f\in L^2(\mathbb{R}^d) \mid \|f\|_{\mathcal{H}^s} := \|\langle X\rangle^s \mathscr{F} f\| < \infty\big\}. \tag{5.13}$$

Note that this space coincide with the completion of $\mathcal{S}(\mathbb{R}^d)$ with the norm $\|\cdot\|_{\mathcal{H}^s}$. For $s>0$, the spaces $\mathcal{H}^{-s}(\mathbb{R}^d)$ can either be defined by duality, namely $\mathcal{H}^{-s}(\mathbb{R}^d) = \mathcal{H}^s(\mathbb{R}^d)^*$, or by the completion of $\mathcal{S}(\mathbb{R}^d)$ with the norm $\|f\|_{\mathcal{H}^{-s}} := \|\langle X\rangle^{-s}\mathscr{F} f\|$. Then, the main link between these spaces and pseudo-differential operators is summarized in the following statement. Recall that the definition of closed operators has been provided in Definition 1.4.6.

Theorem 5.1.15. *Let* $A := a(X,Y,D)$ *be the operator defined on* $\mathcal{S}(\mathbb{R}^d)$ *by an amplitude* $a\in\mathcal{A}^m_{1,0}(\mathbb{R}^d)$ *with* $m\geq 0$.

(i) *A extends continuously to a bounded linear operator from* $\mathcal{H}^s(\mathbb{R}^d)$ *to* $\mathcal{H}^{s-m}(\mathbb{R}^d)$ *for any* $s\in\mathbb{R}$,

(ii) *If* $m>0$ *then the extension of* $A:\mathcal{H}^m(\mathbb{R}^d)\to\mathcal{H}^0(\mathbb{R}^d)\equiv L^2(\mathbb{R}^d)$ *defines a closed operator,*

(iii) *If* $m=0$ *then the extension of* $A: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ *defines an element of* $\mathscr{B}\big(L^2(\mathbb{R}^d)\big)$.

Clearly, the point (iii) in the previous statement is a slight extension of the result already mentioned in Theorem 5.1.6 for symbols instead of for amplitudes Let us now close this section with the notion of compactly supported and compactly based pseudo-differential operators. Such operators have nice extension properties.

Definition 5.1.16. *Let $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be a pseudo-differential operator.*

(i) A is compactly supported *if the exists $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A\psi(X)$,*

(ii) A is compactly based *if the exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A$.*

Based on these definitions one easily infers the following lemma. Recall that the set $\mathcal{S}^{-\infty}(\mathbb{R}^d)$ has been introduced in Definition 5.1.3.

Lemma 5.1.17. *Let A, B be pseudo-differential operators. Then*

(i) If A, B are compactly supported, so are A^*, AB and BA,

(ii) A is compactly supported if and only if A and A^* are compactly based,

(iii) If A is compactly based, so is AB,

(iv) If A is compactly based, then there exists a compactly supported pseudo-differential operator A' such that $A - A' \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$.

Exercise 5.1.18. *Provide a proof of the last statement of the previous lemma.*

In relation with this last statement, let us mention a useful result about the difference $A - A'$. Unfortunately, we can not prove it here because it would require the definition of the so-called Shubin pseudo-differential operators. These operators are defined with slightly different classes of symbols. We refer to the book [Shu] for a different approach to pseudo-differential operators, and especially to Section 27 of this reference for a proof of the subsequent statement.

Lemma 5.1.19. *For any compactly based pseudo-differential operator A of order m there exists a compactly supported pseudo-differential operator A' of order m such that the difference $A - A'$ is trace class,* i.e. *$A - A' \in \mathscr{J}_1$.*

The following statement will play an essential role subsequently. We shall comment about its generality and its proof after the statement.

Theorem 5.1.20. *Let A be a compactly based pseudo-differential operator of order m. If $m < 0$ then the extension of $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defines a compact operator. If $m < -d$ then this extension defines a trace class operator.*

The previous statement is quite well-known for compactly supported pseudo-differential operators. A more general statement for arbitrary Schatten ideals can be found in [Ars], or an approach using operators of the form $f(X)g(D)$ can be borrowed from [Sim, Chap. 4]. The extension to compactly based pseudo-differential operators follows then directly from Lemma 5.1.19.

Remark 5.1.21. *In relation with the paper [Ars] let us mention that there exists several types of quantization of symbols on $\mathbb{R}^{2d}$. The one introduced so far corresponds to the so-called Kohn-Nirenberg quantization. Each of these quantizations has some properties of special interest: for example the Weyl quantization of real-valued functions provides self-adjoint operators, the Berezin quantization of positive functions provides positive operators, etc. Some of these quantizations can be recast in a single quantization (τ-quantization) which depends on an additional parameter τ. We refer the interested reader to Chapter 2 of the reference [Del] which presents the similarities and the differences between some of these quantizations.*

Extension 5.1.22. *Study some alternative quantization, as presented for example in [Del].*

5.2 Noncommutative residue

In this section we introduce the concept of noncommutative residue on the set of classical and compactly based pseudo-differential operators of order $-d$, where n is the space dimension. This concept is also called *Wodzicki residue* after the seminal papers [Wod1, Wod2]. In these papers the general theory is presented in the framework of global analysis on manifolds, and the special case presented here corresponds to the Remark 7.13 of [Wod1].

Before introducing the definition of noncommutative residue let us observe that if a is a classical symbol of order m with $a(X, D)$ compactly based, then one can impose that each term the expansion $a \sim \sum_{k=0}^{\infty} a_{m-k}$ has a compact support for the first variable (the variable x). Note that such symbols will simply be called classical and compactly based symbols. In addition, if m is an integer, then a_{-d} is well-defined and is equal to 0 if $-d > m$, while if m is not an integer, then we set $a_{-d} := 0$.

Definition 5.2.1. *Let $a \in \mathcal{S}^m_{cl}(\mathbb{R}^d)$ be a classical and compactly based symbols of order m. The* noncommutative residue of $a(X, D)$ *is defined by*

$$\mathsf{Res}_W\big(a(X, D)\big) := \frac{1}{d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d} a_{-d}(x, \theta)\,\mathrm{d}x\,\mathrm{d}\theta. \tag{5.14}$$

Before stating and proving some of the properties of this noncommutative residue, let us come back to the Examples 5.1.14. For ϕ in $C_c^\infty(\mathbb{R}^d)$ let us consider the operator $\phi(X)(1-\Delta)^{-d/2}$ which is associated with a classical and compactly based symbol of order $-d$. Its principal symbol is given for $|\xi| \geq 1$ by the map $(x, \xi) \mapsto \phi(x)(2\pi|\xi|)^{-d}$. Thus, we easily get

$$\begin{aligned}\mathsf{Res}_W\big(\phi(X)(1-\Delta)^{-d/2}\big) &= \frac{1}{d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\phi(x)(2\pi)^{-d}\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \frac{\mathsf{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d}\phi(x)\,\mathrm{d}x,\end{aligned}$$

where $\mathsf{Vol}(\mathbb{S}^{d-1})$ denotes the volume of the sphere $\mathbb{S}^{d-1}$.

Proposition 5.2.2. *The noncommutative residue* Res_W

(i) is a linear functional on the set of all classical and compactly based pseudo-differential operators with symbol of order $-d$,

(ii) vanishes on compactly based pseudo-differential operators with symbol of order m *with* $m < -d$ *(and in particular on trace class operators),*

(iii) is a trace in the sense that if A *is a classical and compactly based pseudo-differential operator of order* m_a *and if* B *is a classical and compactly based pseudo-differential operator of order* m_b *with* $m_a + m_b = -d$*, then*

$$\mathsf{Res}_W([A, B]) = 0.$$

Proof. For (i), the linearity of Res_W is a direct consequence of the linearity of the action of taking the principal symbol on $\mathcal{S}_{cl}^{-d}(\mathbb{R}^d)$. For symbols of order $m < -d$ the noncommutative residue is trivial by its definition, which implies the statement (ii). Finally, since AB and BA share the same principal symbol, as mentioned in Examples 5.1.14, it follows that the principal symbol of $[A, B]$ is of order $-d - 1$. The statement (iii) follows then from (ii). □

It was A. Connes who realized in [Con] that this noncommutative residue can be linked to the Dixmier trace, with an equality of the form $\mathsf{Res}_W(A) = \mathrm{Tr}_\omega(A)$ for some states ω. Such an equality is often called *Connes' trace theorem*. Again this was proved in the context of global analysis on manifolds. In order to understand such a result in our context of pseudo-differential operators on $\mathbb{R}^d$ and in the framework developed in Chapter 3, additional information are necessary. In particular, since there exist several different Dixmier traces on operator which are not Dixmier measurable (see Definition 3.4.16) it is important to understand when an equality with the noncommutative residue is possible ?

5.3 Modulated operators

In this section we introduce the concept of modulated operators and study their properties. Most of this material is borrowed from [KLPS] and [LSZ]. In the sequel $\mathcal{H}$ denotes the Hilbert space $L^2(\mathbb{R}^d)$, and we recall that the Hilbert-Schmidt norm is denoted by $\|\cdot\|_2$. Part of the theory can be built with an abstract bounded and positive operator V in $\mathcal{H}$, but for simplicity and for our purpose, we shall only consider the operator $V := (1 - \Delta)^{-d/2}$.

Definition 5.3.1. *An operator* $T \in \mathscr{B}(\mathcal{H})$ *is* Laplacian-modulated *if the operator* $T\big(1 + t(1 - \Delta)^{-d/2}\big)^{-1}$ *is a Hilbert-Schmidt operator for any* $t > 0$*, and*

$$\|T\|_{mod} := \sup_{t>0} t^{1/2}\big\|T\big(1 + t(1 - \Delta)^{-d/2}\big)^{-1}\big\|_2 < \infty.$$

Note that a Laplacian-modulated operator T is automatically Hilbert-Schmidt since one has

$$\begin{aligned}\|T\|_2 &= \big\|T\big(1+(1-\Delta)^{-d/2}\big)^{-1}\big(1+(1-\Delta)^{-d/2}\big)\big\|_2\\ &\leq \big\|1+(1-\Delta)^{-d/2}\big\|\big\|T\big(1+(1-\Delta)^{-d/2}\big)^{-1}\big\|_2\\ &\leq \big(1+\big\|(1-\Delta)^{-d/2}\big\|\big)\|T\|_{mod}.\end{aligned}$$

The following statement can also easily be proved by taking into account the completeness of $\mathscr{J}_2$, see also [LSZ, Prop. 11.2.2].

Proposition 5.3.2. *The set of all Laplacian-modulated operator is a Banach space with the norm $\|\cdot\|_{mod}$. In addition, if B is Laplacian-modulated and $A \in \mathscr{B}(\mathcal{H})$ one has $\|AB\|_{mod} \leq \|A\|\|B\|_{mod}$.*

In order to further study this Banach space, let us come back to some algebras of functions.

Definition 5.3.3. *A function $f \in L^1(\mathbb{R}^d)$ is a* modulated function, *written $f \in L^1_{mod}(\mathbb{R}^d)$, if*

$$\|f\|_{L^1_{mod}} := \sup_{t>0}\,(1+t)^d \int_{|x|>t} |f(x)|\,\mathrm{d}x \ < \infty. \tag{5.15}$$

Clearly, the inequality $\|f\|_{L^1} \leq \|f\|_{L^1_{mod}}$ holds. Observe also that the natural operation on such functions is the convolution, as shown in the next statement.

Lemma 5.3.4. *If $f, g \in L^1_{mod}(\mathbb{R}^d)$ then the convolution $f * g$ belongs to $L^1_{mod}(\mathbb{R}^d)$.*

Proof. For any $t > 0$ observe that for $|y| > |x|/2$ one has

$$\begin{aligned}\int_{|x|>t}\int_{|y|>|x|/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x &\leq \int_{\mathbb{R}^d}\int_{|y|>t/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x\\ &= \|f\|_{L^1}\int_{|y|>t/2}|g(y)|\,\mathrm{d}y.\end{aligned}$$

On the other hand, if $|y| \leq |x|/2$ and $|x| > t$ it follows that $|x-y| \geq |x|/2 \geq t/2$, and then

$$\begin{aligned}\int_{|x|>t}\int_{|y|<|x|/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x &\leq \iint_{|x-y|>t/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x\\ &= \iint_{|x|>t/2} |g(y)||f(x)|\,\mathrm{d}y\,\mathrm{d}x\\ &= \|g\|_{L^1}\int_{|x|>t/2}|f(x)|\,\mathrm{d}x.\end{aligned}$$

By splitting the following integral into two parts and by using the previous estimates one gets

$$\begin{aligned}
\|f*g\|_{L^1_{mod}} &= \sup_{t>0}\,(1+t)^d \int_{|x|>t} |[f*g](x)|\,\mathrm{d}x \\
&= \sup_{t>0}\,(1+t)^d \int_{|x|>t}\int_{\mathbb{R}^d} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \|f\|_{L^1}\|g\|_{L^1_{mod}} + \|g\|_{L^1}\|f\|_{L^1_{mod}} \\
&\leq 2\|f\|_{L^1_{mod}}\|g\|_{L^1_{mod}} \qquad (5.16)
\end{aligned}$$

which leads directly to the result. □

Based on the previous result, one gets:

Lemma 5.3.5. *$L^1_{mod}(\mathbb{R}^d)$ endowed with the convolution product is a Banach algebra.*

Proof. 1) With the definition of $\|f\|_{L^1_{mod}}$ provided in (5.15) the space $L^1_{mod}(\mathbb{R}^d)$ is clearly a normed space. We first show that this space is complete. Since the inequality $\|f\| \leq \|f\|_{L^1_{mod}}$ holds, if $\{f_p\}$ is a Cauchy sequence in the L^1_{mod}-norm it is also a Cauchy sequence in the L^1-norm. Let $f \in L^1(\mathbb{R}^d)$ denote the limit of this Cauchy sequence. Then, for any fixed $\varepsilon > 0$ let us choose $N \in \mathbb{N}$ such that

$$(1+t)^d \int_{|x|>t} |f_n(x) - f_m(x)|\,\mathrm{d}x \leq \varepsilon$$

for any $n, m \geq N$ and every $t > 0$. Then one infers by the dominated convergence theorem that for arbitrary $t > 0$ and $n \geq N$ one has

$$(1+t)^d \int_{|x|>t} |f_n(x) - f(x)|\,\mathrm{d}x = \lim_{q\to\infty} (1+t)^d \int_{|x|>t} |f_n(x) - f_m(x)|\,\mathrm{d}x \leq \varepsilon.$$

Since ε is arbitrary, one concludes that $L^1_{mod}(\mathbb{R}^d)$ is a complete vector space.

2) It has already been proved in the previous lemma that $L^1_{mod}(\mathbb{R}^d)$ is an algebra with the convolution product. In addition, it has been proved in (5.16) that $\|f*g\|_{L^1_{mod}} \leq 2\|f\|_{L^1_{mod}}\|g\|_{L^1_{mod}}$ which proves the continuity of the product, and hence makes $L^1_{mod}(\mathbb{R}^d)$ a Banach algebra. □

Additional properties of this Banach algebra are presented in [LSZ, Sec. 11.3]. For example, it is proved that the set of compactly supported L^1-functions is not dense in $L^1_{mod}(\mathbb{R}^d)$. A similar space with L^2-functions is also introduced and studied, namely

$$L^2_{mod}(\mathbb{R}^d) := \big\{ f \in L^2(\mathbb{R}^d) \mid |f|^2 \in L^1_{mod}(\mathbb{R}^d) \big\}$$

endowed with the norm $\|f\|_{L^2_{mod}} := \||f|^2\|^{1/2}_{L^1_{mod}}$. This space is again a Banach space, but despite the fact that it is made of L^2-functions, this space has not good properties with respect to the Fourier transform.

Extension 5.3.6. *Study the previous statements.*

Our next aim is to connect this Banach algebra $L^1_{mod}(\mathbb{R}^d)$ with the concept of Laplacian-modulated operators. For that purpose, let us recall that there exists a bijective relation between the set of Hilbert-Schmidt operators in $\mathcal{H} = L^2(\mathbb{R}^d)$ and the set of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$-functions, see Theorem 2.5.1. We present below a slightly modified version of this correspondence, which is based on the mentioned theorem and on Plancherel theorem.

Lemma 5.3.7. *For any Hilbert-Schmidt operator $T \in \mathscr{B}(\mathcal{H})$ there exists a unique function $p_T \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that the following relation holds:*

$$[Tf](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, p_T(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi, \qquad \forall f \in L^2(\mathbb{R}^d). \tag{5.17}$$

Definition 5.3.8. *For any Hilbert-Schmidt operator T, the unique function $p_T \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying* (5.17) *is called* the symbol *of the operator T.*

Clearly, the previous definition is slightly ambiguous since it does not require the regularity conditions of the symbols of a pseudo-differential operators. However, the context together with the index $_T$ should prevent any confusion. On the other hand, the very good point of this definition is that if T is a pseudo-differential operator and a Hilbert-Schmidt operator, its symbol as a pseudo-differential operator and its symbol as a Hilbert-Schmidt operator coincide.

The main result linking all these notions is:

Proposition 5.3.9. *A Hilbert-Schmidt operator $T \in \mathscr{B}(\mathcal{H})$ is Laplacian-modulated if and only if its symbol p_T satisfies*

$$\int_{\mathbb{R}^d} |p_T(x,\cdot)|^2\,\mathrm{d}x \ \in L^1_{mod}(\mathbb{R}^d).$$

We provide below a proof of this statement. However, it involves an equivalent definition for Laplacian-modulated operator which is only provided in Lemma 5.4.9 in a slightly more general context.

Proof. It follows from Lemma 5.4.9 that T is Laplacian-modulated if and only if

$$\big\|TE_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)\big\|_2 = O(t^{-1/2}) \qquad \forall t > 0, \tag{5.18}$$

where $E_{(1-\Delta)^{-d/2}}$ denotes the spectral measure associated with the operator $(1-\Delta)^{-d/2}$. The key point is that the spectral projection $E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)$ is explicitly known, namely for suitable f and any $x \in \mathbb{R}^d$

$$\big[E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)f\big](x) = \int_{(1+4\pi^2|\xi|^2)^{-d/2}\le t^{-1}} \mathrm{e}^{2\pi i x\cdot\xi}[\mathscr{F}f](\xi)\,\mathrm{d}\xi.$$

Now, let us define a family of projections P_t by the formula

$$[P_t f](x) := \int_{|\xi|>t} \mathrm{e}^{2\pi i x\cdot\xi}[\mathscr{F} f](\xi)\,\mathrm{d}\xi.$$

By a simple computation we then find that for any $t \geq 1$

$$P_{(c_{min}t)^{1/d}} \leq E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big) \leq P_{(c_{max}t)^{1/d}}$$

with $c_{min} := (4\pi^2+1)^{-d/2}$ and $c_{max} := (4\pi^2)^{-d/2}$. It follows from (5.18) that T is Laplacian-modulated if and only if $\|TP_t\|_2 = O(t^{-d/2})$. The statement can finally easily be obtained by observing that

$$\|TP_t\|_2^2 = \int_{|\xi|>t}\int_{\mathbb{R}^d} |p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi.$$

□

Remark 5.3.10. *By endowing the set of symbols of Hilbert-Schmidt Laplacian-modulated operators with the norm*

$$\|p_T\|_{mod} := \Big(\sup_{t>0}\,(1+t)^d \int_{|\xi|>t}\int_{\mathbb{R}^d} |p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2}, \tag{5.19}$$

it follows from the previous proposition and its proof that there is an isometry between the Banach space of Laplacian-modulated symbols and the Banach space of Laplacian-modulated operators mentioned in Proposition 5.3.2. Both norms have been denoted by $\|\cdot\|_{mod}$ *for that purpose.*

We shall soon show that the set of Laplacian-modulated operators is an extension of the set of compactly based pseudo-differential operators of order $-d$. For that purpose, observe first that the definition of compactly supported or compactly based operators can also be used in the context of bounded operators, namely an operator $A \in \mathscr{B}\big(L^2(\mathbb{R}^d)\big)$ is compactly supported if the exists $\phi,\psi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A\psi(X)$, while A is compactly based if the exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A$. Then, one easgets that a Laplacian-modulated operator T is compactly supported if and only if its Schwartz kernel is compactly supported. On the other hand, this operator is compactly based if and only if its symbol p_T is compactly supported in the first variable. Note that the notion of a compactly supported operator does not really fit well with the notion of the symbol of a pseudo-differential operator or of a Laplacian-modulated operator. On the other hand, this notion can be used for the Schwartz kernel or for the kernel of an amplitude operator.

In the next statement we show that the concept of Laplacian-modulated operator extends the notion of compactly based pseudo-differential operator of degree $-d$.

Theorem 5.3.11. *Let $A = a(X,D)$ be a compactly based pseudo-differential operator with symbol in $a \in \mathcal{S}^{-d}_{1,0}(\mathbb{R}^d)$, Then A and A^* extends continuously to Laplacian-modulated operators.*

We provide below the proof of the statement for the operator A. The proof for A^* is slightly more complicated and involves Shubin pseudo-differential operators already mentioned in Section 5.1. We refer to[LSZ, Thm. 11.3.17] for the details.

Proof. By assumption one has $|a(x,\xi)| \leq C\langle\xi\rangle^{-d}$ for all $x \in \mathbb{R}^d$ and a constant C independent of x and ξ. In addition, since the operator A is compactly based, its symbol a is compactly supported in the first variable. Thus, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $a(x,\xi) = 0$ for any $x \notin \Omega$. We then infer that

$$\begin{aligned}\sup_{t>0}\,(1+t)^d \int_{|\xi|>t}\int_{\mathbb{R}^d} |a(x,\xi)|^2 \,\mathrm{d}x\,\mathrm{d}\xi &= \sup_{t>0}\,(1+t)^d \int_{|\xi|>t}\int_{\Omega} |a(x,\xi)|^2 \,\mathrm{d}x\,\mathrm{d}\xi \\ &\leq C^2\,|\Omega| \sup_{t>0}\,(1+t)^d \int_{|\xi|>t} \langle\xi\rangle^{-2d}\,\mathrm{d}\xi \\ &\leq C'|\Omega| \sup_{t>0}\,(1+t)^d \int_t^\infty r^{-2d} r^{d-1}\,\mathrm{d}r \\ &= \frac{C'}{d}|\Omega| \sup_{t>0}\,(1+t)^d\, t^{-d} \\ &< \infty,\end{aligned}$$

where $|\Omega|$ means the Lebesgue measure of the set Ω, and C' is a constant. It follows from Proposition 5.3.9 that a corresponds to the symbol of a Laplacian-modulated operator. As a consequence, the operator A extends continuously to a Hilbert-Schmidt operator which is Laplacian-modulated. □

In order to extend the noncommutative residue to all compactly based Laplacian-modulated operators, the following rather technical lemma is necessary. For that purpose we recall that any Laplacian-modulated operator T is itself a Hilbert-Schmidt operator.

Lemma 5.3.12. *Let T be a compactly based Laplacian-modulated operator, and let p_T denotes its symbol. Then the map*

$$\mathbb{N} \ni\; n \mapsto \frac{1}{\ln(n+1)} \int_{|\xi|\leq n^{1/d}} \int_{\mathbb{R}^d} p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \;\in \mathbb{C}$$

is bounded

Proof. Recall first that since the operator T is compactly based, its symbol p_T is compactly supported in the first variable. Thus, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $p_T(x,\xi) = 0$ for any $x \notin \Omega$. Observe in addition that there exists a constant C (depending only on the space dimension d) such that for any $k \geq 0$

$$\big|\Omega \times \big\{\xi \in \mathbb{R}^d \mid \mathrm{e}^k \leq |\xi| \leq \mathrm{e}^{k+1}\big\}\big| = C|\Omega|\,\mathrm{e}^{kd}\,.$$

It then follows by an application of Cauchy-Schwartz inequality that

$$\begin{aligned}
&\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\mathbb{R}^d}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi\\
&=\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\Omega}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi\\
&\leq C^{1/2}|\Omega|^{1/2}\Big((e^k)^d\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\Omega}|p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2}\\
&\leq C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod},
\end{aligned}$$

where the definition (5.19) has been used in the last step.

Based on this estimate one infers that for $t>1$

$$\begin{aligned}
&\Big|\int_{|\xi|\leq t}\int_{\mathbb{R}^d}p_T(x;\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big|\\
&\leq\Big|\int_{|\xi|\leq 1}\int_{\mathbb{R}^d}p_T(x;\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big|+\sum_{k=0}^{\lfloor\ln(t)\rfloor}\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\mathbb{R}^d}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi\\
&\leq\big(\ln(t)+1\big)C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+D
\end{aligned}$$

with D independent of t. By setting then $t=n^{1/d}$ for $n>1$ one gets

$$\begin{aligned}
&\frac{1}{\ln(n+1)}\Big|\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d}p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big|\\
&\leq\frac{\ln(n^{1/d})+1}{\ln(n+1)}C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+o(n)\\
&=\frac{\frac{1}{d}\ln(n)+1}{\ln(n+1)}C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+o(n)
\end{aligned}$$

which clearly defines a bounded function of $n\in\mathbb{N}$. □

Based on this result, it is now natural to set:

Definition 5.3.13. *The map* Res, *from the set of compactly based Laplacian-modulated operator to the quotient ℓ_∞/c_0, is defined for any compactly based Laplacian-modulated operator T by*

$$\mathsf{Res}(T):=\Big[\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d}p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}\Big] \tag{5.20}$$

where p_T denotes the symbol associated with T and $[\,\cdot\,]$ denotes the equivalence class in ℓ_∞/c_0. This map is called the generalized residue.

Let us directly check that this notion extends the noncommutative residue introduced in Section 5.2. First of all we need a preliminary lemma, which uses the fact proved in Theorem 5.3.11 that any compactly based pseudodifferential operator of order $-d$ extends to a compactly based Laplacian-modulated operator.

Lemma 5.3.14. *The generalized residue of a compactly based pseudo-differential operator of order $-d$ depends only on its principal symbol.*

Proof. Let A_1 and A_2 be two compactly based pseudo-differential operators of order $-d$ sharing the same principal symbol. Then the difference $A_1 - A_2$ is a compactly based pseudo-differential operator of order $-d-1$, which means that its symbol a satisfies $a|(x,\xi)| \leq C\langle\xi\rangle^{-d-1}$ for all $x, \xi \in \mathbb{R}^d$ and a constant C independent of x and ξ. Since a has compact support in the first variable, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $a(x,\xi) = 0$ if $x \notin \Omega$. Then one has

$$\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big| \leq C|\Omega|\int_{\mathbb{R}^d}\langle\xi\rangle^{-d-1}\,\mathrm{d}\xi < \infty.$$

As a consequence of this estimate, it follows from the definition of Res provided in (5.20) that $\mathsf{Res}(A_1 - A_2) = 0$, and therefore that $\mathsf{Res}(A_1) = \mathsf{Res}(A_2)$. □

In the next statement we clearly identify $\mathbb{C}$ with the set of constant elements of ℓ_∞.

Proposition 5.3.15. *For any $a \in \mathcal{S}^{-d}_{cl}(\mathbb{R}^d)$ with compact support in the first variable of all elements of its asymptotic expansion, or equivalently for any classical and compactly based pseudo-differential operator A of order $-d$ (with $A = a(X,D)$) one has*

$$\mathsf{Res}_W(A) = \mathsf{Res}(A).$$

Proof. Let us denote by a_{-d} the principal symbol of the operator a. By the previous lemma $\mathsf{Res}(A)$ depends only on the symbol a_{-d}, and is determined by the equivalence class in ℓ_∞/c_0 of the sequence

$$\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}.$$

Since a_{-d} is homogeneous of order $-d$ and is compact in its first variable one has

$$\begin{aligned}
\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi &= \int_{\mathbb{R}^d}\int_{1<|\xi|\leq n^{1/d}} |\xi|^{-d} a_{-d}\Big(x,\frac{\xi}{|\xi|}\Big)\,\mathrm{d}\xi\,\mathrm{d}x + C \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \int_1^{n^{1/d}} r^{-d} r^{d-1}\,\mathrm{d}r + C \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \ln\big(n^{1/d}\big) + C \\
&= \frac{\ln(n)}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \ln(n) + C
\end{aligned}$$

with C a constant independent of n. As a consequence one infers that

$$\begin{aligned}
\mathsf{Res}(A) &= \Big[\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}\Big] \\
&= \Big[\Big(\frac{\ln(n)}{\ln(n+1)}\frac{1}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x + \frac{C}{\ln(n+1)}\Big)_{n\in\mathbb{N}}\Big] \\
&= \Big[\Big(\frac{1}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x\Big)_{n\in\mathbb{N}}\Big] \\
&= \mathsf{Res}_W(A)
\end{aligned}$$

with the identification mentioned before the statement of the proposition. □

In reference [LSZ], it is shown that there exist some symbols for which the generalized residue is not a constant sequence. We provide a counterexample in the following exercise, and refer to Example 10.2.10 and Proposition 11.3.22 of that reference for more information.

Exercise 5.3.16. *Consider the smooth function* $a_\star : \{\xi \in \mathbb{R}^d \mid |\xi| > 4\} \to \mathbb{R}$ *given by*

$$a_\star(\xi) := |\xi|^m\Big(\sin\big(\ln(\ln(|\xi|))\big) + \cos\big(\ln(\ln(|\xi|))\big)\Big) \qquad \forall |\xi| > 4.$$

1) Based on this function, show that there exists a symbol $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ *such that its principal symbol can not be a homogeneous function. For that purpose one can show that the map*

$$\xi \mapsto a_\star(2|\xi|) - 2^m a_\star(|\xi|)$$

does not belong to $S^{m-1}_{1,0}(\mathbb{R}^d)$*.*

2) In the special case $m = -d$*, let* $\phi \in C^\infty_c(\mathbb{R}^d)$ *and let* $a \in \mathcal{S}^{-d}_{1,0}(\mathbb{R}^d)$ *satisfying* $a(x,\xi) := \phi(x)a_\star(\xi)$ *for any* $|\xi| > 4$ *and any* $x \in \mathbb{R}^d$*. Show that*

$$\mathsf{Res}\big(a(X,D)\big) = \big[(b_n)_{n\in\mathbb{N}}\big]$$

with $b_n = \frac{1}{d}\sin\big(\ln(\ln(n^{1/d}))\big)$ *for* n *large enough. The sequence* (b_n) *is clearly not a convergent sequence.*

5.4 Connes' trace theorem

In this section we state a generalized version of Connes' trace theorem and sketch the main arguments of its proof. Again, our framework are operators acting $\mathbb{R}^d$ while the original setting was for operators acting on compact manifolds.

Recall that the space $\mathscr{L}_{1,\infty}$ has been introduced in (3.3) and corresponds to

$$\big\{A \in \mathscr{K}(\mathcal{H}) \mid \mu_n(A) \in O\big(n^{-1}\big)\big\}. \tag{5.21}$$

Theorem 5.4.1. *Let T be a compactly supported Laplacian modulated operator with symbol p_T, and let ω be any dilation invariant extended limit on ℓ_∞. Then:*

(i) T belongs to $\mathscr{L}_{1,\infty}$ and

$$\mathrm{Tr}_\omega(T) = \omega\big(\mathsf{Res}(T)\big),$$

(ii) T is Dixmier measurable if and only if $\mathsf{Res}(T)$ is a constant sequence, and then $\mathrm{Tr}_\omega(T) = \mathsf{Res}(T)$.

Remark 5.4.2. *In the corresponding statement [LSZ, Thm. 11.5.1] the dilation invariance of the extended limit ω is not required. Indeed, it is shown in [LSZ, Sec. 9.7] that once applied to operators in $\mathscr{L}_{1,\infty}$ the dilation invariance of ω holds automatically. However, since we have not introduced this material and since our Dixmier traces were introduced on the more general space $\mathscr{M}_{1,\infty}$ we shall not consider this refinement here.*

As mentioned before, the sketch of the proof will be given subsequently. Our aim is to mention some corollaries of the previous statement.

Theorem 5.4.3. *Let A be a compactly based pseudo-differential operator of order $-d$. Then A extends continuously to an element of $\mathscr{L}_{1,\infty}$ and satisfies $\mathrm{Tr}_\omega(A) = \omega\big(\mathsf{Res}(A)\big)$ for any dilation invariant extended limit ω on ℓ_∞.*

Proof. First of all, it follows from Theorem 5.3.11 that the operator A extends continuously to a Laplacian modulated operator. In addition, there exists a function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(X)A = A$. The operator $A' := A\phi(X)$ is then compactly supported and the difference $A - A'$ is a compactly based operator and a pseudodifferential operator of order $-\infty$. Note that the operator A' corresponds to the one already mentioned in the statement (iv) of Lemma 5.1.17 and in Lemma 5.1.19. It then follows from Lemma 5.3.14 that $\mathsf{Res}(A) = \mathsf{Res}(A')$, and from Lemma 5.1.19 that $A - A' \in \mathscr{J}_1$. Thus, one infers from Theorem 5.4.1 that $A' \in \mathscr{L}_{1,\infty}$, and since $\mathscr{J}_1 \subset \mathscr{L}_{1,\infty}$ one also gets that $A \in \mathscr{L}_{1,\infty}$. Finally, again from Theorem 5.4.1 one deduces that

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A') = \omega\big(\mathsf{Res}(A')\big) = \omega\big(\mathsf{Res}(A)\big) \tag{5.22}$$

which corresponds to the statement. □

Note that this result makes the Dixmier trace of any compactly based pseudo-differential operator easily computable. Indeed, for a classical symbol the residue $\mathsf{Res}(A)$ of the corresponding pseudo-differential operator A can be computed by its Wodzicki residue, see Proposition 5.3.15, and the expression $\omega\big(\mathsf{Res}(A)\big)$ does not depend on ω. On the other hand, if the symbol is not classical, then the generalized residue $\mathsf{Res}(A)$ of the corresponding operator can be computed by (5.20) in Definition 5.3.13. Then, if this sequence is not constant, the r.h.s. of (5.22) does depend on ω, but nevertheless it makes the Dixmier trace $\mathrm{Tr}_\omega(A)$ computable. For example, the pseudo-differential operator $a(X, D)$ exhibited in Exercise 5.3.16 is compactly based and possesses a generalized residue $\mathsf{Res}\big(a(X,D)\big)$ which is not a constant sequence. It then follows that the r.h.s. of (5.22) depends on the choice of ω.

By collecting the information obtained so far one directly deduces the following statement:

Corollary 5.4.4. *For any $\phi \in C_c^\infty(\mathbb{R}^d)$ and for any dilation invariant extended limit ω on ℓ_∞ one has*

$$\mathrm{Tr}_\omega\big(\phi(X)(1-\Delta)^{-d/2}\big) = \frac{\mathsf{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d}\phi(x)\,\mathrm{d}x.$$

Let us mention that the above equality still holds if ϕ belongs to $L^2(\mathbb{R}^d)$ and has compact support. We refer to [LSZ, Thm. 11.7.5] for the proof of this extension.

We now come to the proof of Theorem 5.4.1. In fact, its content is a simple consequence of the following two major statements.

Theorem 5.4.5. *Let T be a compactly supported Laplacian-modulated operator with symbol p_T. Then $T \in \mathscr{L}_{1,\infty}$ and the map*

$$\mathbb{N} \ni\; n \mapsto \sum_{j=1}^{n}\lambda_j(T) - \int_{\mathbb{R}^d}\int_{|\xi|<n^{1/d}} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \;\in \mathbb{C} \tag{5.23}$$

is bounded, where $\lambda_j(T)$ denote the eigenvalues of T and these eigenvalues are ordered such that their modulus decrease.

Note that this result should be read with the content of Theorem 2.6.6 in mind. Indeed, in that result and for a trace class operator A its trace was expressed as a integral over its Schwartz symbol. Here, the operator T is not trace class, and p_T is not a Schwartz kernel, but anyway the difference between the partial sum of eigenvalues and a partial integral over the kernel p_T remains bounded, as a function of n.

Theorem 5.4.6 (Lidskii's type formula for the Dixmier trace)**.** *For any $A \in \mathscr{M}_{1,\infty}$ and for any dilation invariant extended limit on ℓ_∞ the following formula holds:*

$$\mathrm{Tr}_\omega(A) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=0}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big)$$

where again $\lambda_j(T)$ denote the eigenvalues of T and these eigenvalues are ordered such that their modulus decrease.

Based on the previous two statements one has:

Proof of Theorem 5.4.1. i) By Theorem 5.4.5 and the definition of the residue $\mathsf{Res}(T)$ one has $T \in \mathscr{L}_{1,\infty}$ and

$$\mathsf{Res}(T) = \Big[\Big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(T)\Big)_{n\in\mathbb{N}}\Big].$$

Since $\mathscr{L}_{1,\infty} \subset \mathscr{M}_{1,\infty}$ we can then apply Theorem 5.4.6 and infer that

$$\mathrm{Tr}_\omega(T) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^n \lambda_j(T)\big)_{n\in\mathbb{N}}\Big) = \omega\big(\mathsf{Res}(T)\big).$$

ii) It is clear that if $\mathsf{Res}(T)$ is a constant sequence, then $\omega\big(\mathsf{Res}(T)\big) = \mathsf{Res}(T)$ for any dilation invariant extended limit ω on ℓ_∞. For the reverse implication, we refer to [LSZ, Thm. 10.1.3.(f)] since the statement is based on the notion of Tauberian operator (see Definition 9.7.1 of that reference) which has not been introduced in these notes. □

In the rest of this section we provide some information about the proofs of Theorems 5.4.5 and 5.4.6. These results are rather deep statements and we shall not be able to prove them in detail. We start with Theorem 5.4.6 which also provides the necessary tools for the proof of the initial Lidskii's theorem. We first prove a necessary estimate.

Lemma 5.4.7. *Let ω be a dilation invariant extended limit on ℓ_∞, and let $A \in \mathscr{M}_{1,\infty}$. Then one has*

$$\omega\Big(\big(\frac{n}{\ln(n+1)}\mu_n(A)\big)_{n\in\mathbb{N}}\Big) = 0$$

Proof. Since $\omega = \omega \circ D_2$ with D_2 the dilation operator introduced in Section 3.1 one infers that

$$\begin{aligned}\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^n \mu_j(A)\big)_{n\in\mathbb{N}}\Big) &= \omega\Big(\big(\frac{1}{\ln(\lfloor n/2\rfloor+1)}\sum_{j=1}^{\lfloor n/2\rfloor} \mu_j(A) + o(n)\big)_{n\in\mathbb{N}}\Big)\\ &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{\lfloor n/2\rfloor} \mu_j(A) + o(n)\big)_{n\in\mathbb{N}}\Big)\end{aligned}$$

where we have used that $\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(\lfloor n/2\rfloor+1)} = 1$. As a consequence, one has

$$\begin{aligned}0 &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^n \mu_j(A)\big)_{n\in\mathbb{N}}\Big) - \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{\lfloor n/2\rfloor} \mu_j(A) + o(n)\big)_{n\in\mathbb{N}}\Big)\\ &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=\lfloor n/2\rfloor+1}^{n} \mu_j(A) + o(n)\big)_{n\in\mathbb{N}}\Big)\\ &\geq \omega\Big(\big(\frac{n}{2\ln(n+1)}\mu_n(A)\big)_{n\in\mathbb{N}}\Big)\end{aligned}$$

from which one deduces the statement. □

Proof of Theorem 5.4.6. 1) First of all, let $A \in \mathscr{M}_{1,\infty}$ be self-adjoint, and recall that $A = A_+ - A_-$ with $A_\pm \geq 0$. By the linearity of the Dixmier trace one has

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A_+) - \mathrm{Tr}_\omega(A_-) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^n \big\{\lambda_j(A_+) - \lambda_j(A_-)\big\}\big)_{n\in\mathbb{N}}\Big).$$

In the point 2) below we shall prove that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j(A_+)+\lambda_j(A_-)\big\}\Big|\leq n\mu_n(A). \tag{5.24}$$

It then follows from Lemma 5.4.7 that $\omega\Big(\big(\frac{1}{\ln(n+1)}n\mu_n(A)\big)_{n\in\mathbb{N}}\Big)=0$, and therefore

$$\operatorname{Tr}_\omega(A)=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big).$$

If $A\in\mathscr{M}_{1,\infty}$ is a normal operator, it follows from the previous paragraph that

$$\begin{aligned}\operatorname{Tr}_\omega(A)&=\operatorname{Tr}_\omega\big(\Re(A)\big)+i\operatorname{Tr}_\omega\big(\Im(A)\big)\\&=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\big\{\lambda_j\big(\Re(A)\big)+i\lambda_j\big(\Im(A)\big)\big\}\big)_{n\in\mathbb{N}}\Big).\end{aligned}$$

Again in the point 2) below we shall prove that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j\big(\Re(A)\big)-i\lambda_j\big(\Im(A)\big)\big\}\Big|\leq 5n\mu_n(A), \tag{5.25}$$

from which one infers with Lemma 5.4.7 that

$$\operatorname{Tr}_\omega(A)=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big).$$

For the general case $A\in\mathscr{M}_{1,\infty}$ one has to rely on a rather deep decomposition of A, namely $A=N+Q$ with $N,Q\in\mathscr{M}_{1,\infty}$, N normal, Q satisfying $\operatorname{Tr}_\omega(Q)=0$, and $\lambda_j(A)=\lambda_j(N)$. This decomposition is provided for example in [LSZ, Thm. 5.5.1] in a more general framework. Note also that this decomposition can be used for proving the usual Lidskii's theorem, see (2.33). With this information at hand, the proof of the statement follows directly.

2) For (5.24) one first observes that for any $n\in\mathbb{N}$

$$\big\{\lambda_j(A)\big\}_{j=1}^{n}\subset\Big\{\big\{\lambda_j(A_+)\big\}_{j=1}^{n}\cup\big\{-\lambda_j(A_-)\big\}_{j=1}^{n}\Big\}.$$

Indeed, this easily follows from the functional calculus of the self-adjoint operator A. In addition, one also observes that

$$\Big\{\big\{\lambda_j(A_+)\big\}_{j=1}^{n}\cup\big\{-\lambda_j(A_-)\big\}_{j=1}^{n}\Big\}\setminus\big\{\lambda_j(A)\big\}_{j=1}^{n}\subset\big\{\lambda\in\mathbb{C}\mid|\lambda|\leq|\lambda_n(A)|\big\}$$

and that the cardinality of the set on the l.h.s. contains at most n elements. It then follows that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j(A_+)+\lambda_j(A_-)\big\}\Big|\leq n|\lambda_n(A)|=n\mu_n(A). \tag{5.26}$$

For (5.25) recall first that since A is a normal compact operator it has the canonical form $A = \sum_j \lambda_j(A)|f_j\rangle\langle f_j|$ for $\lambda_j(A) \in \mathbb{C}$ ordered with a decrease of their modulus. It then follows that for any $n \in \mathbb{N}$

$$\big\{\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\lambda| > \mu_n(A)\}\big\} \subset \{\lambda_j(A)\}_{j=1}^n .$$

One also observes that

$$\{\lambda_j(A)\}_{j=1}^n \setminus \big\{\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\lambda| > \mu_n(A)\}\big\} \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \mu_n(A)\}$$

and that the cardinality of the set on the l.h.s. contains at most n elements. As a consequence one has

$$\Big|\sum_{j=1}^n \lambda_j(A) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \lambda\Big| \leq n\mu_n(A).$$

By a similar argument one also gets that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(\Re(A)),|\lambda|>\mu_n(A)} \lambda\Big| \leq n\mu_n(A).$$

Since $\Re\big(\sigma(A)\big) = \sigma\big(\Re(A)\big)$, by the normality of A, this is equivalent to

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(A),|\Re(\lambda)|>\mu_n(A)} \Re(\lambda)\Big| \leq n\mu_n(A). \tag{5.27}$$

On the other hand one infers that

$$\begin{aligned}
&\Big|\sum_{\lambda\in\sigma(A),|\Re(\lambda)|>\mu_n(A)} \Re(\lambda) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Re(\lambda)\Big| \\
&\leq \sum_{\lambda\in\sigma(A),|\Re(\lambda)|\leq\mu_n(A)| \text{ and } |\lambda|\geq\mu_n(A)} |\Re(\lambda)| \\
&\leq \sum_{\lambda\in\sigma(A),|\lambda|\geq\mu_n(A)} \mu_n(A) \\
&= n\mu_n(A).
\end{aligned}$$

By this estimate together with (5.27) we finally infer that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Re(\lambda)\Big| \leq 2n\mu_n(A).$$

Similarly, one can also deduce that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Im(A)\big) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Im(\lambda)\Big| \leq 2n\mu_n(A).$$

By combining the previous two estimates one infers that

$$\Big|\sum_{j=1}^{n}\lambda_j\big(\Re(A)\big)+i\lambda_j\big(\Im(A)\big)-\sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)}\lambda\Big|\leq 4n\mu_n(A), \tag{5.28}$$

It finally follows from (5.26) and (5.28) that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j\big(\Re(A)\big)-i\lambda_j\big(\Im(A)\big)\big\}\Big|\leq 5n\mu_n(A),$$

as announced. □

Let us now come to the proof of Theorem 5.4.5 which is at the heart of Connes' trace theorem. A first step in the proof consists in studying more deeply the notion of V-modulated operator. As already mentioned at the beginning of Section 5.3 this notion is more general than Laplacian-modulated and has some advantages. For the record:

Definition 5.4.8. *Let $V\in\mathscr{B}(\mathcal{H})$ be positive. An operator $T\in\mathscr{B}(\mathcal{H})$ is V-*modulated *if the operator $T\big(1+tV\big)^{-1}$ is a Hilbert-Schmidt operator for any $t>0$, and*

$$\|T\|_{mod}:=\sup_{t>0}\ t^{1/2}\big\|T\big(1+tV\big)^{-1}\big\|_2\ <\infty. \tag{5.29}$$

Before going on with the main result related to V-modulated operator, let us provide an equivalent definition. It proof involves the functional calculus of the self-adjoint operator V.

Lemma 5.4.9. *Let $V\in\mathscr{J}_2$ be positive. An operator $T\in\mathscr{B}(\mathcal{H})$ is V-*modulated *if and only if*

$$\big\|TE_V\big([0,t^{-1}]\big)\big\|_2=O(t^{-1/2})\qquad\forall t>0, \tag{5.30}$$

where E_V denotes the spectral measure associated with the operator V.

Proof. For fixed $t>0$, observe first that for any $x\in\mathbb{R}_+$ one has $1\leq 2(1+tx)^{-1}$ if and only if $x\leq 1/t$. Since in addition $2(1+tx)^{-1}>0$ one infers that the following inequality holds for functions: $\chi_{[0,1/t]}\leq 2(1+t\,\cdot\,)^{-1}$. By functional calculus for V it follows that

$$E_V\big([0,t^{-1}]\big)\equiv\chi_{[0,1/t]}(V)\leq 2(1+tV)^{-1}.$$

Thus, if we assume that T satisfies (5.29) one infers that

$$\begin{aligned}\big\|TE_V\big([0,t^{-1}]\big)\big\|_2&=\big\|T\big(2(1+tV)^{-1}\big)E_V\big([0,t^{-1}]\big)\big(2(1+tV)^{-1}\big)^{-1}\big\|_2\\&\leq 2\big\|T(1+tV)^{-1}\big\|_2\ \big\|E_V\big([0,t^{-1}]\big)\big(2(1+tV)^{-1}\big)^{-1}\big\|\\&\leq\big\|T(1+tV)^{-1}\big\|_2\\&\leq\|T\|_{mod}\ t^{-1/2}.\end{aligned}$$

For the converse assertion, let us first assume that $\|V\| < 1$. Since the inequality (5.29) is always satisfied for $t \in (0,1)$ we can consider without restriction that $t \geq 1$. Let $k \in \mathbb{N}_0$ such that $t \in [2^k, 2^{k+1})$. Then by assuming (5.30) one has

$$\begin{aligned}
\|T(1+tV)^{-1}\| &\leq \|TE_V([0,2^{-k}])\|_2 + \sum_{j=0}^{k-1} \|TE_V((2^{-j-1},2^{-j}])(1+tV)^{-1}\|_2 \\
&\leq O(t^{-1/2}) + \sum_{j=0}^{k-1}(1+t2^{-j-1})^{-1}\|TE_V((2^{-j-1},2^{-j}])\|_2 \\
&\leq O(t^{-1/2}) + C\sum_{j=0}^{k-1}(1+2^{k-j-1})^{-1}\, 2^{-j/2} \\
&= O(t^{-1/2}) + C\sum_{j=0}^{k-1}\frac{\sqrt{2}\, 2^{-k/2}}{2^{(j-k+1)/2}+2^{-(j-k+1)/2}} \\
&\leq O(t^{-1/2}).
\end{aligned}$$

Note that for the summation in the last term one can use an argument involving the estimate $\int_{\mathbb{R}} \frac{1}{\cosh(x)}\,\mathrm{d}x < 0$.

For arbitrary $V > 0$ one can consider $(1+tV) = (1+\{t\|V\|\}\hat{V})$ with $\hat{V} = \frac{V}{\|V\|}$ which is of norm 1. The adaptation of the proof is then straightforward. ☐

The main result in the present context is provided in [LSZ, Thm. 11.2.3]. We can not provide a proof of this statement without additional efforts, but let us state it and see its role in the proof of Theorem 5.4.5. By a strictly positive operator we denote a positive operator with empty kernel.

Theorem 5.4.10. *Let $V \in \mathcal{L}_{1,\infty}$ be a strictly positive operator, and let $T \in \mathscr{B}(\mathcal{H})$ be a V-modulated operator. Let $\{f_n\}$ be an orthonormal basis of $\mathcal{H}$ ordered such that $Vf_n = \mu_n(V)f_n$ for any $n \in \mathbb{N}$. Then we have:*

(i) $T \in \mathcal{L}_{1,\infty}$ and the sequence $\big(\langle f_n, Tf_n\rangle\big)_{n\in\mathbb{N}}$ belongs to $\ell_{1,\infty}$,

(ii) The map

$$\mathbb{N} \ni \sum_{j=1}^{n}\lambda_j(T) - \sum_{j=1}^{n}\langle f_j, Tf_j\rangle \in \mathbb{C} \tag{5.31}$$

is bounded.

Note that equation (5.31) should be read with the results of Chapter 2 on the usual trace in mind. Indeed, for a trace class operator A, the sum $\sum_n \langle f_n, Af_n\rangle$ gives the same value for an arbitrary orthonormal basis of $\mathcal{H}$, and by Lidskii's theorem this sum is equal to $\sum_j \lambda_j(A)$. In the present situation, the operator T is not trace class, and therefore neither $\sum_j \lambda_j(T)$ nor $\sum_n \langle f_n, Tf_n\rangle$ are well-defined. However, equation (5.31) states that a suitable difference (depending on a parameter n) of these expressions remains bounded for all n. One additional difference with the content of Chapter 2 is that the basis of $\mathcal{H}$ is not arbitrary but is adapted to the operator V to which the operator T is modulated. In a vague sense it means that the chosen basis of $\mathcal{H}$ is made of elements which have a certain regularity with respect to T.

Clearly, the above result can not be applied for any Laplacian-modulated operator since the operator $(1-\Delta)^{-d/2}$ is never a compact operator. However, the trick is to replace the Laplacian operator by the Laplacian on a bounded domain. This change will be possible thanks to the assumption on the support of the operator T. So, for any $m \in \mathbb{Z}^d$ let us set $\mathrm{e}_m \in L^2([0,1]^d)$ by $\mathrm{e}_m(x) := \mathrm{e}^{2\pi i m\cdot x}$ and let us denote by $-\Delta_0$ the Laplacian in $L^2([0,1]^d)$ with domain $\mathsf{D}(-\Delta) := \mathsf{Span}\big(\{\mathrm{e}_m\}_{m\in\mathbb{Z}^d}\big)$. Clearly, $-\Delta_0\,\mathrm{e}_m = 4\pi^2 m^2\,\mathrm{e}_m$. One major interest in this operator is that its resolvent has very good spectral properties, more precisely one has $(1-\Delta_0)^{-d/2} \in \mathcal{L}_{1,\infty}$ as a consequence of Weyl law. In addition, this operator is strictly positive, and therefore satisfies the assumptions of Theorem 5.4.10

Exercise 5.4.11. *By using the Weyl asymptotic provided in the theorem on page 30 of [Cha] show that $(1-\Delta_0)^{-d/2} \in \mathcal{L}_{1,\infty}$. Show also that such an inclusion holds for the Laplacian Δ_0 for any bounded rectangular domain in $\mathbb{R}^d$.*

In the sequel, we shall consider the functions e_m as periodic functions on $\mathbb{R}^2$. Clearly, these functions are not in $L^2(\mathbb{R}^d)$, but nevertheless they are going to play an important role.

For the next statement, recall that if T is a Hilbert-Schmidt operator and if $\{f_n\}$ is an orthogonal basis of $\mathcal{H}$, then the summation $\sum_n \|Tf_n\|^2$ is finite, see Proposition 2.5.4. Clearly, the family of functions $\{\mathrm{e}_m\}$ is not suitable for such an estimate, but once multiplied by a nice function one gets:

Lemma 5.4.12. *Let $T \in \mathscr{B}(\mathcal{H})$ be a Laplacian-modulated operator, and let ϕ be an arbitrary element of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Then one has*

$$\sum_{|m|>t} \|T(\phi\,\mathrm{e}_m)\|^2 = O(t^{-d}), \qquad \forall t > 0.$$

The rather lengthy proof of this lemma is provided in [LSZ, Lem. 11.4.2]. It is only based on the properties of the Schwartz functions and makes an extensive used of the algebra $L^1_{mod}(\mathbb{R}^d)$.

With the previous result we can show that any compactly supported Laplacian-modulated operator is also Δ_0-modulated operator. For the compactly supported operator, we shall assume from now on that the support is inside $[0,1]^d$. Obviously, this is not a loss of generality since other arbitrary cubes could have been chosen, see also Exercise 5.4.11.

Theorem 5.4.13. *Let $T \in \mathscr{B}(\mathcal{H})$ be a compactly supported Laplacian-modulated operator with support in $[0,1]^d$. Then the operator T, considered from $L^2([0,1]^d)$ to $L^2([0,1]^d)$ is Δ_0-modulated.*

Proof. For any $m \in \mathbb{Z}^d$ let e_m be the functions introduced above, seen either as element of $L^2([0,1]^d)$ or as continuous and periodic functions on $\mathbb{R}^d$. Let also $\phi \in \mathcal{S}(\mathbb{R}^d)$ be positive and such that $\phi(x) = 1$ for any $x \in [0,1]^d$. Since T is compactly supported, one has $T\,\mathrm{e}_m = T(\phi\,\mathrm{e}_m)$.

Now, for any $t > 0$ and in the Hilbert space $L^2([0,1]^d)$ one has

$$\begin{aligned}\big\|TE_{(1-\Delta_0)^{-d/2}}([0,t^{-1}])\big\|_2^2 &= \sum_{1+4\pi^2|m|^2 \geq t^{2/d}} \|T\,\mathrm{e}_m\|^2 \\ &\leq \sum_{|m|\geq t^{1/d}/2\pi} \|T(\phi\,\mathrm{e}_m)\|^2 \\ &= O(t^{-1})\end{aligned}$$

where the last estimate is provided by Lemma 5.4.12. The statement follows now directly from Lemma 5.4.9. □

Based on Theorem 5.4.10 let us finally provide a sketch of the proof of Theorem 5.4.5.

Proof of Theorem 5.4.5. 1) We shall assume without loss of generality that the compactly supported operator T has support in $[0,1]^d$. As already observed, the operator $V := (1-\Delta_0)^{-d/2}$ belongs to $\mathscr{L}_{1,\infty}$ and is strictly positive. In addition, one has shown in Theorem 5.4.13 that T is Δ_0-modulated, or more precisely that $T : L^2([0,1]^d) \to L^2([0,1]^d)$ is V-modulated. As a consequence of Theorem 5.4.10 one infers that $T \in \mathscr{L}_{1,\infty}\big(L^2([0,1]^d)\big)$, and then by the inclusion of $L^2([0,1]^d)$ into $L^2(\mathbb{R}^d)$ that $T \in \mathscr{L}_{1,\infty}\big(L^2(\mathbb{R}^d)\big)$ as well. In that respect it is worth noting that the eigenvalues of $T : L^2([0,1]^d) \to L^2([0,1]^d)$ and of $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ coincide since the subspace $L^2([0,1]^d)$ is left invariant by T.

Now, let $\{f_n\}$ be a rearrangement of the eigenfunctions $\{\mathrm{e}_m\}$ according to an increase of $|m|$. More precisely for any given $n \in \mathbb{N}$ we have $f_n = \mathrm{e}_{m_n}$ with $|m_n| \geq |m_{n'}|$ whenever $n > n'$. One can also observe that $|m_n| \cong n^{1/d}$. Then Theorem 5.4.10 implies that

$$\sum_{j=1}^{n} \lambda_j(T) = \sum_{j=1}^{n} \langle f_j, T f_j\rangle + O(1) = \sum_{|m|\leq n^{1/d}} \langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle + O(1). \tag{5.32}$$

2) For the initial statement, it remains to show that for any $t > 0$

$$\int_{|\xi|<t}\int_{\mathbb{R}^d} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x - \sum_{|m|\leq t} \langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle = O(1). \tag{5.33}$$

For that purpose, let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be positive and such that $\phi(x) = 1$ for any $x \in [0,1]^d$. We then have $T\,\mathrm{e}_m = T(\phi\,\mathrm{e}_m)$ and $[\mathscr{F}(\phi\,\mathrm{e}_m)](x) = [\mathscr{F}\phi](x-m)$. It then follows from the explicit formula (5.17) that

$$\langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot(\xi-m)}\, p_T(x,\xi)[\mathscr{F}\phi](\xi-m)\,\mathrm{d}\xi\,\mathrm{d}x.$$

By taking into account that $p_T(x,\xi)=0$ if $x\notin[0,1]^d$, one gets

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\int_{|\xi|<t} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x-\sum_{|m|\leq t}\langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle\Big|\\ &=\Big|\int_{\mathbb{R}^d}\int_{[0,1]^d} p_T(x,\xi)\Big(\sum_{|m|\leq t}\mathrm{e}^{2\pi i x\cdot(\xi-m)}[\mathscr{F}\phi](\xi-m)-\xi_{[0,t]}(|\xi|)\Big)\,\mathrm{d}x\,\mathrm{d}\xi\Big|\end{aligned}$$

Now, it has been shown in [LSZ, Lem. 11.4.4] that the term inside the big parenthesis can be further estimated and one gets

$$\sum_{|m|\leq t}\mathrm{e}^{2\pi i x\cdot(\xi-m)}[\mathscr{F}\phi](\xi-m)-\xi_{[0,t]}(|\xi|)=O\big(\langle t-|\xi|\rangle^{-d}\big)$$

for any $t>0$ and $\xi\in\mathbb{R}^d$, and uniformly in $x\in[0,1]^d$. It only remains then to estimate the term

$$\int_{\mathbb{R}^d}\int_{[0,1]^d}|p_T(x,\xi)|\langle t-|\xi|\rangle^{-d}\,\mathrm{d}x\,\mathrm{d}\xi.$$

It is again shown in the technical statement [LSZ, Lem. 11.4.5] that this term is uniformly bounded for $t>0$. By setting $t=n^{1/d}$ in (5.33) and by using (5.32) one directly obtains the statement contained in (5.23). □