Dixmier traces

S. Richard

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Contents

Chapter 1

Hilbert space and linear operators

The purpose of this first chapter is to introduce (or recall) many standard definitions related to the study of operators on a Hilbert space. Its content is mainly based on the first two chapters of the book [Amr].

1.1 Hilbert space

Definition 1.1.1. A (complex) Hilbert space $\mathcal{H}$ *is a vector space on* $\mathbb{C}$ *with a strictly positive scalar product (or inner product) which is complete for the associated norm*[1] *and which admits a countable orthonormal basis. The scalar product is denoted by* $\langle \cdot, \cdot \rangle$ *and the corresponding norm by* $\| \cdot \|$.

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the following properties hold:

(i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,

(ii) $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$,

(iii) $\|f\|^2 = \langle f, f \rangle \geq 0$, and $\|f\| = 0$ if and only if $f = 0$.

Note that $\overline{\langle g, f \rangle}$ means the complex conjugate of $\langle g, f \rangle$. Note also that the linearity in the second argument in (ii) is a matter of convention, many authors define the linearity in the first argument. In (iii) the norm of f is defined in terms of the scalar product $\langle f, f \rangle$. We emphasize that the scalar product can also be defined in terms of the norm of $\mathcal{H}$, this is the content of the *polarisation identity*:

$$4\langle f, g \rangle = \|f + g\|^2 - \|f - g\|^2 - i\|f + ig\|^2 + i\|f - ig\|^2. \tag{1.1}$$

From now on, the symbol $\mathcal{H}$ will always denote a Hilbert space.

[1] Recall that $\mathcal{H}$ is said to be complete if any Cauchy sequence in $\mathcal{H}$ has a limit in $\mathcal{H}$. More precisely, $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is a Cauchy sequence if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for any $n, m \geq N$. Then $\mathcal{H}$ is complete if for any such sequence there exists $f_\infty \in \mathcal{H}$ such that $\lim_{n \to \infty} \|f_n - f_\infty\| = 0$.

Examples 1.1.2. *(i)* $\mathcal{H} = \mathbb{C}^d$ *with* $\langle \alpha, \beta \rangle = \sum_{j=1}^d \overline{\alpha_j}\,\beta_j$ *for any* $\alpha, \beta \in \mathbb{C}^d$,

(ii) $\mathcal{H} = \ell_2(\mathbb{Z})$ *with* $\langle a, b \rangle = \sum_{j\in\mathbb{Z}} \overline{a_j}\, b_j$ *for any* $a, b \in \ell_2(\mathbb{Z})$,

(iii) $\mathcal{H} = L^2(\mathbb{R}^d)$ *with* $\langle f, g \rangle = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\,\mathrm{d}x$ *for any* $f, g \in L^2(\mathbb{R}^d)$.

Let us recall some useful inequalities: For any $f, g \in \mathcal{H}$ one has

$$|\langle f, g\rangle| \le \|f\|\,\|g\| \qquad \text{Schwarz inequality,} \tag{1.2}$$
$$\|f + g\| \le \|f\| + \|g\| \qquad \text{triangle inequality,} \tag{1.3}$$
$$\|f + g\|^2 \le 2\|f\|^2 + 2\|g\|^2, \tag{1.4}$$
$$\big|\|f\| - \|g\|\big| \le \|f - g\|. \tag{1.5}$$

The proof of these inequalities is standard and is left as a free exercise, see also [Amr, p. 3-4]. Let us also recall that $f, g \in \mathcal{H}$ are said to be *orthogonal* if $\langle f, g\rangle = 0$.

Definition 1.1.3. *A sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is* strongly convergent *to* $f_\infty \in \mathcal{H}$ *if* $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$*, or is* weakly convergent *to* $f_\infty \in \mathcal{H}$ *if for any* $g \in \mathcal{H}$ *one has* $\lim_{n\to\infty}\langle g, f_n - f_\infty\rangle = 0$*. One writes* $\text{s-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is strongly convergent, and* $\text{w-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is weakly convergent.*

Clearly, a strongly convergent sequence is also weakly convergent. The converse is not true.

Exercise 1.1.4. *In the Hilbert space* $L^2(\mathbb{R})$*, exhibit a sequence which is weakly convergent but not strongly convergent.*

Lemma 1.1.5. *Consider a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$*. One has*

$$\text{s-}\lim_{n\to\infty} f_n = f_\infty \iff \text{w-}\lim_{n\to\infty} f_n = f_\infty \ \textit{ and } \ \lim_{n\to\infty}\|f_n\| = \|f_\infty\|.$$

Proof. Assume first that $\text{s-}\lim_{n\to\infty} f_n = f_\infty$. By the Schwarz inequality one infers that for any $g \in \mathcal{H}$:

$$|\langle g, f_n - f_\infty\rangle| \le \|f_n - f_\infty\|\,\|g\| \to 0 \qquad \text{as } n \to \infty,$$

which means that $\text{w-}\lim_{n\to\infty} f_n = f_\infty$. In addition, by (1.5) one also gets

$$\big|\|f_n\| - \|f_\infty\|\big| \le \|f_n - f_\infty\| \to 0 \quad \text{as} \quad n \to \infty,$$

and thus $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$.

For the reverse implication, observe first that

$$\|f_n - f_\infty\|^2 = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle. \tag{1.6}$$

If $\text{w-}\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$, then the right-hand side of (1.6) converges to $\|f_\infty\|^2 + \|f_\infty\|^2 - \|f_\infty\|^2 - \|f_\infty\|^2 = 0$, so that $\text{s-}\lim_{n\to\infty} f_n = f_\infty$. □

Let us also note that if $\text{s-}\lim_{n\to\infty} f_n = f_\infty$ and $\text{s-}\lim_{n\to\infty} g_n = g_\infty$ then one has

$$\lim_{n\to\infty} \langle f_n, g_n\rangle = \langle f_\infty, g_\infty\rangle$$

by a simple application of the Schwarz inequality.

Exercise 1.1.6. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of an infinite dimensional Hilbert space. Show that* $\text{w-}\lim_{n\to\infty} e_n = 0$*, but that* $\text{s-}\lim_{n\to\infty} e_n$ *does not exist.*

Exercise 1.1.7. *Show that the limit of a strong or a weak Cauchy sequence is unique. Show also that such a sequence is bounded,* i.e. *if $\{f_n\}_{n\in\mathbb{N}}$ denotes this Cauchy sequence, then* $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$*.*

For the weak Cauchy sequence, the boundedness can be obtained from the following quite general result which will be useful later on. Its proof can be found in [Kat, Thm. III.1.29]. In the statement, Λ is simply a set.

Theorem 1.1.8 (Uniform boundedness principle)**.** *Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous maps*[2] *$\varphi_\lambda : \mathcal{H} \to [0,\infty)$ satisfying*

$$\varphi_\lambda(f+g) \leq \varphi_\lambda(f) + \varphi_\lambda(g) \qquad \forall f, g \in \mathcal{H}.$$

If the set $\{\varphi_\lambda(f)\}_{\lambda\in\Lambda} \subset [0,\infty)$ is bounded for any fixed $f \in \mathcal{H}$, then the family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded, i.e. *there exists $c > 0$ such that* $\sup_\lambda \varphi_\lambda(f) \leq c$ *for any $f \in \mathcal{H}$ with $\|f\| = 1$.*

In the next definition, we introduce the notion of subspace of a Hilbert space.

Definition 1.1.9. A subspace $\mathcal{M}$ *of a Hilbert space $\mathcal{H}$ is a linear subset of $\mathcal{H}$, or more precisely $\forall f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$ one has $f + \alpha g \in \mathcal{M}$. The subspace $\mathcal{M}$ is* closed *if any Cauchy sequence in $\mathcal{M}$ converges strongly in $\mathcal{M}$.*

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$.

Examples 1.1.10. *(i) If $f_1, \dots, f_n \in \mathcal{H}$, then $\mathsf{Span}(f_1, \dots, f_n)$ is the closed vector space generated by the linear combinations of $f_1, \dots f_n$. $\mathsf{Span}(f_1, \dots, f_n)$ is a closed subspace.*

(ii) If $\mathcal{M}$ is a subset of $\mathcal{H}$, then

$$\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0, \forall g \in \mathcal{M}\} \tag{1.7}$$

is a closed subspace of $\mathcal{H}$.

Exercise 1.1.11. *Check that in the above example the set $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$.*

[2] φ_λ is continuous if $\varphi_\lambda(f_n) \to \varphi_\lambda(f_\infty)$ whenever $\text{s-}\lim_{n\to\infty} f_n = f_\infty$.

Exercise 1.1.12. *Check that a subspace $\mathcal{M} \subset \mathcal{H}$ is dense in $\mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$.*

If $\mathcal{M}$ is a subset of $\mathcal{H}$ the closed subspace $\mathcal{M}^\perp$ is called *the orthocomplement of $\mathcal{M}$ in $\mathcal{H}$*. The following result is important in the setting of Hilbert spaces. Its proof is not complicated but a little bit lengthy, we thus refer to [Amr, Prop. 1.7].

Proposition 1.1.13 (Projection Theorem)**.** *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then, for any $f \in \mathcal{H}$ there exist a unique $f_1 \in \mathcal{M}$ and a unique $f_2 \in \mathcal{M}^\perp$ such that $f = f_1 + f_2$.*

Let us close this section with the so-called Riesz Lemma. For that purpose, recall first that the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ consists in the set of all bounded linear functionals on $\mathcal{H}$, *i.e.* $\mathcal{H}^*$ consists in all mappings $\varphi : \mathcal{H} \to \mathbb{C}$ satisfying for any $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$

(i) $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g)$, (linearity)

(ii) $|\varphi(f)| \leq c\|f\|$, (boundedness)

where c is a constant independent of f. One then sets

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{0 \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{\|f\|}.$$

Clearly, if $g \in \mathcal{H}$, then g defines an element φ_g of $\mathcal{H}^*$ by setting $\varphi_g(f) := \langle g, f\rangle$. Indeed φ_g is linear and one has

$$\|\varphi_g\|_{\mathcal{H}^*} := \sup_{0 \neq f \in \mathcal{H}} \frac{1}{\|f\|} |\langle g, f\rangle| \leq \sup_{0 \neq f \in \mathcal{H}} \frac{1}{\|f\|} \|g\| \, \|f\| = \|g\|.$$

In fact, note that $\|\varphi_g\|_{\mathcal{H}^*} = \|g\|$ since $\frac{1}{\|g\|}\varphi_g(g) = \frac{1}{\|g\|}\|g\|^2 = \|g\|$.

The following statement shows that any element $\varphi \in \mathcal{H}^*$ can be obtained from an element $g \in \mathcal{H}$. It corresponds thus to a converse of the previous construction.

Lemma 1.1.14 (Riesz Lemma)**.** *For any $\varphi \in \mathcal{H}^*$, there exists a unique $g \in \mathcal{H}$ such that for any $f \in \mathcal{H}$*

$$\varphi(f) = \langle g, f\rangle.$$

In addition, g satisfies $\|\varphi\|_{\mathcal{H}^*} = \|g\|$.

Since the proof is quite standard, we only sketch it and leave the details to the reader, see also [Amr, Prop. 1.8].

Sketch of the proof. If $\varphi \equiv 0$, then one can set $g := 0$ and observe trivially that $\varphi = \varphi_g$.

If $\varphi \neq 0$, let us first define $\mathcal{M} := \{f \in \mathcal{H} \mid \varphi(f) = 0\}$ and observe that $\mathcal{M}$ is a closed subspace of $\mathcal{H}$. One also observes that $\mathcal{M} \neq \mathcal{H}$ since otherwise $\varphi \equiv 0$. Thus, let $h \in \mathcal{H}$ such that $\varphi(h) \neq 0$ and decompose $h = h_1 + h_2$ with $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^\perp$ by Proposition 1.1.13. One infers then that $\varphi(h_2) = \varphi(h) \neq 0$.

For arbitrary $f \in \mathcal{H}$ one can consider the element $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{H}$ and observe that $\varphi\big(f - \frac{\varphi(f)}{\varphi(h_2)} h_2\big) = 0$. One deduces that $f - \frac{\varphi(f)}{\varphi(h_2)} h_2$ belongs to $\mathcal{M}$, and since $h_2 \in \mathcal{M}^\perp$ one infers that

$$\varphi(f) = \frac{\varphi(h_2)}{\|h_2\|^2} \langle h_2, f\rangle.$$

One can thus set $g := \frac{\overline{\varphi(h_2)}}{\|h_2\|^2} h_2 \in \mathcal{H}$ and easily obtain the remaining parts of the statement. □

As a consequence of the previous statement, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

Exercise 1.1.15. *Check that this identification is not linear but* anti-linear.

1.2 Bounded linear operators

First of all, let us recall that a linear map B between two complex vector spaces $\mathcal{M}$ and $\mathcal{N}$ satisfies $B(f + \alpha g) = Bf + \alpha Bg$ for all $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Definition 1.2.1. *A map $B : \mathcal{H} \to \mathcal{H}$ is* a bounded linear operator *if $B : \mathcal{H} \to \mathcal{H}$ is a linear map, and if there exists $c > 0$ such that $\|Bf\| \leq c\|f\|$ for all $f \in \mathcal{H}$. The set of* all bounded linear operators on $\mathcal{H}$ *is denoted by $\mathscr{B}(\mathcal{H})$.*

For any $B \in \mathscr{B}(\mathcal{H})$, one sets

$$\begin{aligned} \|B\| &:= \inf\{c > 0 \mid \|Bf\| \leq c\|f\| \ \forall f \in \mathcal{H}\} \\ &= \sup_{0 \neq f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}. \end{aligned} \tag{1.8}$$

and call it *the norm of* B. Note that the same notation is used for the norm of an element of $\mathcal{H}$ and for the norm of an element of $\mathscr{B}(\mathcal{H})$, but this does not lead to any confusion. Let us also introduce the *range* of an operator $B \in \mathscr{B}(\mathcal{H})$, namely

$$\mathsf{Ran}(B) := B\mathcal{H} = \{f \in \mathcal{H} \mid f = Bg \text{ for some } g \in \mathcal{H}\}. \tag{1.9}$$

This notion will be important when the inverse of an operator will be discussed.

Exercise 1.2.2. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two dense subspaces of $\mathcal{H}$, and let $B \in \mathscr{B}(\mathcal{H})$. Show that*

$$\|B\| = \sup_{f \in \mathcal{M}_1, g \in \mathcal{M}_2 \text{ with } \|f\| = \|g\| = 1} |\langle f, Bg\rangle|. \tag{1.10}$$

Exercise 1.2.3. *Show that $\mathscr{B}(\mathcal{H})$ is a complete normed algebra and that the inequality*

$$\|AB\| \leq \|A\| \, \|B\| \tag{1.11}$$

holds for any $A, B \in \mathscr{B}(\mathcal{H})$.

An additional structure can be added to $\mathscr{B}(\mathcal{H})$: an involution. More precisely, for any $B \in \mathscr{B}(\mathcal{H})$ and any $f, g \in \mathcal{H}$ one sets

$$\langle B^* f, g\rangle := \langle f, Bg\rangle. \tag{1.12}$$

Exercise 1.2.4. *For any $B \in \mathscr{B}(\mathcal{H})$ show that*

(i) B^* is uniquely defined by (1.12) and satisfies $B^* \in \mathscr{B}(\mathcal{H})$ with $\|B^*\| = \|B\|$,

(ii) $(B^*)^* = B$,

(iii) $\|B^*B\| = \|B\|^2$,

(iv) If $A \in \mathscr{B}(\mathcal{H})$, then $(AB)^* = B^*A^*$.

The operator B^* is called *the adjoint of* B, and the proof the unicity in (i) involves the Riesz Lemma. A complete normed algebra endowed with an involution for which the property (iii) holds is called a C^*-algebra. In particular $\mathscr{B}(\mathcal{H})$ is a C^*-algebra. Such algebras have a well-developed and deep theory, see for example [Mur]. However, we shall not go further in this direction in this course.

We have already considered two distinct topologies on $\mathcal{H}$, namely the strong and the weak topology. On $\mathscr{B}(\mathcal{H})$ there exist several topologies, for the time being we consider only three of them.

Definition 1.2.5. *A sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is* uniformly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$, is* strongly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f \in \mathcal{H}$ one has $\lim_{n\to\infty} \|B_n f - B_\infty f\| = 0$, or is* weakly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f, g \in \mathcal{H}$ one has $\lim_{n\to\infty}\langle f, B_n g - B_\infty g\rangle = 0$. In these cases, one writes respectively $u - \lim_{n\to\infty} B_n = B_\infty$, $s - \lim_{n\to\infty} B_n = B_\infty$ and $w - \lim_{n\to\infty} B_n = B_\infty$.*

Clearly, uniform convergence implies strong convergence, and strong convergence implies weak convergence. The reverse statements are not true. Note that if $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is weakly convergent, then the sequence $\{B_n^*\}_{n\in\mathbb{N}}$ of its adjoint operators is also weakly convergent. However, the same statement does not hold for a strongly convergent sequence. Finally, we shall not prove but often use that $\mathscr{B}(\mathcal{H})$ is also weakly and strongly closed. In other words, any weakly (or strongly) Cauchy sequence in $\mathscr{B}(\mathcal{H})$ converges in $\mathscr{B}(\mathcal{H})$.

Exercise 1.2.6. *Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ be two strongly convergent sequence in $\mathscr{B}(\mathcal{H})$, with limits A_∞ and B_∞ respectively. Show that the sequence $\{A_nB_n\}_{n\in\mathbb{N}}$ is strongly convergent to the element $A_\infty B_\infty$.*

Let us close this section with some information about the inverse of a bounded operator.

Definition 1.2.7. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* invertible *if the equation $Bf = 0$ only admits the solution $f = 0$. In such a case, there exists a linear map $B^{-1} : \mathsf{Ran}(B) \to \mathcal{H}$ which satisfies $B^{-1}Bf = f$ for any $f \in \mathcal{H}$, and $BB^{-1}g = g$ for any $g \in \mathsf{Ran}(B)$. If B is invertible and $\mathsf{Ran}(B) = \mathcal{H}$, then $B^{-1} \in \mathscr{B}(\mathcal{H})$ and B is said to be* invertible in $\mathscr{B}(\mathcal{H})$ *(or* boundedly invertible*).*

Note that the two conditions B invertible and $\mathsf{Ran}(B) = \mathcal{H}$ imply $B^{-1} \in \mathscr{B}(\mathcal{H})$ is a consequence of the Closed graph Theorem[3]. In the sequel, we shall use the notation $\mathbf{1} \in \mathscr{B}(\mathcal{H})$ for the operator defined on any $f \in \mathcal{H}$ by $\mathbf{1}f = f$, and $\mathbf{0} \in \mathscr{B}(\mathcal{H})$ for the operator defined by $\mathbf{0}f = 0$.

The next statement is very useful in applications, and holds in a much more general context. Its proof is classical and can be found in every textbook.

Lemma 1.2.8 (Neumann series)**.** *If $B \in \mathscr{B}(\mathcal{H})$ and $\|B\| < 1$, then the operator $(\mathbf{1} - B)$ is invertible in $\mathscr{B}(\mathcal{H})$, with*

$$(\mathbf{1} - B)^{-1} = \sum_{n=0}^{\infty} B^d,$$

and with $\big\|(\mathbf{1} - B)^{-1}\big\| \leq (1 - \|B\|)^{-1}$. The series converges in the uniform norm of $\mathscr{B}(\mathcal{H})$.

Note that we have used the identity $B^0 = \mathbf{1}$.

1.3 Special classes of bounded linear operators

In this section we provide some information on some subsets of $\mathscr{B}(\mathcal{H})$. We start with some operators which will play an important role in the sequel.

Definition 1.3.1. *An operator $B \in \mathscr{B}(\mathcal{H})$ is called* self-adjoint *if $B^* = B$, or equivalently if for any $f, g \in \mathcal{H}$ one has*

$$\langle f, Bg\rangle = \langle Bf, g\rangle. \tag{1.13}$$

For these operators the computation of their norm can be simplified (see also Exercise 1.2.2) :

Exercise 1.3.2. *If $B \in \mathscr{B}(\mathcal{H})$ is self-adjoint and if $\mathcal{M}$ is a dense subspace in $\mathcal{H}$, show that*

$$\|B\| = \sup_{f\in\mathcal{M},\ \|f\|=1} |\langle f, Bf\rangle|. \tag{1.14}$$

A special set of self-adjoint operators is provided by the set of orthogonal projections:

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[3]Closed graph theorem: If $(B, \mathcal{H})$ is a closed operator (see further on for this definition), then $B \in \mathscr{B}(\mathcal{H})$, see for example [Kat, Sec. III.5.4]. This can be studied as an Extension.

Definition 1.3.3. *An element $P \in \mathscr{B}(\mathcal{H})$ is* a projection *if $P = P^2$. This projection is* orthogonal *if in addition $P = P^*$. The set of all orthogonal projections is denoted by $\mathscr{P}(\mathcal{H})$.*

It not difficult to check that there is a one-to-one correspondence between the set of closed subspaces of $\mathcal{H}$ and the set of orthogonal projections in $\mathscr{B}(\mathcal{H})$. Indeed, any orthogonal projection P defines a closed subspace $\mathcal{M} := P\mathcal{H}$. Conversely by taking the projection Theorem (Proposition 1.1.13) into account one infers that for any closed subspace $\mathcal{M}$ one can define an orthogonal projection P with $P\mathcal{H} = \mathcal{M}$.

We gather in the next exercise some easy relations between orthogonal projections and the underlying closed subspaces. For that purpose we use the notation $P_\mathcal{M}, P_\mathcal{N}$ for the orthogonal projections on the closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$.

Exercise 1.3.4. *Show the following relations:*

(i) If $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M}$, then $P_\mathcal{M} P_\mathcal{N}$ is an orthogonal projection and the associated closed subspace is $\mathcal{M} \cap \mathcal{N}$,

(ii) If $\mathcal{M} \subset \mathcal{N}$, then $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M} = P_\mathcal{M}$,

(iii) If $\mathcal{M} \perp \mathcal{N}$, then $P_\mathcal{M} P_\mathcal{N} = P_\mathcal{N} P_\mathcal{M} = \mathbf{0}$, and $P_{\mathcal{M}\oplus\mathcal{N}} = P_\mathcal{M} + P_\mathcal{N}$,

(iv) If $P_\mathcal{M} P_\mathcal{N} = \mathbf{0}$, then $\mathcal{M} \perp \mathcal{N}$.

Note that the operators introduced so far are special instances of normal operators:

Definition 1.3.5. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* normal *if the equality $BB^* = B^*B$ holds.*

Clearly, bounded self-adjoint operators are normal. Other examples of normal operators are unitary operators, as considered now. In fact, we introduce not only unitary operators, but also isometries and partial isometries. For that purpose, we recall that $\mathbf{1}$ denotes the identify operator in $\mathscr{B}(\mathcal{H})$.

Definition 1.3.6. *An element $U \in \mathscr{B}(\mathcal{H})$ is* a unitary operator *if $UU^* = \mathbf{1}$ and if $U^*U = \mathbf{1}$.*

Note that if U is unitary, then U is invertible in $\mathscr{B}(\mathcal{H})$ with $U^{-1} = U^*$. Indeed, observe first that $Uf = 0$ implies $f = U^*(Uf) = U^*0 = 0$. Secondly, for any $g \in \mathcal{H}$, one has $g = U(U^*g)$, and thus $\mathsf{Ran}(U) = \mathcal{H}$. Finally, the equality $U^{-1} = U^*$ follows from the unicity of the inverse.

More generally, an element $V \in \mathscr{B}(\mathcal{H})$ is called *an isometry* if the equality

$$V^*V = \mathbf{1} \tag{1.15}$$

holds. Clearly, a unitary operator is an instance of an isometry. For isometries the following properties can easily be obtained.

Proposition 1.3.7. *a) Let $V \in \mathscr{B}(\mathcal{H})$ be an isometry. Then*

(i) *V preserves the scalar product, namely $\langle Vf, Vg\rangle = \langle f, g\rangle$ for any $f, g \in \mathcal{H}$,*

(ii) *V preserves the norm, namely $\|Vf\| = \|f\|$ for any $f \in \mathcal{H}$,*

(iii) *If $\mathcal{H} \neq \{0\}$ then $\|V\| = 1$,*

(iv) *VV^* is the orthogonal projection on $\mathsf{Ran}(V)$,*

(v) *V is invertible (in the sense of Definition 1.2.7),*

(vi) *The adjoint V^* satisfies $V^*f = V^{-1}f$ if $f \in \mathsf{Ran}(V)$, and $V^*g = 0$ if $g \perp \mathsf{Ran}(V)$.*

b) An element $W \in \mathscr{B}(\mathcal{H})$ is an isometry if and only if $\|Wf\| = \|f\|$ for all $f \in \mathcal{H}$.

Exercise 1.3.8. *Provide a proof for the previous proposition (as well as the proof of the next proposition).*

More generally one defines *a partial isometry* as an element $W \in \mathscr{B}(\mathcal{H})$ such that

$$W^*W = P \tag{1.16}$$

with P an orthogonal projection. Again, unitary operators or isometries are special examples of partial isometries.

As before the following properties of partial isometries can be easily proved.

Proposition 1.3.9. *Let $W \in \mathscr{B}(\mathcal{H})$ be a partial isometry as defined in* (1.16)*. Then*

(i) *One has $WP = W$ and $\langle Wf, Wg\rangle = \langle Pf, Pg\rangle$ for any $f, g \in \mathcal{H}$,*

(ii) *If $P \neq \mathbf{0}$ then $\|W\| = 1$,*

(iii) *WW^* is the orthogonal projection on $\mathsf{Ran}(W)$.*

For a partial isometry W one usually calls *initial set projection* the orthogonal projection defined by W^*W and by *final set projection* the orthogonal projection defined by WW^*.

Let us now introduce a last subset of bounded operators, namely the ideal of *compact operators*. For that purpose, consider first any family $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$ and for any $f \in \mathcal{H}$ one sets

$$Af := \sum_{j=1}^{N} \langle g_j, f\rangle\, h_j. \tag{1.17}$$

Then $A \in \mathscr{B}(\mathcal{H})$, and $\mathsf{Ran}(A) \subset \mathsf{Span}(h_1, \dots, h_N)$. Such an operator A is called *a finite rank operator*. In fact, any operator $B \in \mathscr{B}(\mathcal{H})$ with $\dim\big(\mathsf{Ran}(B)\big) < \infty$ is a finite rank operator.

Exercise 1.3.10. *For the operator A defined in* (1.17)*, give an upper estimate for $\|A\|$ and compute A^*.*

Definition 1.3.11. *An element* $B \in \mathscr{B}(\mathcal{H})$ *is* a compact operator *if there exists a family* $\{A_n\}_{n\in\mathbb{N}}$ *of finite rank operators such that* $\lim_{n\to\infty}\|B - A_n\| = 0$*. The set of all compact operators is denoted by* $\mathscr{K}(\mathcal{H})$*.*

The following proposition contains sone basic properties of $\mathscr{K}(\mathcal{H})$. Its proof can be obtained by playing with families of finite rank operators.

Proposition 1.3.12. *The following properties hold:*

(i) $B \in \mathscr{K}(\mathcal{H}) \Longleftrightarrow B^* \in \mathscr{K}(\mathcal{H})$,

(ii) $\mathscr{K}(\mathcal{H})$ *is a* $*$*-algebra, complete for the norm* $\|\cdot\|$,

(iii) If $B \in \mathscr{K}(\mathcal{H})$ *and* $A \in \mathscr{B}(\mathcal{H})$*, then* AB *and* BA *belong to* $\mathscr{K}(\mathcal{H})$*.*

As a consequence, $\mathscr{K}(\mathcal{H})$ is a C^*-algebra and an ideal of $\mathscr{B}(\mathcal{H})$. In fact, compact operators have the nice property of improving some convergences, as shown in the next statement.

Proposition 1.3.13. *Let* $K \in \mathscr{K}(\mathcal{H})$*.*

(i) If $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is a weakly convergent sequence with limit* $f_\infty \in \mathcal{H}$*, then the sequence* $\{Kf_n\}_{n\in\mathbb{N}}$ *strongly converges to* Kf_∞*,*

(ii) If the sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ *strongly converges to* $B_\infty \in \mathscr{B}(\mathcal{H})$*, then the sequences* $\{B_nK\}_{n\in\mathbb{N}}$ *and* $\{KB_n^*\}_{n\in\mathbb{N}}$ *converge in norm to* $B_\infty K$ *and* KB_∞^**, respectively.*

Proof. a) Let us first set $\varphi_n := f_n - f_\infty$ and observe that $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$. By an application of the uniform boundedness principle, see Theorem 1.1.8, it follows that $\{\|\varphi_n\|\}_{n\in\mathbb{N}}$ is bounded, *i.e.* there exists $M > 0$ such that $\|\varphi_n\| \leq M$ for any $n \in \mathbb{N}$. Since K is compact, for any $\varepsilon > 0$ there exists a finite rank operator A of the form given in (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then one has

$$\|K\varphi_n\| \leq \|(K-A)\varphi_n\| + \|A\varphi_n\| \leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |\langle g_j, \varphi_n\rangle|\,\|h_j\|.$$

Since $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$ there exists $n_0 \in \mathbb{N}$ such that $|\langle g_j, \varphi_n\rangle|\,\|h_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1,\dots,N\}$ and all $n \geq n_0$. As a consequence, one infers that $\|K\varphi_n\| \leq \varepsilon$ for all $n \geq n_0$, or in other words $\mathrm{s-}\lim_{n\to\infty} K\varphi_n = 0$.

b) Let us set $C_n := B_n - B_\infty$ such that $\mathrm{s-}\lim_{n\to\infty} C_n = \mathbf{0}$. As before, there exists $M > 0$ such that $\|C_n\| \leq M$ for any $n \in \mathbb{N}$. For any $\varepsilon > 0$ consider a finite rank operator A of the form (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then observe that for any

$f \in \mathcal{H}$

$$\begin{aligned}\|C_n K f\| &\leq M\|(K-A)f\| + \|C_n A f\| \\ &\leq M\|K-A\|\,\|f\| + \sum_{j=1}^{N} |\langle g_j, f\rangle|\,\|C_n h_j\| \\ &\leq \Big\{M\|K-A\| \ + \sum_{j=1}^{N} \|g_j\|\,\|C_n h_j\|\Big\}\|f\|.\end{aligned}$$

Since C_n strongly converges to $\mathbf{0}$ one can then choose $n_0 \in \mathbb{N}$ such that $\|g_j\|\,\|C_n h_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1, \ldots N\}$ and all $n \geq n_0$. One then infers that $\|C_n K\| \leq \varepsilon$ for any $n \geq n_0$, which means that the sequence $\{C_n K\}_{n\in\mathbb{N}}$ uniformly converges to $\mathbf{0}$. The statement about $\{KB_n^*\}_{n\in\mathbb{N}}$ can be proved analogously by taking the equality $\|KB_n^* - KB_\infty^*\| = \|B_n K^* - B_\infty K^*\|$ into account and by remembering that K^* is compact as well. □

Exercise 1.3.14. *Check that an orthogonal projection P is a compact operator if and only if $P\mathcal{H}$ is of finite dimension.*

There are various subalgebras of $\mathscr{K}(\mathcal{H})$, for example the algebra of Hilbert-Schmidt operators, the algebra of trace class operators, and more generally the Schatten classes. Note that these algebras are not closed with respect to the norm $\|\cdot\|$ but with respect to some stronger norms $|||\cdot|||$. These algebras are ideals in $\mathscr{B}(\mathcal{H})$. In the following chapter these subalgebras will be extensively studied.

1.4 Unbounded, closed, and self-adjoint operators

Even if unbounded operators will not play an important role in the sequel, they might appear from time to time. For that reason, we gather in this section a couple of important definitions related to them. Obviously, the following definitions and results are also valid for bounded linear operators.

Definition 1.4.1. *A* linear operator *on $\mathcal{H}$ is a pair $\big(A, \mathsf{D}(A)\big)$, where $\mathsf{D}(A)$ is a subspace of $\mathcal{H}$ and A is a linear map from $\mathsf{D}(A)$ to $\mathcal{H}$. $\mathsf{D}(A)$ is called* the domain of A. *One says that the operator $\big(A, \mathsf{D}(A)\big)$ is* densely defined *if $\mathsf{D}(A)$ is dense in $\mathcal{H}$.*

Note that one often just says *the linear operator A*, but that its domain $\mathsf{D}(A)$ is implicitly taken into account. For such an operator, its range $\mathsf{Ran}(A)$ is defined by

$$\mathsf{Ran}(A) := A\,\mathsf{D}(A) = \{f \in \mathcal{H} \mid f = Ag \text{ for some } g \in \mathsf{D}(A)\}.$$

In addition, one defines the kernel $\mathsf{Ker}(A)$ of A by

$$\mathsf{Ker}(A) := \{f \in \mathsf{D}(A) \mid Af = 0\}.$$

Let us also stress that the sum $A + B$ for two linear operators is *a priori* only defined on the subspace $\mathsf{D}(A) \cap \mathsf{D}(B)$, and that the product AB is *a priori* defined only on the subspace $\{f \in \mathsf{D}(B) \mid Bf \in \mathsf{D}(A)\}$. These two sets can be very small.

Example 1.4.2. *Let* $\mathcal{H} := L^2(\mathbb{R})$ *and consider the operator* X *defined by* $[Xf](x) = xf(x)$ *for any* $x \in \mathbb{R}$. *Clearly,* $\mathsf{D}(X) = \{f \in \mathcal{H} \mid \int_{\mathbb{R}} |xf(x)|^2 \,\mathrm{d}x < \infty\} \subsetneq \mathcal{H}$. *In addition, by considering the family of functions* $\{f_y\}_{y\in\mathbb{R}} \subset \mathsf{D}(X)$ *with* $f_y(x) := 1$ *in* $x \in [y, y+1]$ *and* $f_y(x) = 0$ *if* $x \notin [y, y+1]$, *one easily observes that* $\|f_y\| = 1$ *but* $\sup_{y\in\mathbb{R}} \|Xf_y\| = \infty$, *which can be compared with* (1.8).

Clearly, a linear operator A can be defined on several domains. For example the operator X of the previous example is well-defined on the Schwartz space $\mathcal{S}(\mathbb{R})$, or on the set $C_c(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support, or on the space $\mathsf{D}(X)$ mentioned in the previous example. More generally, one has:

Definition 1.4.3. *For any pair of linear operators* $\big(A, \mathsf{D}(A)\big)$ *and* $\big(B, \mathsf{D}(B)\big)$ *satisfying* $\mathsf{D}(A) \subset \mathsf{D}(B)$ *and* $Af = Bf$ *for all* $f \in \mathsf{D}(A)$, *one says that* $\big(B, \mathsf{D}(B)\big)$ *is* an extension of $\big(A, \mathsf{D}(A)\big)$ *to* $\mathsf{D}(B)$, *or that* $\big(A, \mathsf{D}(A)\big)$ *is the restriction of* $\big(B, \mathsf{D}(B)\big)$ *to* $\mathsf{D}(A)$.

Let us now note that if $\big(A, \mathsf{D}(A)\big)$ is densely defined and if there exists $c \in \mathbb{R}$ such that $\|Af\| \leq c\|f\|$ for all $f \in \mathsf{D}(A)$, then there exists a natural continuous extension $\overline{A}$ of A with $\mathsf{D}(\overline{A}) = \mathcal{H}$. This extension satisfies $\overline{A} \in \mathscr{B}(\mathcal{H})$ with $\|\overline{A}\| \leq c$, and is called *the closure of the operator* A.

Exercise 1.4.4. *Work on the details of this extension.*

Let us now consider a similar construction but in the absence of a constant $c \in \mathbb{R}$ such that $\|Af\| \leq c\|f\|$ for all $f \in \mathsf{D}(A)$. More precisely, consider an arbitrary densely defined operator $\big(A, \mathsf{D}(A)\big)$. Then for any $f \in \mathcal{H}$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ strongly converging to f. Note that the sequence $\{Af_n\}_{n\in\mathbb{N}}$ will not be Cauchy in general. However, let us assume that this sequence is strongly Cauchy, *i.e.* for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|Af_n - Af_m\| < \varepsilon$ for any $n, m \geq N$. Since $\mathcal{H}$ is complete, this Cauchy sequence has a limit, which we denote by h, and it would then be natural to set $\overline{A}f = h$. In short, one would have $\overline{A}f := \mathrm{s-}\lim_{n\to\infty} Af_n$. It is easily observed that this definition is meaningful if and only if by choosing a different sequence $\{f_n'\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ strongly convergent to f and also defining a Cauchy sequence $\{Af_n'\}_{n\in\mathbb{N}}$ then $\mathrm{s-}\lim_{n\to\infty} Af_n' = \mathrm{s-}\lim_{n\to\infty} Af_n$. If this condition holds, then $\overline{A}f$ is well-defined. Observe in addition that the previous equality can by rewritten as $\mathrm{s-}\lim_{n\to\infty} A(f_n - f_n') = 0$, which leads naturally to the following definition.

Definition 1.4.5. *A linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* closable *if for any sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathsf{D}(A)$ *satisfying* $\mathrm{s-}\lim_{n\to\infty} f_n = 0$ *and such that* $\{Af_n\}_{n\in\mathbb{N}}$ *is strongly Cauchy, then* $\mathrm{s-}\lim_{n\to\infty} Af_n = 0$.

As shown before this definition, in such a case one can define an extension $\overline{A}$ of A with $\mathsf{D}(\overline{A})$ given by the sets of $f \in \mathcal{H}$ such that there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ with $\mathrm{s-}\lim_{n\to\infty} f_n = f$ and such that $\{Af_n\}_{n\in\mathbb{N}}$ is strongly Cauchy. For such an element f one sets $\overline{A}f = \mathrm{s-}\lim_{n\to\infty} Af_n$, and the extension $\big(\overline{A}, \mathsf{D}(\overline{A})\big)$ is called the closure of A.

In relation with the previous construction the following definition is now natural:

Definition 1.4.6. *An linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* closed *if for any sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A)$ *with* $\mathrm{s-}\lim_{n\to\infty} f_n = f \in \mathcal{H}$ *and such that* $\{Af_n\}_{n\in\mathbb{N}}$ *is strongly Cauchy, then one has* $f \in \mathsf{D}(A)$ *and* $\mathrm{s-}\lim_{n\to\infty} Af_n = Af$.

Let us now come back to the notion of the adjoint of an operator. This concept is slightly more subtle for unbounded operators than in the bounded case.

Definition 1.4.7. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. The adjoint A^* of A *is the operator defined by*

$$\mathsf{D}(A^*) := \big\{ f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f^*, g\rangle = \langle f, Ag\rangle \text{ for all } g \in \mathsf{D}(A)\big\}$$

and $A^* f := f^*$ *for all* $f \in \mathsf{D}(A^*)$.

Let us note that the density of $\mathsf{D}(A)$ is necessary to ensure that A^* is well-defined. Indeed, if f_1^*, f_2^* satisfy for all $g \in \mathsf{D}(A)$

$$\langle f_1^*, g\rangle = \langle f, Ag\rangle = \langle f_2^*, g\rangle,$$

then $\langle f_1^* - f_2^*, g\rangle = 0$ for all $g \in \mathsf{D}(A)$, and this equality implies $f_1^* = f_2^*$ only if $\mathsf{D}(A)$ is dense in $\mathcal{H}$. Note also that once $\big(A^*, \mathsf{D}(A^*)\big)$ is defined, one has

$$\langle A^* f, g\rangle = \langle f, Ag\rangle \qquad \forall f \in \mathsf{D}(A^*) \text{ and } \forall g \in \mathsf{D}(A).$$

Exercise 1.4.8. *Show that if* $\big(A, \mathsf{D}(A)\big)$ *is closable, then* $\mathsf{D}(A^*)$ *is dense in* $\mathcal{H}$.

Some relations between A and its adjoint A^* are gathered in the following lemma.

Lemma 1.4.9. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. *Then*

(i) $\big(A^*, \mathsf{D}(A^*)\big)$ *is closed,*

(ii) One has $\mathsf{Ker}(A^*) = \mathsf{Ran}(A)^\perp$,

(iii) If $\big(B, \mathsf{D}(B)\big)$ *is an extension of* $\big(A, \mathsf{D}(A)\big)$, *then* $\big(A^*, \mathsf{D}(A^*)\big)$ *is an extension of* $\big(B^*, \mathsf{D}(B^*)\big)$.

Proof. a) Consider $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(A^*)$ such that $\mathrm{s-}\lim_{n\to\infty} f_n = f \in \mathcal{H}$ and such that $\mathrm{s-}\lim_{n\to\infty} A^* f_n = h \in \mathcal{H}$. Then for each $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = \lim_{n\to\infty} \langle f_n, Ag\rangle = \lim_{n\to\infty} \langle A^* f_n, g\rangle = \langle h, g\rangle.$$

Hence $f \in \mathsf{D}(A^*)$ and $A^* f = h$, which proves that A^* is closed.

b) Let $f \in \mathsf{Ker}(A^*)$, *i.e.* $f \in \mathsf{D}(A^*)$ and $A^* f = 0$. Then, for all $g \in \mathsf{D}(A)$, one has

$$0 = \langle A^* f, g\rangle = \langle f, Ag\rangle$$

meaning that $f \in \mathsf{Ran}(A)^\perp$. Conversely, if $f \in \mathsf{Ran}(A)^\perp$, then for all $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = 0 = \langle 0, g\rangle$$

meaning that $f \in \mathsf{D}(A^*)$ and $A^* f = 0$, by the definition of the adjoint of A.

c) Consider $f \in \mathsf{D}(B^*)$ and observe that $\langle B^* f, g\rangle = \langle f, Bg\rangle$ for any $g \in \mathsf{D}(B)$. Since $\big(B, \mathsf{D}(B)\big)$ is an extension of $\big(A, \mathsf{D}(A)\big)$, one infers that $\langle B^* f, g\rangle = \langle f, Ag\rangle$ for any $g \in \mathsf{D}(A)$. Now, this equality means that $f \in \mathsf{D}(A^*)$ and that $A^* f = B^* f$, or more explicitly that A^* is defined on the domain of B^* and coincide with this operator on this domain. This means precisely that $\big(A^*, \mathsf{D}(A^*)\big)$ is an extension of $\big(B^*, \mathsf{D}(B^*)\big)$. □

Let us finally introduce the analogue of the bounded self-adjoint operators but in the unbounded setting. These operators play a key role in quantum mechanics and their study is very well developed.

Definition 1.4.10. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* self-adjoint *if* $\mathsf{D}(A^*) = \mathsf{D}(A)$ *and* $A^* f = Af$ *for all* $f \in \mathsf{D}(A)$.

Note that as a consequence of Lemma 1.4.9.(i) a self-adjoint operator is always closed. Recall also that in the bounded case, a self-adjoint operator was characterized by the equality

$$\langle Af, g\rangle = \langle f, Ag\rangle \tag{1.18}$$

for any $f, g \in \mathcal{H}$. In the unbounded case, such an equality still holds if $f, g \in \mathsf{D}(A)$. However, let us emphasize that (1.18) does not completely characterize a self-adjoint operator. In fact, a densely defined operator $\big(A, \mathsf{D}(A)\big)$ satisfying (1.18) is called *a symmetric operator*, and self-adjoint operators are special instances of symmetric operators (but not all symmetric operators are self-adjoint). For a symmetric operator the adjoint operator $\big(A^*, \mathsf{D}(A^*)\big)$ is an extension of $\big(A, \mathsf{D}(A)\big)$, but the equality of these two operators holds only if $\big(A, \mathsf{D}(A)\big)$ is self-adjoint. Note also that for any symmetric operator the scalar $\langle f, Af\rangle$ is real for any $f \in \mathsf{D}(A)$.

Exercise 1.4.11. *Show that a symmetric operator is always closable.*

Let us add one more definition related to self-adjoint operators.

Definition 1.4.12. *A symmetric operator* $\big(A, \mathsf{D}(A)\big)$ *is* essentially self-adjoint *if its closure* $\big(\overline{A}, \mathsf{D}(\overline{A})\big)$ *is self-adjoint. In this case* $\mathsf{D}(A)$ *is called* a core *for* $\overline{A}$.

A following *fundamental criterion for self-adjointness* is important in this context, and its proof can be found in [Amr, Prop. 3.3].

Proposition 1.4.13. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a symmetric operator in a Hilbert space* $\mathcal{H}$. *Then*

(i) $\big(A, \mathsf{D}(A)\big)$ *is self-adjoint if and only if* $\mathsf{Ran}(A + i) = \mathcal{H}$ *and* $\mathsf{Ran}(A - i) = \mathcal{H}$,

(ii) $\big(A, \mathsf{D}(A)\big)$ *is essentially self-adjoint if and only if* $\mathsf{Ran}(A + i)$ *and* $\mathsf{Ran}(A - i)$ *are dense in* $\mathcal{H}$.

For completeness, let us recall the definitions of a spectral family and a spectral measure, and mention one version of the spectral theorem for self-adjoint operators. We do not provide more explanations here and refer to [Amr, Chap. 4] for a thorough introduction to this important result of spectral theory. Later on, it will be useful to have these definitions and this statement at hand.

Definition 1.4.14. A spectral family*, or* a resolution of the identity*, is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of orthogonal projections in $\mathcal{H}$ satisfying:*

(i) The family is non-decreasing, i.e. $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$,

(ii) The family is strongly right continuous, i.e. $E_\lambda = E_{\lambda+0} = s-\lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$,

(iii) $s-\lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ *and* $s-\lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$,

Given such a family, one first defines

$$E\big((a,b]\big) := E_b - E_a, \qquad a,b\in\mathbb{R}, \tag{1.19}$$

and extends this definition to all sets $V\in\mathcal{A}_B$, where $\mathcal{A}_B$ denotes the set of Borel sets on $\mathbb{R}$. Thus one ends up with the notion of a *spectral measure*, which consists in a projection-valued map $E:\mathcal{A}_B\to\mathscr{P}(\mathcal{H})$ which satisfies $E(\emptyset)=\mathbf{0}$, $E(\mathbb{R})=\mathbf{1}$, $E(V_1)E(V_2)=E(V_1\cap V_2)$ for any Borel sets V_1,V_2.

Theorem 1.4.15 (Spectral Theorem)**.** *With any self-adjoint operator $\big(A,\mathsf{D}(A)\big)$ on a Hilbert space $\mathcal{H}$ one can associate a unique spectral family $\{E_\lambda\}$, called* the spectral family of A*, such that $\mathsf{D}(A)=\mathsf{D}_{\mathrm{id}}$ with*

$$\mathsf{D}_\varphi := \Big\{f\in\mathcal{H} \mid \int_{-\infty}^{\infty}|\varphi(\lambda)|^2\,\langle E(\mathrm{d}\lambda)f,f\rangle<\infty\Big\}.$$

and $A=\int_{-\infty}^{\infty}\lambda\,E(\mathrm{d}\lambda)$. Conversely any spectral family or any spectral measure defines a self-adjoint operator in $\mathcal{H}$ by the previous formulas.

1.5 Resolvent and spectrum

We come now to the important notion of the spectrum of an operator. As already mentioned in the previous section we shall often speak about a linear operator A, its domain $\mathsf{D}(A)$ being implicitly taken into account. Recall also that the notion of a closed linear operator has been introduced in Definition 1.4.6.

The notion of the inverse of a bounded linear operator has already been introduced in Definition 1.2.7. By analogy we say that any linear operator A is *invertible* if $\mathsf{Ker}(A)=\{0\}$. In this case, the inverse A^{-1} gives a bijection from $\mathsf{Ran}(A)$ onto $\mathsf{D}(A)$. More precisely $\mathsf{D}(A^{-1})=\mathsf{Ran}(A)$ and $\mathsf{Ran}(A^{-1})=\mathsf{D}(A)$. It can then be checked that if A is closed and invertible, then A^{-1} is also closed. Note also if A is closed and if $\mathsf{Ran}(A)=\mathcal{H}$ then $A^{-1}\in\mathscr{B}(\mathcal{H})$. In fact, the boundedness of A^{-1} is a consequence of the closed graph theorem and one says in this case that A is *boundedly invertible* or *invertible in $\mathscr{B}(\mathcal{H})$*.

Definition 1.5.1. *For a closed linear operator A its* resolvent set $\rho(A)$ *is defined by*

$$\begin{aligned}\rho(A) :=& \big\{z \in \mathbb{C} \mid (A - z) \text{ is invertible in } \mathscr{B}(\mathcal{H})\big\} \\ =& \big\{z \in \mathbb{C} \mid \mathsf{Ker}(A - z) = \{0\} \text{ and } \mathsf{Ran}(A - z) = \mathcal{H}\big\}.\end{aligned}$$

For $z \in \rho(A)$ the operator $(A-z)^{-1} \in \mathscr{B}(\mathcal{H})$ is called the resolvent of A at the point z. The spectrum $\sigma(A)$ of A *is defined as the complement of $\rho(A)$ in $\mathbb{C}$,* i.e.

$$\sigma(A) := \mathbb{C} \setminus \rho(A). \tag{1.20}$$

The following statement summarized several properties of the resolvent set and of the resolvent of a closed linear operator.

Proposition 1.5.2. *Let A be a closed linear operator on a Hilbert space $\mathcal{H}$. Then*

(i) The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$,

(ii) If $z_1, z_2 \in \rho(A)$ then the first resolvent equation *holds, namely*

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1} \tag{1.21}$$

(iii) If $z_1, z_2 \in \rho(A)$ then the operators $(A - z_1)^{-1}$ and $(A - z_2)^{-1}$ commute,

(iv) In each connected component of $\rho(A)$ the map $z \mapsto (A - z)^{-1}$ is holomorphic.

As a consequence of the previous proposition, the spectrum of a closed linear operator is always closed. In particular, $z \in \sigma(A)$ if $A-z$ is not invertible or if $\mathsf{Ran}(A-z) \neq \mathcal{H}$. The first situation corresponds to the definition of an eigenvalue:

Definition 1.5.3. *For a closed linear operator A, a value $z \in \mathbb{C}$ is* an eigenvalue of A *if there exists $f \in \mathsf{D}(A)$, $f \neq 0$, such that $Af = zf$. In such a case, the element f is called* an eigenfunction of A associated with the eigenvalue z. *The dimension of the vector space generated by all eigenfunctions associated with an eigenvalue z is called* the geometric multiplicity of z. *The set of all eigenvalues of A is denoted by $\sigma_p(A)$, and is often called* the point spectrum *of A.*

Let us still provide two properties of the spectrum of an operator in the special cases of a bounded operator or of a self-adjoint operator.

Exercise 1.5.4. *By using the Neumann series, show that for any $B \in \mathscr{B}(\mathcal{H})$ its spectrum is contained in the ball in the complex plane of center 0 and of radius $\|B\|$.*

Lemma 1.5.5. *Let A be a self-adjoint operator in $\mathcal{H}$.*

(i) Any eigenvalue of A is real,

(ii) More generally, the spectrum of A is real, i.e. $\sigma(A) \subset \mathbb{R}$,

(iii) Eigenvectors associated with different eigenvalues are orthogonal to one another.

Proof. a) Assume that there exists $z \in \mathbb{C}$ and $f \in \mathsf{D}(A)$, $f \neq 0$ such that $Af = zf$. Then one has

$$z\|f\|^2 = \langle f, zf\rangle = \langle f, Af\rangle = \langle Af, f\rangle = \langle zf, f\rangle = \overline{z}\|f\|^2.$$

Since $\|f\| \neq 0$, one deduces that $z \in \mathbb{R}$.

b) Let us consider $z = \lambda + i\varepsilon$ with $\lambda, \varepsilon \in \mathbb{R}$ and $\varepsilon \neq 0$, and show that $z \in \rho(A)$. Indeed, for any $f \in \mathsf{D}(A)$ one has

$$\begin{aligned}\|(A-z)f\|^2 &= \|(A-\lambda)f - i\varepsilon f\|^2 \\ &= \big\langle (A-\lambda)f - i\varepsilon f, (A-\lambda)f - i\varepsilon f\big\rangle \\ &= \|(A-\lambda)f\|^2 + \varepsilon^2\|f\|^2.\end{aligned}$$

It follows that $\|(A-z)f\| \geq |\varepsilon|\|f\|$, and thus $A - z$ is invertible.

Now, for any for any $g \in \mathsf{Ran}(A-z)$ let us observe that

$$\|g\| = \big\|(A-z)(A-z)^{-1}g\big\| \geq |\varepsilon|\big\|(A-z)^{-1}g\big\|.$$

Equivalently, it means for all $g \in \mathsf{Ran}(A-z)$, one has

$$\big\|(A-z)^{-1}g\big\| \leq \frac{1}{|\varepsilon|}\|g\|. \tag{1.22}$$

Let us finally observe that $\mathsf{Ran}(A-z)$ is dense in $\mathcal{H}$. Indeed, by Lemma 1.4.9 one has

$$\mathsf{Ran}(A-z)^{\perp} = \mathsf{Ker}\big((A-z)^*\big) = \mathsf{Ker}(A^* - \overline{z}) = \mathsf{Ker}(A - \overline{z}) = \{0\}$$

since all eigenvalues of A are real. Thus, the operator $(A-z)^{-1}$ is defined on the dense domain $\mathsf{Ran}(A-z)$ and satisfies the estimate (1.22). As explained just before the Exercise 1.4.4, it means that $(A-z)^{-1}$ continuously extends to an element of $\mathscr{B}(\mathcal{H})$, and therefore $z \in \rho(A)$.

c) Assume that $Af = \lambda f$ and that $Ag = \mu g$ with $\lambda, \mu \in \mathbb{R}$ and $\lambda \neq \mu$, and $f, g \in \mathsf{D}(A)$, with $f \neq 0$ and $g \neq 0$. Then

$$\lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, Ag\rangle = \mu\langle f, g\rangle,$$

which implies that $\langle f, g\rangle = 0$, or in other words that f and g are orthogonal. □

1.6 Positivity and polar decomposition

The notion of positive operators can be introduced either in a Hilbert space setting or in a C^*-algebraic setting. The next definition is based on the former framework, and its analog in the latter framework will be mentioned subsequently.

Definition 1.6.1. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *in* $\mathcal{H}$ *is positive if*

$$\langle f, Af\rangle \geq 0 \qquad \text{for any } f \in \mathsf{D}(A). \tag{1.23}$$

Clearly, such an operator is symmetric, see the paragraph following Definition 1.4.10. If A is bounded one also infers that A is self-adjoint, but this might not be the case if A is unbounded. It is then a natural question to check whether there exists some self-adjoint extensions of A which are still positive. We shall not go further in this direction and stick to the self-adjoint case. More precisely, a self-adjoint operator $\big(A, \mathsf{D}(A)\big)$ is positive if (1.23) holds.

For positive self-adjoint operators, the following consequences of the spectral theorem are very useful, see Theorem 1.4.15.

Proposition 1.6.2. *For any positive and self-adjoint operator* $\big(A, \mathsf{D}(A)\big)$ *in* $\mathcal{H}$ *the following properties hold:*

(i) $\sigma(A) \subset [0, \infty)$,

(ii) There exists a unique self-adjoint and positive operator $\big(B, \mathsf{D}(B)\big)$ *such that* $A = B^2$ *on* $\mathsf{D}(A)$. *The operator* B *is called* the positive square root *of* A *and is denoted by* $A^{1/2}$

Since in a purely C^*-algebraic the scalar product in (1.23) does not exist (note that this statement is not really correct because of the GNS representation) one usually says that a bounded operator A is positive if $A = A^*$ and $\sigma(A) \subset [0, \infty)$. However, this definition coincides with the one mentioned above as long as one considers bounded operators only.

Remark 1.6.3. *If* A *is an arbitrary element of* $\mathscr{B}(\mathcal{H})$, *observe that* A^*A *and* AA^* *are positive operators. Indeed, self-adjointness follows easily from Exercise 1.2.4 while positivity is obtained by the equalities*

$$\langle f, A^*Af\rangle = \langle Af, Af\rangle = \|Af\|^2 \geq 0$$

and similarly for AA^*. *In fact, the set* $\{A^*A \mid A \in \mathscr{B}(\mathcal{H})\}$ *is equal to the set of all positive operators in* $\mathscr{B}(\mathcal{H})$.

Let us add one statement which contains several properties of bounded positive operators. It can be stated in a purely C^*-algebraic framework, but we present it for simplicity for $\mathscr{B}(\mathcal{H})$ only. Note that if $A \in \mathscr{B}(\mathcal{H})$ we often denote its positivity by writing $A \geq 0$. Now, if A_1, A_2 are bounded and self-adjoint operators, one writes $A_1 \geq A_2$ if $A_1 - A_2 \geq 0$. We shall also use the notation $\mathscr{B}(\mathcal{H})_+$ for the set of positive elements of $\mathscr{B}(\mathcal{H})$.

Proposition 1.6.4. *(i) The sum of two positive elements of* $\mathscr{B}(\mathcal{H})$ *is a positive element of* $\mathscr{B}(\mathcal{H})$, *(ii)* *The set* $\mathscr{B}(\mathcal{H})_+$ *is equal to* $\{A^*A \mid A \in \mathscr{B}(\mathcal{H})\}$,

(iii) *If* A, B *are self-adjoint elements of* $\mathscr{B}(\mathcal{H})$ *and if* $C \in \mathscr{B}(\mathcal{H})$, *then* $A \geq B \Rightarrow C^*AC \geq C^*BC$,

(iv) *If* $A \geq B \geq 0$, *then* $A^{1/2} \geq B^{1/2}$,

(v) *If* $A \geq B \geq 0$, *then* $\|A\| \geq \|B\|$,

(vi) *If* A, B *are positive and invertible elements of* $\mathscr{B}(\mathcal{H})$, *then* $A \geq B \Rightarrow B^{-1} \geq A^{-1} \geq 0$,

(vii) *For any* $A \in \mathscr{B}(\mathcal{H})$ *there exist* $A_1, A_2, A_3, A_4 \in \mathscr{B}(\mathcal{H})_+$ *such that*

$$A = A_1 - A_2 + iA_3 - iA_4.$$

Proof. See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur]. □

We finally state and prove a very useful result for arbitrary element of $B \in \mathscr{B}(\mathcal{H})$. For that purpose we first introduce

$$|B| := (B^*B)^{1/2}. \tag{1.24}$$

Theorem 1.6.5 (Polar decomposition)**.** *For any* $B \in \mathscr{B}(\mathcal{H})$ *there exists a unique partial isometry* $W \in \mathscr{B}(\mathcal{H})$ *such that*

$$W|B| = B \qquad \text{and} \qquad \mathsf{Ker}(W) = \mathsf{Ker}(B). \tag{1.25}$$

In addition, $W^*B = |B|$.

Proof. For any $f \in \mathcal{H}$ one has

$$\big\||B|f\big\|^2 = \langle |B|f, |B|f\rangle = \langle f, |B|^2 f\rangle = \langle f, B^*Bf\rangle = \|Bf\|^2,$$

which means that the map

$$W_0 : |B|\mathcal{H} \ni |B|f \mapsto Bf \in \mathcal{H}$$

is well-defined, isometric, and also linear. It can then be uniquely extended to a linear isometric map from the closure $\overline{|B|\mathcal{H}}$ to $\mathcal{H}$. This extension is still denoted by W_0. We can thus define the operator $W \in \mathscr{B}(\mathcal{H})$ by $W = W_0$ on $\overline{|B|\mathcal{H}}$ and $W = \mathbf{0}$ on its orthocomplement. It then follows that $W|B| = B$, and W is isometric on $\mathsf{Ker}(W)^\perp$ since $\mathsf{Ker}(W) = \overline{|B|\mathcal{H}}^\perp$. Thus, W is a partial isometry and $\mathsf{Ker}(W) = \mathsf{Ker}(|B|)$. Now, since for any $f, g \in \mathcal{H}$ one has

$$\langle W^*Bf, |B|g\rangle = \langle Bf, Bg\rangle = \langle f, B^*Bg\rangle = \langle |B|f, |B|g\rangle,$$

one deduces that $\langle W^*Bf, h\rangle = \langle |B|f, h\rangle$ for any $h \in \overline{|B|\mathcal{H}}$, and then for any $h \in \mathcal{H}$. Thus $W^*B = |B|$, and since $\mathsf{Ker}(W^*) = \mathsf{Ran}(W)^\perp = \mathsf{Ran}(B)^\perp$, one infers that $\mathsf{Ker}(B) = \mathsf{Ker}(|B|) = \mathsf{Ker}(W)$.

For the uniqueness, suppose that there exists another partial isometry $W' \in \mathscr{B}(\mathcal{H})$ such that $W'|B| = B$ and $\mathsf{Ker}(W') = \mathsf{Ker}(B)$. Then W' is equal to W on $\overline{|B|\mathcal{H}}$ and on $\overline{|B|\mathcal{H}}^\perp$ both operators are equal to $\mathbf{0}$. As a consequence, $W' = W$. □

Chapter 2

Normed ideals of $\mathscr{K}(\mathcal{H})$

In this chapter we review the classical theory related to compact operators: their singular values and their eigenvalues, some operator ideals, etc. We shall mainly follow [Sim] but an alternative reference is [GK]. Note that some results might be improved in the subsequent chapters. Before starting with any new material, let us emphasize one tiny but important point.

Remark 2.0.6. *Up to now, choosing* $\mathbb{N} = \{0, 1, 2, \dots\}$ *or* $\mathbb{N} = \{1, 2, 3, \dots\}$ *was not relevant and we did not impose any choice. However, in some of the subsequent formulas starting with* $n = 0$ *or with* $n = 1$ *makes a difference. So from now on we shall take the convention that* $\mathbb{N} := \{1, 2, 3, \dots\}$ *and stress that some formulas look different with the other convention. Thus, without further notice all sequences* $\{f_n\}$ *or* (a_n) *will start with* $n = 1$*. Relatedly, we shall use the convenient notation* N *either for a finite number or for* ∞*.*

2.1 Compact operators and the canonical expansion

In order to study the ideal of compact operators $\mathscr{K}(\mathcal{H})$, a standard result on analytic operator-valued functions has to be recalled. Its proof is provided for example in [RS1, Thm VI.14]. For its statement, we recall that a subset S of an open set Ω is discrete if it has no limit points in Ω.

Theorem 2.1.1 (Analytic Fredholm theorem)**.** *Let* Ω *be an open connected subset of* $\mathbb{C}$*. Let* $\Psi : \Omega \to \mathscr{K}(\mathcal{H})$ *be an analytic operator-valued function. Then one of the following alternative holds:*

(i) $\big(\mathbf{1} - \Psi(z)\big)^{-1}$ *exists for no* $z \in \Omega$*,*

(ii) $\big(\mathbf{1} - \Psi(z)\big)^{-1}$ *exists for all* $z \in \Omega \setminus S$ *where* S *is a discrete subset of* Ω*. In this case,* $\big(\mathbf{1} - \Psi(z)\big)^{-1}$ *is meromorphic in* Ω*, analytic in* $\Omega \setminus S$*, the residue at the poles are finite rank operators, and if* $z \in S$ *then the equation* $\Psi(z)f = f$ *has a nonzero solution in* $\mathcal{H}$*.*

This theorem has several important consequences. We state a few of them.

Corollary 2.1.2 (Fredholm alternative)**.** *If A belongs to $\mathscr{K}(\mathcal{H})$, either $(\mathbf{1}-A)^{-1}$ exists or $Af=f$ has a solution in $\mathcal{H}$.*

Proof. Set $\Psi(z)=zA$ and apply the previous theorem at $z=1$. □

Theorem 2.1.3 (Riesz-Schauder theorem)**.** *If A belongs to $\mathscr{K}(\mathcal{H})$, then its spectrum $\sigma(A)$ is a discrete set having no limit points except perhaps 0. In addition, any non-zero $\lambda\in\sigma(A)$ is an eigenvalue of finite geometric multiplicity.*

Proof. Let us set $\Psi(z)=zA$, which makes Ψ an analytic $\mathscr{K}(\mathcal{H})$-valued function on $\mathbb{C}$. Thus from Theorem 2.1.1 one infers that the set $\{z\in\mathbb{C}\mid\Psi(z)f=f$ for some $f\in\mathcal{H}, f\neq 0\}$ is a discrete set. Now, if $\lambda\neq 0$ and if $\frac{1}{\lambda}$ is not in this discrete set then

$$(\lambda-A)^{-1}=\frac{1}{\lambda}\big(\mathbf{1}-\tfrac{1}{\lambda}A\big)^{-1}$$

exists, which means that $\lambda\notin\sigma(A)$. From this, one deduces that the spectrum of A consists in the discrete set mentioned above, and possibly in the value 0. Finally, the fact that the non-zero eigenvalues have finite geometric multiplicity follows directly from the compactness of A. □

The following statement is a direct consequence of Riesz-Schauder theorem together with some information deduced from the spectral theorem for self-adjoint operators, see Theorem 1.4.15.

Theorem 2.1.4 (Hilbert-Schmidt theorem)**.** *If A is self-adjoint and belongs to $\mathscr{K}(\mathcal{H})$ then there exists a complete orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ such that $Af_n=\lambda_n f_n$ and $\lim_{n\to\infty}\lambda_n=0$.*

If A is not self-adjoint, a "canonical" description of A can still be provided. For its statement, we shall use the convenient notation $|f\rangle\langle g|$ for the rank-one operator defined by

$$|f\rangle\langle g|h:=\langle g,h\rangle f,\qquad \text{for any } f,g,h\in\mathcal{H}. \tag{2.1}$$

Theorem 2.1.5. *If A belongs to $\mathscr{K}(\mathcal{H})$ then A has a norm convergent expansion*

$$A=\sum_{n=1}^{N}\mu_n(A)|g_n\rangle\langle f_n| \tag{2.2}$$

with N either a finite number or equal to ∞, with each $\mu_n(A)>0$ and satisfying $\mu_n(A)\geq\mu_{n+1}(A)$, and with each family $\{f_n\}$ and $\{g_n\}$ orthonormal but not necessarily complete. Moreover, each $\mu_n(A)$ is uniquely determined while the f_n and g_n are usually not uniquely defined.

Proof. From the polar decomposition provided in Theorem 1.6.5 one infers that there exists a partial isometry W such that $|A| = W^*A$. Thus, $|A|$ is a compact and self-adjoint operator to which Theorem 2.1.4 applies. With the notations introduced above, this reads

$$|A| = \sum_{n=1}^{N} \mu_n(A)|f_n\rangle\langle f_n|$$

where the $\mu_n(A)$ are the eigenvalues of $|A|$ and f_n the corresponding eigenfunctions. Clearly, the family $\{f_n\}$ is orthonormal. Since W is an isometry on $\mathsf{Ran}(|A|)$ and by setting $g_n := Wf_n$ one also infers that $\{g_n\}$ is orthonormal. Since the relation $W|A| = A$ holds, one directly deduces the equality (2.2). The uniqueness follows if one observes that if (2.2) holds, then $\{\mu_n(A)^2\}$ are the eigenvalues of A^*A, $\{f_n\}$ the eigenvectors of A^*A and $\{g_n\}$ the eigenvectors of AA^*. The lack of uniqueness of f_n and g_n comes from the possible degeneracy of the eigenvalues of A^*A and AA^*. □

In the previous result, the real values $\mu_n(A)$ are called *the singular values of* A and the equality (2.2) is called *the canonical expansion of* A. Let us also emphasize that

$$\mu_n(A^*) = \mu_n(A), \tag{2.3}$$

as it can be directly deduced from (2.2) or from the fact that the spectrum of A^*A and AA^* coincide (multiplicity counted) with the possible exception of the eigenvalue 0.

Let us still add one more useful result which can be easily deduced from the construction provided in [Kat, Sec. III.6.4].

Lemma 2.1.6. *If A belongs to $\mathscr{K}(\mathcal{H})$ and $\lambda \in \sigma(A)$ is not equal to 0, then there exists a finite rank projection P_λ such that $AP_\lambda = P_\lambda A$, $\sigma\big(A \upharpoonright P_\lambda\mathcal{H}\big) = \{\lambda\}$ and $\sigma\big(A \upharpoonright (\mathbf{1} - P_\lambda)\mathcal{H}\big) = \sigma(A) \setminus \{\lambda\}$.*

Note that a possible expression for P_λ is provided by the formula

$$P_\lambda := -\frac{1}{2\pi i}\int_{|z-\lambda|=\varepsilon} (A - z)^{-1}\,\mathrm{d}z$$

for $\varepsilon > 0$ small enough. The dimension of $\mathsf{Ran}(P_\lambda)$ is called *the algebraic multiplicity* of λ. We still recall that the geometric and the algebraic multiplicity of an eigenvalue can be different, but the geometric multiplicity can never exceed the algebraic multiplicity.

2.2 Eigenvalues and singular values

In this section we begin the study of the singular values of any compact operator A, and then state some relations between singular values and eigenvalues. The proofs for most of these relations are not provided but references are given.

We start with some results on singular values. Since these values can be computed by an application of the min-max principle, we first introduce this principle for positive compact operator. Note that a similar statement holds for the negative eigenvalues of any self-adjoint compact operator.

Theorem 2.2.1 (Min-max principle). *Let B be a positive compact operator in $\mathcal{H}$, and let $\{\lambda_n\}$ be the set of its eigenvalues (counting multiplicity) and ordered such that $\lambda_n \geq \lambda_{n+1}$. Then*

$$\lambda_n = \min\big\{ \sup\{\langle f, Bf\rangle \mid f \in \mathcal{M}_n^\perp, \|f\| = 1\} \mid \mathcal{M}_n \subset \mathcal{H}, \dim(\mathcal{M}_n) = n-1\big\}. \quad (2.4)$$

Proof. For any $n \in \mathbb{N}$ let us set $F_n := \mathsf{Span}(f_1, \dots, f_n)$ with f_j a normalized eigenvector corresponding to the eigenvalue λ_j of B. Let us also consider any subspace $\mathcal{M}_n \subset \mathcal{H}$ with $\dim(\mathcal{M}_n) = n-1$. Clearly, $F_n \cap \mathcal{M}_n^\perp \neq \{0\}$, and thus one can choose $f \in F_n \cap \mathcal{M}_n^\perp$ with $\|f\| = 1$. More precisely, $f = \sum_{j=1}^d c_j f_j$ with $\sum_{j=1}^d |c_j|^2 = 1$. It then follows that

$$\langle f, Bf\rangle = \sum_{j=1}^d \langle c_j f_j, \lambda_j c_j f_j\rangle = \sum_{j=1}^d \lambda_j |c_j|^2 \geq \lambda_n \sum_{j=1}^d |c_j|^2 = \lambda_n$$

since the eigenvalues of B are ordered. Hence we have obtained that

$$\sup\big\{\langle f, Bf\rangle \mid f \in \mathcal{M}_n^\perp, \|f\| = 1\big\} \geq \lambda_n.$$

For the converse inequality, one can choose $\mathcal{M}_n := \mathsf{Span}(f_1, \dots, f_{n-1})$ and then

$$\sup\big\{\langle f, Bf\rangle \mid f \in \mathcal{M}_n^\perp, \|f\| = 1\big\} = \lambda_n,$$

which implies the statement. □

By setting $B := A^*A$ in the previous statement and by recalling that $\mu_n(A)^2$ corresponds to the n-eigenvalue of A^*A one directly obtains a characterization of the singular values of any compact operator A, namely:

Proposition 2.2.2. *Let A belong to $\mathscr{K}(\mathcal{H})$ and let $\{\mu_n(A)\}$ denote its singular values ordered such that $\mu_n(A) \geq \mu_{n+1}(A)$. Then*

$$\mu_n(A) = \min\big\{ \sup\{\|Af\| \mid f \in \mathcal{M}_n^\perp, \|f\| = 1\} \mid \mathcal{M}_n \subset \mathcal{H}, \dim(\mathcal{M}_n) = n-1\big\}. \quad (2.5)$$

As a consequence one directly infers the following estimates:

Corollary 2.2.3. *For any $A \in \mathscr{K}(\mathcal{H})$ and $B \in \mathscr{B}(\mathcal{H})$ one has*

$$\mu_n(AB) \leq \mu_n(A)\|B\| \quad \textit{and} \quad \mu_n(BA) \leq \mu_n(A)\|B\|. \quad (2.6)$$

Proof. Observe that the first inequality can be deduced from the second one and from (2.3). Indeed one has

$$\mu_n(AB) = \mu_n(B^*A^*) \leq \mu_n(A^*)\|B^*\| = \mu_n(A)\|B\|.$$

For the second equality, one uses (2.5) together with the inequality $\|BAf\| \leq \|B\|\|Af\|$ for any $f \in \mathcal{H}$. □

In the next statement, we mention some generalizations of the previous result. Proofs can be found in [Fan, Thm. 2].

Proposition 2.2.4. *Let A, B belong to $\mathscr{K}(\mathcal{H})$. Then the following inequalities hold for any $m, n \in \mathbb{N}$:*

$$\mu_{m+n-1}(AB) \leq \mu_m(A)\,\mu_n(B), \tag{2.7}$$
$$\mu_{n+m-1}(A+B) \leq \mu_m(A) + \mu_n(B). \tag{2.8}$$

Note that (2.6) corresponds to the special case $m = 1$ since $\mu_1(B) = \|B\|$. One additional relation between the singular values of A, B and AB is given by:

Lemma 2.2.5. *For any A, B in $\mathscr{K}(\mathcal{H})$ and for any $n \in \mathbb{N}$ one has*

$$\prod_{j=1}^{d} \mu_j(AB) \leq \prod_{j=1}^{d} \mu_j(A)\mu_j(B). \tag{2.9}$$

Proof. See [Hor], Theorem 3 and its proof. □

Let us still mention some relations linking singular values and eigenvalues. Note that the eigenvalues of a compact operator are not enumerated arbitrarily but according to the following definition.

Definition 2.2.6. *If A belongs to $\mathscr{K}(\mathcal{H})$ its eigenvalues $\lambda_1(A), \lambda_2(A), \ldots$ are ordered such that $|\lambda_j(A)| \geq |\lambda_{j+1}(A)|$ for any $j \in \mathbb{N}$, and each eigenvalue is counted up to its algebraic multiplicity.*

The following result comes from the paper [Wey].

Lemma 2.2.7. *For any A in $\mathscr{K}(\mathcal{H})$ and for any $n \in \mathbb{N}$ one has*

$$\prod_{j=1}^{d} |\lambda_j(A)| \leq \prod_{j=1}^{d} \mu_j(A).$$

As a consequence of the previous two results one has:

Proposition 2.2.8. *For any A, B in $\mathscr{K}(\mathcal{H})$ and for any monotone increasing function $\phi : [0, \infty) \to \mathbb{R}_+$ such that $t \mapsto \phi(\mathrm{e}^t)$ is convex one has*

(i)

$$\sum_j \phi\big(|\lambda_j(A)|\big) \leq \sum_j \phi\big(\mu_j(A)\big), \tag{2.10}$$

and in particular for any $p \geq 1$

$$\sum_j |\lambda_j(A)|^p \leq \sum_j \mu_j(A)^p, \tag{2.11}$$

(ii)

$$\sum_j \phi\big(\mu_j(AB)\big) \leq \sum_j \phi\big(\mu_j(A)\mu_j(B)\big). \tag{2.12}$$

Proof. These inequalities directly follow from Lemmas 2.2.5 and 2.2.7 together with Corollary 2.3.4 introduced in the next section. □

Note that the results mentioned above are usually proved for finite matrices, and then a limiting procedure is applied in order to extend the result to certain compact operators. As mentioned in the proof of Proposition 2.2.8 some additional technicalities are now required. Some of them are introduced in the next section.

2.3 Technical interlude

Let us start by introducing some ideas and results about *rearrangement* or *(sub)majorization*. This concept plays an important role when dealing with the spectrum of matrices or compact operators, and has been extensively studied in the book [MOB]. In the next definition we use the notation c_0 for the set of complex sequences $a = (a_j)_{j=1}^\infty$ satisfying $\lim_{j\to\infty} a_j = 0$.

Definition 2.3.1. *For any $a = (a_j)$ in $\mathbb{C}^d$ or in c_0 we denote by a^* the element of $\mathbb{R}^d$ or c_0 obtained by a non-increasing rearrangement of $\{|a_j|\}_j$.*

In other words, if means that $a_j^* \geq a_{j+1}^*$ and that the sets $\{a_j^*\}$ and $\{|a_j|\}$ are identical, counting multiplicity. For simplicity, we shall say that an element $a \in \mathbb{R}^d$ or $a \in c_0$ is *positive and ordered* if $a_j \geq 0$ and $a_j \geq a_{j+1}$ for any j. Clearly, a^* is always positive and ordered.

Now, based on the *rearrangement inequality*[1], as presented for example in [HLP, Thm 368] one infers that for two sequences a and b as in the previous definition one has

$$\sum_j |a_j b_j| \leq \sum_j a_j^* b_j^* \tag{2.13}$$

as long as the r.h.s. is meaningful (if a and b belong to $\mathbb{C}^d$ it is obviously the case). The following result, stated first in [Ma1, Lem. 1] and proved in [Ma2, Thm. 1.2], will be important later on. The version presented here is taken from [Sim, Thm. 1.9] where a proof is provided.

[1] For any $a \in \mathbb{R}^d$ let us set $a^\star$ for the non-increasing rearrangement of $\{a_j\}$ (without the absolute value). If $a, b \in \mathbb{R}^d$ the *rearrangement inequality* reads

$$\sum_{j=1}^d a_j^\star b_{n+1-j}^\star \leq \sum_{j=1}^d a_j b_j \leq \sum_{j=1}^d a_j^\star b_j^\star.$$

Theorem 2.3.2. *Let $a, b \in \mathbb{C}^d$ and assume that*

$$\sum_{j=1}^{k} b_j^* \leq \sum_{j=1}^{k} a_j^* \qquad \text{for any } k \in \{1, \dots, n\}. \tag{2.14}$$

Then there exist m points $a^{(1)}, \dots, a^{(m)}$ in $\mathbb{C}^d$ with $(a^{(\ell)})^* = a^*$ for $\ell \in \{1, \dots, m\}$ and there exist $\{\lambda_\ell\} \subset [0,1]$ satisfying $\sum_{\ell=1}^m \lambda_\ell = 1$ such that

$$b = \sum_{\ell=1}^{m} \lambda_\ell a^{(\ell)}. \tag{2.15}$$

In addition if Φ is a positive valued function on $[0,\infty)^d$ and if the function $\phi : \mathbb{C}^d \to \mathbb{R}_+$, defined by $\phi(c) := \Phi(c_1^*, \dots, c_n^*)$, is convex on $\mathbb{C}^d$, then

$$\phi(b) \leq \phi(a). \tag{2.16}$$

Note that condition (2.14) is often denoted by $b \lll a$ in the literature. In addition, what (2.15) really says is that b belongs to the convex hull of a family of vectors of the form $(\varepsilon_k a_{j_k})_{k=1}^d$ with $|\varepsilon_k| = 1$ and j_k is an arbitrary permutation of the numbers $1, 2, \dots, n$. The elements $a^{(\ell)}$ are the points which define the convex hull. This, together with the fact that $\phi(a^{(\ell)}) = \phi(a)$ and the convexity of the function ϕ, directly implies the inequality (2.16).

Exercise 2.3.3. *Provide a proof of Theorem 2.3.2.*

Before mentioning two results linked to the previous statement, let us show how one can construct examples of functions Φ. Consider any function $f : [0,\infty) \to \mathbb{R}_+$ which is convex and increasing and let us set $\Phi(x) := \sum_{j=1}^d f(x_j)$ for any $x \in [0,\infty)^d$. Then one observes that for any $\theta \in [0,1]$ and $b, c \in \mathbb{C}^d$ one has

$$\begin{aligned}
\phi\big(\theta b + (1-\theta)c\big) &= \Phi\big((\theta b + (1-\theta)c)^*\big) \\
&= \sum_{j=1}^{d} f\big(\big|\theta b_j + (1-\theta)c_j\big|\big) \\
&\leq \sum_{j=1}^{d} \Big(\theta f(|b_j|) + (1-\theta) f(|c_j|)\Big) \\
&= \theta\phi(b) + (1-\theta)\phi(c)
\end{aligned}$$

which means that ϕ is convex on $\mathbb{C}^d$. As a consequence the function Φ satisfies the requirement of Theorem 2.3.2.

The next statement is an application of Theorem 2.3.2 for transforming estimates on products to estimates on sums.

Corollary 2.3.4. *Let $a, b \in \mathbb{R}^d$ be positive and ordered, and suppose that*

$$\prod_{j=1}^{k} b_j \leq \prod_{j=1}^{k} a_j \qquad \text{for any } k \in \{1, \ldots, n\}.$$

Then, for any continuous, monotone increasing function $g : [0, \infty) \to \mathbb{R}_+$ with $t \mapsto g(\mathrm{e}^t)$ convex, we have that

$$\sum_{j=1}^{k} g(b_j) \leq \sum_{j=1}^{k} g(a_j) \qquad \text{for any } k \in \{1, \ldots, n\}. \tag{2.17}$$

In particular, (2.14) *can be obtained by taking $g(x) = x$.*

Proof. Assume without loss of generality that a_j and b_j are all non-zero. By setting then $\tilde{a}_j := \gamma a_j$ and $\tilde{b}_j := \gamma b_j$ for γ large enough, we get that all $\tilde{a}_j$, $\tilde{b}_j$ are bigger than 1. By considering then $\ln(\tilde{a}_j)$ and $\ln(\tilde{b}_j)$, one observes that the condition (2.14) is satisfied for these numbers. By setting then $f(t) := g\big(\gamma^{-1}\,\mathrm{e}^t\big)$, the function f is convex and increasing, and by the observation made above, the function Φ defined by $\Phi(x) := \sum f(x_j)$ for any $x \in [0, \infty)^d$ satisfies the assumption of Theorem 2.3.2. The inequality (2.17) follows then directly from (2.16). □

Exercise 2.3.5. *Check the details of the previous proof.*

The second domain linked with Theorem 2.3.2 is related to the notion of doubly substochastic matrices.

Definition 2.3.6. *A matrix $\alpha = (a_{jk}) \in M_N(\mathbb{C})$ is called* doubly substochastic *(in short dss) if $\sum_{j=1}^{N} |\alpha_{jk}| \leq 1$ for all $k \in \{1, \ldots, N\}$ and $\sum_{k=1}^{N} |\alpha_{jk}| \leq 1$ for all $j \in \{1, \ldots, N\}$.*

Note that such matrices can be constructed from elements of any Hilbert space $\mathcal{H}$. Indeed, if for any $\ell \in \{1, 2, 3, 4\}$ the family $\{f_j^\ell\}_{j=1}^N \subset \mathcal{H}$ is orthonormal, then the matrix α defined by $\alpha_{jk} := |\langle f_j^1, f_k^2\rangle|^2$ is a dss matrix, and the matrix β defined by $\beta_{jk} := \langle f_j^1, f_k^2\rangle\langle f_k^3, f_j^4\rangle$ is a dss matrix. The fact that these matrices are doubly substochastic can be obtained by applying Bessel and Schwartz inequalities.

The next statement is borrowed from [Sim, Prop. 1.12] to which we refer for its proof.

Proposition 2.3.7. *Let $\alpha \in M_n(\mathbb{C})$ be a dss matrix and let $c \in \mathbb{C}^d$. If one sets $a := c^*$ and $b := \alpha c$, then $a, b \in \mathbb{C}^d$ satisfy condition* (2.14) *of Theorem 2.3.2.*

We now introduce the notion of *symmetric normed spaces*. Note that a simple introduction to the subject can be found in [Sch, Sec. V.3]. For that purpose, let us denote by ℓ_∞ the set of all bounded sequences $(a_j)_{j=1}^\infty$ endowed with the sup norm (also denoted ℓ_∞-norm), and let us denote by c_c the set of complex sequences $a = (a_j)_{j=1}^\infty$

with compact support. Clearly, c_c is dense in c_0 for the ℓ_∞-norm. Recall that a norm on c_c is a map $\Phi : c_c \to \mathbb{R}_+$ which satisfies for any $a, b \in c_c$ and $\lambda \in \mathbb{C}$ the following properties: i) $\Phi(\lambda a) = |\lambda|\Phi(a)$, ii) $\Phi(a+b) \leq \Phi(a) + \Phi(b)$, iii) $\Phi(a) = 0$ if and only if $a = 0$.

Definition 2.3.8. *A norm Φ on c_c is* symmetric *if $\Phi(a) = \Phi(a^*)$ for any $a \in c_c$.*

Let us observe that a norm on c_c is symmetric if and only if it is invariant under permutations and under the map $a_j \mapsto \mathrm{e}^{i\theta_j}\, a_j$ for any $\theta_j \in \mathbb{R}$.

Definition 2.3.9. *Let Φ be a norm on c_c. The* maximal space *J_Φ consists in the set of sequence $a = (a_j)_{j=1}^\infty$ such that the limit $\lim_{n\to\infty} \Phi\big((a_1, a_2, \dots, a_n, 0, 0 \dots)\big)$ exists (we denote it then by $\Phi(a)$). The* minimal space *$J_\Phi^{(0)}$ consists in the closure of c_c with the norm Φ. If $J_\Phi = J_\Phi^{(0)}$, that is if c_c is dense in J_Φ the norm Φ is called* regular *(or* mononormalizing *in some references).*

Examples 2.3.10. *1) For $p \geq 1$, if $\Phi(a) \equiv \|a\|_p := \big(\sum_j |a_j|^p\big)^{1/p}$ then J_Φ corresponds to the usual ℓ_p space. We also set $\|a\|_\infty := \sup_j |a_j|$. Note that if $p < \infty$ the norm $\|\cdot\|_p$ is regular.*

2) For $p > 1$ let us set

$$\|a\|_{p,w} := \sup_n \Big(n^{-1+\frac{1}{p}} \sum_{j=1}^d a_j^*\Big), \tag{2.18}$$

which is a symmetric norm, called Calderón norm. The maximal space associated with this norm is denoted by $\ell_{p,w}$ and called weak *ℓ_p-space. The minimal space corresponds to the elements $a \in \ell_{p,w}$ satisfying the additional condition $\lim_{j\to\infty} j^{\frac{1}{p}} a_j^* = 0$, which means that the Calderón norms are not regular. Note also that the following inequalities hold*

$$\|a\|'_{p,w} \leq \|a\|_{p,w} \leq \frac{p}{p-1}\|a\|'_{p,w}$$

with $\|a\|'_{p,w} := \sup_j(j^{\frac{1}{p}} a_j^*)$. Clearly, this expression is simpler than (2.18) but $\|\cdot\|'_{p,w}$ does not define a norm. However, the set of $a \in \ell_\infty$ satisfying $\|a\|'_{p,w} < \infty$ corresponds to $\ell_{p,w}$, and this expression can also be used for $p = 1$.

In the following statement, several properties of maximal and minimal spaces are summarized. Note that Theorem 2.3.2 and Proposition 2.3.7 play an important role in the proof, and that condition (2.14) is explicitly mentioned in the point (b). For the proof of these statements, we refer to [Sim, Thm. 1.16].

Theorem 2.3.11. *Let Φ be a symmetric norm on c_c, then*

(a) If $a \in J_\Phi$ and $\lim_{j\to\infty} a_j = 0$, then $\Phi(a) = \Phi(a^*)$,

(b) If $a, b \in J_\Phi$ with $\lim_{j\to\infty} a_j = 0$ and $\lim_{j\to\infty} b_j = 0$, and if $\sum_{j=1}^d b_j^* \leq \sum_{j=1}^d a_j^*$ for any $n \in \mathbb{N}$, then $\Phi(b) \leq \Phi(a)$,

(c) If $\Phi\big((1,0,0,\dots)\big) = \mathrm{c}$*, then* $\mathrm{c}\|a\|_\infty \leq \Phi(a) \leq \mathrm{c}\|a\|_1$ *for any* $a \in J_\Phi$*,*

(d) Both J_Φ *and* $J_\Phi^{(0)}$ *are Banach spaces,*

(e) If α *is a substochastic matrix and* $a \in J_\Phi$*, resp.* $a \in J_\Phi^{(0)}$*, then* αa *is in* J_Φ*, resp. in* $J_\Phi^{(0)}$*, and* $\Phi(\alpha a) \leq \Phi(a)$*,*

(f) If Φ *is inequivalent to* $\|\cdot\|_\infty$*, then* J_Φ *consists only of sequences which vanish at infinity,*

(g) If $J_\Phi = J_\Psi$*, then* Φ *and* Ψ *are equivalent norms.*

For each symmetric norm Φ on c_c one can define a conjugate norm Φ' on c_c by the following construction: For any $b \in c_c$ one sets

$$\Phi'(b) := \sup\Big\{\big|\sum_j a_j b_j\big| \mid a \in c_c, \Phi(a) \leq 1\Big\}. \tag{2.19}$$

As a consequence of (2.13) one easily infers that for $b, c \in c_c$ with $c = c^*$

$$\sup\Big\{\big|\sum_j a_j b_j\big| \mid a^* = c\Big\} = \sum_j b_j^* c_j$$

and then that Φ' is a symmetric norm on c_c. Some additional standard duality results are gathered in the following statement.

Theorem 2.3.12. *Let* Φ *be a symmetric norm on* c_c*. Then*

(a) $\sum_j |a_j b_j| \leq \Phi(a)\Phi'(b)$*,*

(b) $\big(J_\Phi^{(0)}\big)^* = J_{\Phi'}$ *in the sense that any continuous linear functional on* $J_\Phi^{(0)}$ *has the form* $a \mapsto \sum_j a_j b_j$ *for some* $b \in J_{\Phi'}$*,*

(c) $J_\Phi^{(0)}$*, resp.* J_Φ*, is reflexive if and only if both* Φ *and* Φ' *are regular.*

The proof of the above statement is provided in [Sim, Thm. 1.17] and is based on standard duality arguments.

Exercise 2.3.13. *Provide the proofs of Theorems 2.3.11 and 2.3.12.*

We close this section with a few results related to singular values of pairs of compact operators. Proofs can be found in [Sim, Sec. 1.8 & 1.9].

Proposition 2.3.14. *a) For any pair of compact operators* A *and* B *one has*

$$\mu_n(A) - \mu_n(B) = \sum_m \alpha_{nm}\mu_m(A-B) \tag{2.20}$$

for a dss matrix α.

b) For any pair of finite dimensional self-adjoint matrices A, B, *let* $\lambda_n^\star(A)$ *denote the eigenvalues of* A *listed in decreasing order. Then*

$$\lambda_n^\star(A) - \lambda_n^\star(B) = \sum_m \beta_{nm}\lambda_m^\star(A-B) \tag{2.21}$$

for a dss matrix β.

For any $A \in \mathscr{K}(\mathcal{H})$ we set $\|A\|_p := \big(\sum_n \mu_n(A)^p\big)^{1/p}$ whenever the summation is meaningful.

Proposition 2.3.15 (Clarkson-McCarthy inequalities)**.** *a) For* $2 \leq p < \infty$ *one has*

$$\|A+B\|_p^p + \|A-B\|_p^p \leq 2^{p-1}\big(\|A\|_p^p + \|B\|_p^p\big). \tag{2.22}$$

b) For $1 < p \leq 2$ *and for* $p' = p/(p-1)$ *one has*

$$\|A+B\|_p^{p'} + \|A-B\|_p^{p'} \leq 2\big(\|A\|_p^p + \|B\|_p^p\big)^{p'/p}. \tag{2.23}$$

Note that for A, B positive an additional relation holds:

Proposition 2.3.16. *For* $p \geq 1$ *and for* A, B *positive compact operators, one has*

$$2^{1-p}\|A+B\|_p^p \leq \|A\|_p^p + \|B\|_p^p \leq \|A+B\|_p^p. \tag{2.24}$$

Exercise 2.3.17. *Consider the matrices* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ *and compute* $\|A\|_1$, $\|B\|_1$ *and* $\|A+iB\|_1$. *What do you observe and can you compare this result with the commutative case ?*

2.4 Normed ideals of $\mathscr{B}(\mathcal{H})$

In this section we begin the study of two-sided ideals in $\mathscr{B}(\mathcal{H})$. By definition, as linear subspace $\mathscr{J}$ of $\mathscr{B}(\mathcal{H})$ is *a two-sided ideal of* $\mathscr{B}(\mathcal{H})$ if AB and BA belong to $\mathscr{J}$ whenever $A \in \mathscr{J}$ and $B \in \mathscr{B}(\mathcal{H})$. Some of these spaces will be linked to sequences introduced in the previous sections. We begin with two standard results about operator ideals. The first one state that the biggest ideal of $\mathscr{B}(\mathcal{H})$ is $\mathscr{K}(\mathcal{H})$.

Proposition 2.4.1. *Let* $\mathscr{J}$ *be a two-sided ideal of* $\mathscr{B}(\mathcal{H})$ *containing an element* A *which is not compact. Then* $\mathscr{J} = \mathscr{B}(\mathcal{H})$.

Proof. By the polar decomposition of Theorem 1.6.5, there exists a partial isometry W such that $|A| = W^*A$. If follows that $\mathscr{J}$ contains the positive self-adjoint operator $|A|$ which is not compact. By the spectral theorem, for any $a > 0$ let us set $P_a := \chi_{[a,\infty)}(|A|) \equiv E\big([a,\infty)\big)$ where E denotes the spectral measure associated with the operator $|A|$. If each P_a is a finite dimensional projection, then $|A| = u\!-\!\lim_{a\to 0}|A|P_a$

would be compact (as a norm limit of finite dimensional operator) which is a contradiction with the fact that $|A|$ is not compact. Thus there exists at least one P_a which is not finite dimensional. In addition, since $|A|^{-1}P_a$ is bounded (by functional calculus), then $P_a = |A|\,(|A|^{-1}P_a)$ is an element of $\mathscr{J}$ (as a product of an element in $\mathscr{J}$ and a bounded operator). Thus there exists an infinite dimensional projection P_a which belongs to $\mathscr{J}$. Then, by a general argument there exists an isometry V from $\mathcal{H}$ to $P_a\mathcal{H}$, and then $V^*P_aV = \mathbf{1} \in \mathscr{J}$, since $P_a \in \mathscr{J}$. The fact that $\mathbf{1} \in \mathscr{J}$ automatically implies that $\mathscr{J} = \mathscr{B}(\mathcal{H})$. □

As a consequence of the previous statement, any two-sided ideal in $\mathscr{B}(\mathcal{H})$ consists in compact operators. On the other hand, one easily shows that if this ideal $\mathscr{J}$ contains at least one rank one projection and is norm closed, then it is automatically equal to $\mathscr{K}(\mathcal{H})$. However, without this assumption of closeness, more possibilities exist. Let us first add a short but rather astonishing lemma of comparison between elements of $\mathscr{K}(\mathcal{H})$.

Lemma 2.4.2. *Let $\mathscr{J}$ be a two-sided proper ideal in $\mathscr{B}(\mathcal{H})$, and let $A, B \in \mathscr{K}(\mathcal{H})$ with $\mu_n(B) \leq \mu_n(A)$ for any $n \in \mathbb{N}$. If $A \in \mathscr{J}$, then $B \in \mathscr{J}$.*

Proof. Let $A = \sum_n \mu_n(A)|g_n\rangle\langle f_n|$ and $B = \sum_n \mu_n(B)|k_n\rangle\langle h_n|$ be the canonical expansion of A and B, as introduced in (2.2). Since these respective families of vectors are orthonormal there exist a partial isometry D with $D^*f_n = h_n$ and a contraction C with $Cg_h = \mu_n(B)\mu_n(A)^{-1}k_n$. Since $B = CAD$ it follows that B belongs to A, as stated. □

Corollary 2.4.3. *Every two-sided ideal $\mathscr{J}$ of $\mathscr{B}(\mathcal{H})$ is invariant under taking the adjoint,* i.e. *if $A \in \mathscr{J}$ then $A^* \in \mathscr{J}$.*

Proof. Since $\mu_n(A^*) = \mu_n(A)$ for any $n \in \mathbb{N}$, the statement follows from the previous lemma. □

Another consequence of the previous lemma is that two-sided ideals of $\mathscr{B}(\mathcal{H})$ are completely described by a set of sequences. Let us be more precise about such a statement, by following the adaptation of the main result of [Cal, Sec. 1] provided in [Sim, Chap. 2].

Definition 2.4.4. *A vector subspace J of c_0 is called* a Calkin space *if it possesses the* Calkin property, *namely whenever $a \in J$ and $b \in c_0$ with $b^*_j \leq a^*_j$ for any $j \in \mathbb{N}$, then $b \in J$.*

Theorem 2.4.5 (Calkin correspondence)**.** *There exists a bijective relation between the set of Calkin spaces and the set of two-sided ideals of $\mathscr{B}(\mathcal{H})$.*

Exercise 2.4.6. *Provide a proof of this theorem, and provide a construction for this correspondence as explicitly as possible.*

The previous result together with Theorem 2.3.11 establish a relation between symmetric norms discussed in Section 2.3 and two-sided ideals. Indeed, if Φ is a symmetric norm on c_c which is not equivalent to ℓ_∞ then the maximal space J_Φ and the minimal space $J_\Phi^{(0)}$ are Calkin spaces. The corresponding two-sided ideals of $\mathscr{B}(\mathcal{H})$ will be denoted respectively by $\mathscr{J}_\Phi$ and $\mathscr{J}_\Phi^{(0)}$. More precisely $\mathscr{J}_\Phi$, resp. $\mathscr{J}_\Phi^{(0)}$, are defined by the set of compact operators whose singular values belong to J_Φ, resp. $J_\Phi^{(0)}$. Then, for any $A \in \mathscr{J}_\Phi$ we set

$$\Phi(A) := \Phi\big((\mu_1(A), \mu_2(A), \dots)\big). \tag{2.25}$$

Let us mention a different way of computing this number, see [Sim, Prop. 2.6] for its proof. For that purpose we let $\mathcal{L}$ represent the set of all orthonormal sets $\{f_n\} \subset \mathcal{H}$.

Proposition 2.4.7. *If $A \in \mathscr{J}_\Phi$, then*

$$\Phi(A) = \sup_{\{f_n\},\{g_n\}\in\mathcal{L}} \Phi\big((\langle g_n, A f_n\rangle)\big).$$

The links between Φ and $\mathscr{J}_\Phi$ are summarized in the following statement.

Theorem 2.4.8. *Let Φ be a symmetric norm on c_c, and let $\mathscr{J}_\Phi$ be the corresponding two-sided ideal of $\mathscr{B}(\mathcal{H})$.*

(a) Φ defines a norm on $\mathscr{J}_\Phi$ by the relation (2.25) *and satisfies for all $B \in \mathscr{J}_\Phi$ and $A, C \in \mathscr{B}(\mathcal{H})$:*

$$\Phi(ABC) \le \|A\|\,\|C\|\,\Phi(B) \tag{2.26}$$

$$\Phi(B) \ge \|B\|\,\Phi\big((1,0,\dots)\big). \tag{2.27}$$

(b) $\mathscr{J}_\Phi$ and $\mathscr{J}_\Phi^{(0)}$ are Banach spaces with the norm Φ, and $\mathscr{J}_\Phi^{(0)}$ is the closure in $\mathscr{J}_\Phi$ of the finite rank operators. For any $A \in \mathscr{J}_\Phi^{(0)}$ the canonical decomposition provided in (2.2) *converges in the Φ-norm.*

(c) Any norm on a two-sided ideal $\mathscr{J}$ obeying (2.26) *agrees, on the finite rank operators, with a norm $\tilde{\Phi}$ defined by a symmetric norm on c_c. In addition one has $\mathscr{J} \subset \mathscr{J}_{\tilde{\Phi}}$, and if $\mathscr{J}$ is a Banach space with its norm then $\mathscr{J}_{\tilde{\Phi}}^{(0)} \subset \mathscr{J}$.*

(d) (non-commutative Fatou Lemma) If $A_m \in \mathscr{J}_\Phi$ with $w\!-\!\lim_{m\to\infty} A_m = A \in \mathscr{K}(\mathcal{H})$ and if $\sup_m \Phi(A_m) < \infty$, then $A \in \mathscr{J}_\Phi$ and $\Phi(A) \le \sup_m \Phi(A_m)$. If Φ is not equivalent to ℓ_∞, then A need not be assumed to be compact a priori.

As a consequence of the point (a) we shall call $\mathscr{J}_\Phi$ *a normed ideal of $\mathscr{B}(\mathcal{H})$*.

Exercise 2.4.9. *Provide a proof of the above statement. Note that the material introduced in Section 2.3 and in particular Theorem 2.3.11 are extensively used for this proof.*

Let us still mention and prove some general results which apply to arbitrary norms Φ. More detailed investigations on certain normed ideal will be realized in the next section. For the time being, let us just mention the spaces $\mathscr{I}_p$ and $\mathscr{I}_{p,w}$ which correspond to the normed ideals constructed from the symmetric norms $\|\cdot\|_p$ and $\|\cdot\|_{p,w}$ exhibited in Examples 2.3.10.

Theorem 2.4.10 (Abstract Hölder inequality)**.** *Let Φ_1, Φ_2 and Φ_3 be three symmetric norms on c_c and let J_{Φ_1}, J_{Φ_2} and J_{Φ_3} denote the corresponding maximal spaces. If for any $a \in J_{\Phi_2}$ and $b \in J_{\Phi_3}$ one has $ab \in J_{\Phi_1}$ (pointwise product) and*

$$\Phi_1(ab) \leq \Phi_2(a)\,\Phi_3(b),$$

then if $A \in \mathscr{I}_{\Phi_2}$ and $B \in \mathscr{I}_{\Phi_3}$ it follows that $AB \in \mathscr{I}_{\Phi_1}$ and

$$\Phi_1(AB) \leq \Phi_2(A)\,\Phi_3(B).$$

If either $A \in \mathscr{I}^{(0)}_{\Phi_2}$ or $B \in \mathscr{I}^{(0)}_{\Phi_3}$, then $AB \in \mathscr{I}^{(0)}_{\Phi_1}$.

Proof. By the inequality (2.9) together with Corollary 2.3.4 one infers that

$$\sum_{j=1}^{d} \mu_j(AB) \leq \sum_{j=1}^{d} \mu_j(A)\,\mu_j(B).$$

Then, by Theorem 2.3.11.(b) one deduces that

$$\begin{aligned}\Phi_1(AB) = \Phi_1\Big(\big(\mu_n(AB)\big)\Big) &\leq \Phi_1\Big(\big(\mu_n(A)\,\mu_n(B)\big)\Big)\\ &\leq \Phi_2\Big(\big(\mu_n(A)\big)\Big)\,\Phi_3\Big(\big(\mu_n(B)\big)\Big) = \Phi_2(A)\Phi_3(B).\end{aligned}$$

The second part of the statement is straightforward. □

Corollary 2.4.11. *Let $p, q, r \geq 1$ satisfy $p^{-1} = q^{-1} + r^{-1}$. If $A \in \mathscr{I}_q$ and $B \in \mathscr{I}_r$, then $AB \in \mathscr{I}_p$ with*

$$\|AB\|_p \leq \|A\|_q\,\|B\|_r.$$

For $p > 1$, if $A \in \mathscr{I}_{q,w}$ and $B \in \mathscr{I}_{r,w}$, then $AB \in \mathscr{I}_{p,w}$ with

$$\|AB\|_{p,w} \leq \frac{p}{p-1}\|A\|_{q,w}\,\|B\|_{r,w}.$$

Proof. The first part of the statement is a direct application of the previous theorem together with Hölder inequality while the second one follows from the inequality

$$\frac{(p-1)}{p}\|ab\|_{p,w} \leq \|a\|'_{q,w}\,\|b\|'_{r,w} \leq \|a\|_{q,w}\,\|b\|_{r,w}$$

with the notations introduced in Examples 2.3.10. □

Extension 2.4.12. *Study the complex interpolation in this general framework as introduced in Theorem 2.9 and 2.10 of [Sim]. Provide some applications of these abstract results.*

2.5 The Schatten ideals $\mathscr{J}_p$

In this section we focus on the normed ideals $\mathscr{J}_p$ which are closely related to the commutative ℓ_p-spaces. As mentioned earlier, the norm on $\mathscr{J}_p$ is constructed from the usual ℓ_p-norm. This material is very classical and can be found in any textbook on operator theory. Note that $\mathscr{J}_2$ is usually called *the set of Hilbert-Schmidt operators* while $\mathscr{J}_1$ corresponds to the set of *trace class operators*. More generally, the space $\mathscr{J}_p$ is called the p-Schatten ideal.

The first result about Hilbert-Schmidt operators is useful in applications. We refer for example to [Sim, Thm. 2.11] or [Amr, Prop. 2.15] for its proof.

Theorem 2.5.1. *Let (Ω, μ) be a measure space such that $\mathcal{H} := L^2(\Omega, \mu)$ is separable. An operator A belongs to $\mathscr{J}_2$ if and only if there exists a measurable function $k \in L^2(\Omega \times \Omega, \mu \otimes \mu)$ such that*

$$[Af](x) = \int_\Omega k(x, y) f(y) \mu(\mathrm{d}y). \tag{2.28}$$

In addition the following relation holds

$$\|A\|_{HS} := \|A\|_2 = \|k\|_{L^2(\Omega\times\Omega)}. \tag{2.29}$$

Note that we have used the convenient notation $\|\cdot\|_{HS}$ for the norm $\|\cdot\|_2$ which is often used for Hilbert-Schmidt operators. Now, such a simple characterization of trace class operators does not exist, and this is quite unfortunate since trace class operators often play an important role. Nevertheless, some partial results exist, as presented in the next statement for positive operators.

Theorem 2.5.2. *Let μ be a Baire measure*[2] *on a locally compact Hausdorff space Ω. Let $\mathcal{H} := L^2(\Omega, \mu)$ and let k be a continuous function on $\Omega \times \Omega$. Assume that the following two conditions hold:*

(i) For any $f \in C_c(\Omega)$ one has

$$\iint_{\Omega\times\Omega} \bar{f}(x) f(y) k(x, y) \mu(\mathrm{d}x)\mu(\mathrm{d}y) \geq 0,$$

(ii) $\int_\Omega k(x, x)\mu(\mathrm{d}x) < \infty$.

Then there exists a positive operator A defined by (2.28) which belongs to $\mathscr{J}_1$ and the following relation holds:

$$\|A\|_1 = \int_\Omega k(x, x)\mu(\mathrm{d}x). \tag{2.30}$$

[2] Recall that the Baire sets form a σ-algebra of a topological space that avoids some of the pathological properties of Borel sets. However, in Euclidean spaces the concept of a Baire set coincides with that of a Borel set.

For the proof, we refer to page 65 of [RS3, Sec. XI.4], or to [GK, Sec. III.10] for a more comprehensive approach to the subject.

Extension 2.5.3. *Study the more recent results obtained by C. Brislawn in [Bri] and mentioned in the Addendum D of [Sim, page 128].*

Let us add two results which are often used as a definition of trace class and Hilbert-Schmidt operators. Since we have introduced $\mathscr{I}_1$ and $\mathscr{I}_2$ through a different approach, one has to show that the two definitions coincide. We refer for example to [Mur, Sec. 2.4] for this alternative approach. Note that in the approach used for example in [Mur] various properties of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have to be shown independently, while in our approach all these results follow from the general theory of symmetric norms on c_c.

Proposition 2.5.4. *1) Let $A \in \mathscr{B}(\mathcal{H})$ be positive and let $\{f_n\}$ be an orthonormal basis of $\mathcal{H}$. Then $\sum_n \langle f_n, Af_n\rangle$ is independent of the choice of basis, and it is finite if and only if $A \in \mathscr{I}_1$, with $\sum_n \langle f_n, Af_n\rangle = \|A\|_1$.*

2) Let $B \in \mathscr{B}(\mathcal{H})$ and let $\{f_n\}, \{g_n\}$ be orthonormal bases of $\mathcal{H}$. Then $\sum_n \|Bf_n\|^2$ and $\sum_{n,m} |\langle g_m, Bf_n\rangle|^2$ are independent of the choice of bases and equal. They are finite if and only if $B \in \mathscr{I}_2$, and in this case are equal to $\|B\|_2^2$.

Proof. Let us first observe that 1) follows from 2). Indeed, by setting $B := A^{1/2}$, one infers from 2) that for any orthonormal basis $\{f_n\}$

$$\sum_n \langle f_n, Af_n\rangle = \sum_n \|Bf_n\|^2 = \|B\|_2^2 = \sum_j \mu_j(B)^2 = \sum_j \mu_j(A) = \|A\|_1$$

where we have used that $\mu_j(B)^2 = \mu_j(A)$ which is a direct consequence of the spectral theorem for self-adjoint operators.

For the proof of 2), observe first by Parseval's identity one has

$$\sum_n \|Bf_n\|^2 = \sum_{n,m} |\langle g_m, Bf_n\rangle|^2 = \sum_{n,m} |\langle B^* g_m, f_n\rangle|^2 = \sum_m \|B^* g_m\|^2.$$

By symmetry, one directly gets the required equality and the independence with respect to the choice of a basis. Now, if $B \in \mathscr{I}_2$, *i.e.* if $\sum_n \mu_n(B)^2 = \|B\|_2^2 < \infty$, one easily gets from the canonical decomposition of B provided in (2.2) that $\sum_n \|Bg_n\|^2$ is finite and equal to $\sum_n \mu_n(B)^2$. Conversely, if $\sum_{n,m} |\langle g_m, Bf_n\rangle|^2 < \infty$ one has

$$\sum_n |\langle g_n, Bf_n\rangle|^2 \leq \sum_{n,m} |\langle g_m, Bf_n\rangle|^2 < \infty$$

and $B \in \mathscr{I}_2$ by Proposition 2.4.7. □

In the next statement we emphasize once more the relation between Hilbert-Schmidt operators and trace class operators. Its proof follows easily from what has been introduced so far, see also [Mur, Thm. 2.4.13].

Proposition 2.5.5. *Let A be an element of $\mathscr{B}(\mathcal{H})$. The following conditions are equivalent:*

(i) A is a trace class operator,

(ii) $|A|$ is a trace class operator,

(iii) $|A|^{1/2}$ is a Hilbert-Schmidt operator,

(iv) There exists Hilbert-Schmidt operators B_1, B_2 such that $A = B_1 B_2$.

We close this section with several results on convergence in $\mathscr{J}_p$. The first one is clearly an analog of the dominated convergence theorem. Note that we shall use the convenient notation $A^{(*)}$ for A and for its adjoint A^*. For example, the condition $|A^{(*)}| \leq B$ means $|A| \leq B$ and $|A^*| \leq B$.

Theorem 2.5.6. *Let $A_m, A, B \in \mathscr{B}(\mathcal{H})$ with B self-adjoint. Assume that $|A_m^{(*)}| \leq B$, $|A^{(*)}| \leq B$ and that $w - \lim_{m\to\infty} A_m = A$. Then, if $B \in \mathscr{J}_p$ for some $p < \infty$, then $\|A_m - A\|_p \to 0$ as $m \to \infty$.*

A proof of this statement is provided in [Sim, Thm. 2.16]. The following result is also proved at the end of chapter 2 of [Sim].

Theorem 2.5.7. *Let p belongs to $[1, \infty)$, and let $\{A_n\} \subset \mathscr{J}_p$ and $A \in \mathscr{J}_p$. If $w - \lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} \|A_n\|_p = \|A\|_p$, then $\lim_{n\to\infty} \|A_n - A\|_p = 0$.*

Let us mention that more generally, results like the previous one are a consequence of the uniform convexity of some Banach spaces. We shall not go further in this direction here.

Extension 2.5.8. *Study the notion of uniform convexity for Banach spaces and deduce from this notion the content of the previous theorem.*

2.6 Usual trace

We can finally define the notion of trace, which extends the usual one on matrices. Based on Proposition 2.5.4.(1) one infers that the domain for the "trace" which is closed under $A \mapsto |A|$ can only be $\mathscr{J}_1$. More precisely one has:

Theorem 2.6.1. *Let $A \in \mathscr{J}_1$ with its canonical decomposition $A = \sum_n \mu_n(A)|g_n\rangle\langle f_n|$. Then for any orthonormal basis $\{h_m\}$ of $\mathcal{H}$ one has $\sum_m |\langle h_m, Ah_m\rangle| < \infty$ and*

$$\sum_m \langle h_m, Ah_m\rangle = \sum_n \mu_n(A)\langle f_n, g_n\rangle =: \mathrm{Tr}(A) \tag{2.31}$$

is independent of this basis. Moreover

$$|\mathrm{Tr}(A)| \leq \|A\|_1, \tag{2.32}$$

the map $A \mapsto \mathrm{Tr}(A)$ is a bounded linear functional on $\mathscr{J}_1$, and for any $A \in \mathscr{J}_1$ and $B \in \mathscr{B}(\mathcal{H})$ one has $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Proof. Let us set $\alpha_{nm} := \langle f_n, h_m\rangle\langle h_m, g_n\rangle$ which is a dss matrix. Then $\sum_m |\alpha_{nm}| \leq 1$ for each n, and one has

$$\sum_m |\langle h_m, Ah_m\rangle| = \sum_{m,n} |\alpha_{nm}|\mu_n(A) \leq \|A\|_1$$

which directly proves (2.32). In addition, the absolute convergence of the last double sum justifies an interchange in

$$\sum_m \langle h_m, Ah_m\rangle = \sum_{m,n} \alpha_{nm}\mu_n(A) = \sum_n \Big(\mu_n(A)\sum_m \alpha_{nm}\Big) = \sum_n \mu_n(A)\langle f_n, g_n\rangle.$$

The linearity of the map $A \mapsto \mathrm{Tr}(A)$ follows again from the absolute convergence of the sums. Finally, if $B \in \mathscr{B}(\mathcal{H})$ one has

$$\mathrm{Tr}(AB) = \sum_m \langle h_m, ABh_m\rangle = \sum_n \mu_n(A)\langle B^* f_n, g_n\rangle = \sum_m \langle h_m, BAh_m\rangle = \mathrm{Tr}(BA). \quad \square$$

Corollary 2.6.2. *If $A \in \mathscr{J}_1$ and $B \in \mathscr{B}(\mathcal{H})$, then one has*

$$|\mathrm{Tr}(AB)| \leq \|B\|\mathrm{Tr}(|A|).$$

Proof. By the previous theorem one has

$$|\mathrm{Tr}(AB)| \leq \|AB\|_1 \leq \|B\|\,\|A\|_1 = \|B\|\,\mathrm{Tr}(|A|). \quad \square$$

From the duality theory for symmetric norm introduced just before Theorem 2.3.12 and from the results contained in this statement one easily gets:

Theorem 2.6.3. *Let Φ and Φ' be conjugate symmetric norms on c_c. If $A \in \mathscr{J}_\Phi$ and $B \in \mathscr{J}_{\Phi'}$, then $AB \in \mathscr{J}_1$. Moreover, for each fixed $B \in \mathscr{J}_{\Phi'}$ the map $A \mapsto \mathrm{Tr}(AB)$ is a bounded linear functional in $\mathscr{J}_\Phi$ with norm $\Phi'(B)$. If Φ' is not equivalent to the ℓ_∞-norm, then every functional on $\mathscr{J}_\Phi^{(0)}$ is of this form, that is $\big(\mathscr{J}_\Phi^{(0)}\big)^* = \mathscr{J}_{\Phi'}$. In particular, $\mathscr{J}_\Phi$ is a reflexive space if and only if both Φ and Φ' are regular. If Φ is the ℓ_1-norm, then $\mathscr{J}_\Phi^* = \mathscr{B}(\mathcal{H})$.*

Since the trace on elements of $\mathscr{J}_1$ has now been defined in (2.31), a natural question is about the equality

$$\mathrm{Tr}(A) = \sum_n \lambda_n(A) \tag{2.33}$$

where $\{\lambda_n(A)\}$ corresponds to the set of eigenvalues of A, multiplicity counted. This equality is indeed correct, but as emphasized in any textbook on the subject its proof is surprisingly difficult. It has only been proved in 1959 by Lidskii. Note that the main difficulty comes from nilpotent or quasinilpotent operators (an operator A satisfying respectively $A^d = \mathbf{0}$ for some $n \in \mathbb{N}$ or $\sigma(A) = \{0\}$). We do not provide the proof of the equality (2.33) but suggest to study it as an extension:

Extension 2.6.4. *Study the proof the Lidskii's theorem, namely the equality* (2.33), *either from the information provided in [Sim, Chap. 3] or from any other reference.*

One direct consequence of the equality (2.33) is contained in the following statement.

Corollary 2.6.5. *If $A, B \in \mathscr{B}(\mathcal{H})$ have the property that both AB and BA belong to $\mathscr{J}_1$ (as for example if $A \in \mathscr{J}_\Phi$ and $B \in \mathscr{J}_{\Phi'}$ for any conjugate symmetric norms on c_c), then*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{2.34}$$

Proof. As well known, and shown for example in [Sak, Prop. 1.1.8], the operators AB and BA share the same spectrum, including the algebraic multiplicity, with the only possible exception of 0. Thus, the equality (2.34) follows directly from this fact and from (2.33). □

Let us add one more result related to integral operators which are trace class. Note that the following statement does not contradict Theorem 2.5.2 since it is assumed from the beginning that the operator is trace class.

Theorem 2.6.6. *Let $\mathcal{H} := L^2([a,b])$ and let $A \in \mathscr{J}_1$ be of the form $[Af](x) = \int_a^b k(x,y)f(y)\,\mathrm{d}y$ for some continuous function $k : [a,b] \times [a,b] \to \mathbb{C}$ and all $f \in \mathcal{H}$. Then*

$$\mathrm{Tr}(A) = \int_a^b k(x,x)\,\mathrm{d}x.$$

The proof of this statement is provided in [Sim, Thm. 3.9] and is based on the construction of an explicit basis for $\mathcal{H} := L^2([a,b])$. Many applications of the theory developed so far could be presented. Quite a lot of them are presented in the subsequent chapters of [Sim].

Up to this point, the uniqueness of the above trace has not been discussed. In fact, this uniqueness holds under an additional condition which we are going to introduce. The following material is borrowed from [Les], and we start by recalling an extension of the notion of trace. Recall that if V is a real vector space, then a map $\Phi : V \to [0,\infty]$ is *positive homogeneous* if $\Phi(\lambda v) = \lambda\Phi(v)$ for any $\lambda \geq 0$ and $v \in V$, and is *additive* if $\Phi(v+w) = \Phi(v) + \Phi(w)$ for any $v, w \in V$.

Definition 2.6.7. *A* weight *on $\mathscr{B}(\mathcal{H})$ is a map $\tau : \mathscr{B}(\mathcal{H})_+ \to [0,\infty]$ which is positive homogeneous and additive. Such a weight is* tracial *if $\tau(BB^*) = \tau(B^*B)$ for any $B \in \mathscr{B}(\mathcal{H})$.*

Note that in some references a tracial weight is simply called a trace. However, let us emphasize that a weight is only defined on the positive cone of $\mathscr{B}(\mathcal{H})$, and it can take the value ∞. Now, the trace Tr defined in (2.31) for any $A \in \mathscr{B}(\mathcal{H})_+$ is clearly a tracial weight on $\mathscr{B}(\mathcal{H})$, see also Proposition 3.4.3 and Corollary 3.4.4 in [Ped] for the proof of this statement. In addition, if we define the subset of $\mathscr{B}(\mathcal{H})_+$ on which Tr is finite, one gets $(\mathscr{J}_1)_+$, and then its linear span leads to $\mathscr{J}_1$, as introduced in the previous section.

Let us now show that up to a normalization constant this trace is the unique one on the set $\mathscr{F}(\mathcal{H})$ of finite rank operators in $\mathcal{H}$. In the present context, a *trace* τ on a complex algebra $\mathscr{A}$ is a linear functional $\mathscr{A} \to \mathbb{C}$ satisfying $\tau(AB) = \tau(BA)$ for any $A, B \in \mathscr{A}$.

Lemma 2.6.8. *Any trace on* $\mathscr{F}(\mathcal{H})$ *is proportional to* Tr.

Proof. Let $P, Q \in \mathscr{F}(\mathcal{H})$ be rank one orthogonal projections, or in other words $P = |f\rangle\langle f|$ and $Q = |g\rangle\langle g|$ for some $f, g \in \mathcal{H}$ with $\|f\| = \|g\| = 1$. We now set $T := |g\rangle\langle f|$ which is still a finite rank operator and satisfies $TT^* = |g\rangle\langle g| = Q$ and $T^*T = |f\rangle\langle f| = P$. Thus, if τ is a trace on $\mathscr{F}(\mathcal{H})$ one has

$$\tau(P) = \tau(T^*T) = \tau(TT^*) = \tau(Q),$$

which means that τ takes the same value $\lambda_\tau \in \mathbb{C}$ on all rank one orthogonal projections. Thus for any rank one orthogonal projection P one has

$$\tau(P) = \lambda_\tau = \lambda_\tau \mathrm{Tr}(P).$$

Since any $T \in \mathscr{F}(\mathcal{H})$ is a linear combination of rank one orthogonal projections, the result follows by linearity of τ and Tr. □

The properties of Tr mentioned so far are not sufficient for showing that any tracial weight on $\mathscr{B}(\mathcal{H})$ is proportional to Tr. The necessary additional property is *normality*, as introduced below. Note that we shall also impose that $\tau(B) \geq 0$ if $B \geq 0$, which is a natural requirement.

Definition 2.6.9. *A tracial weight* τ *on* $\mathscr{B}(\mathcal{H})$ *is* normal *if for any increasing sequence* $\{B_n\} \subset \mathscr{B}(\mathcal{H})_+$ *such that* $s-\lim_{n\to\infty} B_n = B \in \mathscr{B}(\mathcal{H})_+$ *one has* $\tau(B) = \sup_n \tau(B_n)$.

One can now prove the following statement:

Theorem 2.6.10. *(i) The usual trace* Tr *on* $\mathscr{B}(\mathcal{H})$ *is normal,*

(ii) If τ *is any normal tracial weight on* $\mathscr{B}(\mathcal{H})$ *then there exists a constant* $\lambda_\tau \in [0, \infty)$ *such that* $\tau(B) = \lambda_\tau \mathrm{Tr}(B)$ *for any* $B \in \mathscr{B}(\mathcal{H})_+$.

Note that an additional pathological case also exists: The tracial weight τ_∞ is defined by $\tau_\infty(B) = \infty$ for any $B \in \mathscr{B}(\mathcal{H})_+ \setminus \{\mathbf{0}\}$ and $\tau_\infty(\mathbf{0}) = 0$. In such a case one has $\lambda_{\tau_\infty} = \infty$. We shall not consider this case subsequently.

Proof. i) Let $\{f_m\}$ be an orthonormal basis of $\mathcal{H}$ and let $s-\lim_{n\to\infty} B_n = B$ in $\mathscr{B}(\mathcal{H})_+$ be an increasing sequence. Then one has $\langle f_m B_n f_m\rangle \nearrow \langle f_m, B f_m\rangle$ for any m, and therefore $\sup_n \langle f_m B_n f_m\rangle = \langle f_m, B f_m\rangle$. It follows then from the monotone convergence theorem for the discrete measure on $\mathbb{N}$ that

$$\mathrm{Tr}(B) = \sum_m \langle f_m, B f_m\rangle = \sum_m \sup_n \langle f_m, B_n f_m\rangle = \sup_n \sum_m \langle f_m, B_n f_m\rangle = \sup_n \mathrm{Tr}(B_n).$$

ii) As in the proof of Lemma 2.6.8 one observes that $\tau \restriction \mathscr{F}(\mathcal{H}) = \lambda_\tau \mathrm{Tr} \restriction \mathscr{F}(\mathcal{H})$ for some $\lambda_\tau \in [0, \infty)$. Let us now choose any increasing sequence $\{P_n\}_{n\in\mathbb{N}}$ of orthogonal projections with the dimension of $\mathsf{Ran}(P_n)$ equal to n. Then, given any $B \in \mathscr{B}(\mathcal{H})_+$ one can consider the increasing sequence $\{B^{1/2}P_nB^{1/2}\}_{n\in\mathbb{N}} \subset \mathscr{F}(\mathcal{H})$ which converges strongly to B. Since τ is assumed to be normal we get

$$\tau(B) = \sup_n \tau\big(B^{1/2}P_nB^{1/2}\big) = \sup_n \lambda_\tau \mathrm{Tr}\big(B^{1/2}P_nB^{1/2}\big) = \lambda_\tau \mathrm{Tr}(B). \qquad \square$$

The conclusion of the previous construction is that on $\mathscr{B}(\mathcal{H})$ and up to a multiplicative constant, the only tracial normal weight is Tr. If we drop the condition of normality, this is not the case, as shown in the next chapter. Note finally that the previous proof is based on the fact that the strong closure of $\mathscr{K}(\mathcal{H})$ is $\mathscr{B}(\mathcal{H})$ itself. In other contexts, such an approximation argument might not be available.

Chapter 3

The Dixmier trace

The aim of this chapter is to present the construction of Dixmier of a non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. Even if the paper of Dixmier [Dix] is only 2 pages long, we will use more pages for understanding and explaining the details. One reason for devoting so much time for this construction is that this trace had an enormous impact on the program of A. Connes in non-commutative geometry, and also several interesting applications. Such developments will be presented in the following chapters.

Before starting with the construction, let us just mention another non-trivial (but non-interesting) non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. For any $B \in \mathscr{B}(\mathcal{H})_+$ we set

$$\tau(B) := \begin{cases} \mathrm{Tr}(B) & \text{if } B \in \mathscr{F}(\mathcal{H}) \\ \infty & \text{if } B \notin \mathscr{F}(\mathcal{H}). \end{cases}$$

Note that the Dixmier trace will not be of this form. One of its special features is to vanish on the usual trace class elements of $\mathscr{B}(\mathcal{H})$.

3.1 Invariant states

The construction of the Dixmier trace relies on an invariant state on $\ell_\infty \equiv \ell_\infty(\mathbb{N})$. We provide now some information on such a state, following closely the paper [CS1] to which we refer for part of the proofs.

A *state* on ℓ_∞ consists in a positive linear functional $\omega : \ell_\infty \to \mathbb{C}$ satisfying $\omega(\mathbf{1}) = 1$. Here we use the notation $\mathbf{1}$ for the element $(1, 1, 1, \dots) \in \ell_\infty$, and recall that the last condition implies that $\|\omega\| = 1$, see [Mur, Corol. 3.3.5]. Clearly, $\|\omega\|$ denotes the norm of ω as an element of $\ell_\infty(\mathbb{N})^*$. We also recall that positivity means that $\omega(a) \geq 0$ for any $a = (a_n) \in \ell_\infty$ satisfying $a_n \geq 0$ for any $n \in \mathbb{N}$. The set of all states on ℓ_∞ is denoted by $\mathcal{S}(\ell_\infty)$.

By the positivity of ω and its normalization, let us already observe that for any real-valued $a \in \ell_\infty$ one has

$$\inf_n a_n \leq \omega(a) \leq \sup_n a_n. \tag{3.1}$$

We refer to [Lor] for a general introduction on states on ℓ_∞.

Let us now introduce three operations on ℓ_∞, namely the shift operator $S : \ell_\infty \to \ell_\infty$, the Cesàro operator $H : \ell_\infty \to \ell_\infty$, and the dilation operators $D_n : \ell_\infty \to \ell_\infty$ for any $n \in \mathbb{N}$ defined by

$$\begin{aligned} S\big((a_1, a_2, a_3, \dots)\big) &= (a_2, a_3, a_4, \dots), \\ H\big((a_1, a_2, a_3, \dots)\big) &= \Big(a_1, \frac{a_1 + a_2}{2}, \frac{a_1 + a_2 + a_3}{3}, \dots\Big), \\ D_n\big((a_1, a_2, a_3, \dots)\big) &= \big(\underbrace{a_1, \dots, a_1}_{n}, \underbrace{a_2, \dots, a_2}_{n}, \dots\big). \end{aligned}$$

The following properties of these operations can easily be checked:

Exercise 3.1.1. *The three operators $S, H, D_n : \ell_\infty \to \ell_\infty$ leave the positive cone $(\ell_\infty)_+$ invariant, leave $\mathbf{1}$ invariant and have norm 1. In addition, $\{D_n\}$ is an Abelian semi-group.*

More interesting relations can also be shown:

Extension 3.1.2. *The following properties hold:*

(i) $D_n S = S^d D_n$ *for any* $n \in \mathbb{N}$*,*

(ii) $\big(HS - SH\big)(a) \in c_0$ *for any* $a \in \ell_\infty$*,*

(iii) $\big(HD_n - D_n H\big)(a) \in c_0$ *for any* $a \in \ell_\infty$*.*

The shift operator allows us to introduce an important concept on $\mathcal{S}(\ell_\infty)$: A state ω on ℓ_∞ is called *a Banach limit* if it is invariant under translations, namely if $\omega(Sa) = \omega(a)$ for any $a \in \ell_\infty$. As a consequence of this property a Banach limit always satisfies $\omega(a) = 0$ if $a \in c_0$. The set all Banach limits will be denoted by $\mathcal{BL}(\ell_\infty)$. Note that for Banach limits the inequalities (3.1) can be slightly improved, namely

$$\liminf_{n\to\infty} a_n \le \omega(a) \le \limsup_{n\to\infty} a_n. \tag{3.2}$$

Subsequently we shall prove the existence of invariant states. The main argument in the proof is the Markov-Kakutani fixed point theorem, that we first recall.

Theorem 3.1.3 (Markov-Kakutani)**.** *Let $\mathcal{M}$ be a locally convex Hausdorff space and let Ω be a non-empty compact and convex subset of $\mathcal{M}$. Let $\mathscr{F}$ be an Abelian semigroup of continuous linear operators on $\mathcal{M}$ which satisfies $F(\Omega) \subset \Omega$ for any $F \in \mathscr{F}$. Then there exists an element $x \in \Omega$ such that $F(x) = x$ for all $F \in \mathscr{F}$.*

We shall now use this theorem for the space $\big(\ell_\infty(\mathbb{N})\big)^*$ endowed with the weak*-topology. The following statement and proof is borrowed from [CS1, Thm. 4.3].

Theorem 3.1.4. *There exists a state $\tilde{\omega}$ on ℓ_∞ such that for all $n \ge 1$ one has*

$$\tilde{\omega} \circ S = \tilde{\omega} \circ H = \tilde{\omega} \circ D_n = \tilde{\omega}.$$

In the following proof, we shall use the convenient notations

$$S^*\omega := \omega \circ S, \quad H^*\omega = \omega \circ H, \quad D_n^*\omega = \omega \circ D_n \qquad \forall \omega \in \mathcal{S}(\ell_\infty).$$

Proof. Let us set $\Omega_0 := \mathcal{BL}(\ell_\infty)$, which is the set of Banach limits, and observe that it is convex and weak*-compact, by Banach-Alaoglu theorem. Let us also observe that $D_n^*(\Omega_0) \subset \Omega_0$. Indeed by the content of the Extension 3.1.2 one infers that $S^* D_n^* = D_n^*(S^*)^d$, and therefore for any $\omega \in \Omega_0$

$$S^*(D_n^*\omega) = D_n^*(S^*)^d\omega = D_n^*\omega$$

which implies that $D_n^*\omega$ belongs to Ω_0, bt its definition. As a consequence, one can apply Theorem 3.1.3 to the set Ω_0 and to the Abelian semi-group $\{D_n^*\}$. The resulting set of fixed points will be denoted by Ω_1, namely

$$\Omega_1 := \big\{\omega \in \mathcal{S}(\ell_\infty) \mid S^*\omega = \omega \text{ and } D_n^*\omega = \omega \ \forall n \in \mathbb{N}\big\}.$$

This set is non-empty, and again it is convex and weak*-compact.

Let us now show that $H^*(\Omega_1) \subset \Omega_1$. Recall that for any $\omega \in \mathcal{BL}(\ell_\infty)$ and any $a \in c_0$ one has $\omega(a) = 0$. One then infers again from Extension 3.1.2 that for $\omega \in \Omega_1$ and any $a \in \ell_\infty$ one has

$$\big(D_n^*H^*\omega - H^*D_n^*\omega\big)(a) = \omega\Big(\big(HD_n - D_nH\big)(a)\Big) = 0.$$

As a consequence it follows that $D_n^*H^*\omega = H^*D_n^*\omega = H^*\omega$. Similarly, one also gets from Extension 3.1.2 that $S^*H^*\omega = H^*\omega$ for any $\omega \in \Omega_1$. These two properties imply that $H^*\omega$ belong to Ω_1, or equivalently $H^*(\Omega_1) \subset \Omega_1$. By applying once again Theorem 3.1.3 to the set Ω_1 and to the semi-group $\{(H^*)^d\}$ we conclude that there exists $\tilde{\omega} \in \Omega_1$ such that $H^*\tilde{\omega} = \tilde{\omega}$. Such a state $\tilde{\omega}$ satisfies all the requirements of the statement. □

3.2 Additional sequence spaces

Let us still introduce some additional sequence spaces which complement the ones already introduced in Examples 2.3.10. These spaces were not mentioned in the paper [Dix] but one of them will appear naturally in this context. Note that in Chapter 2 we concentrated on normed ideals. However, the Calkin correspondence in Theorem 2.4.5 is much stronger since it does not require to speak about norms. Here we take advantage of this fact.

First of all, for any $p \geq 1$ recall that

$$\ell_{p,w} = \big\{a \in c_0 \mid a_n^* \in O\big(n^{-1/p}\big)\big\}.$$

This clearly defines a Calkin space, see Definition 2.4.4. The corresponding two-sided ideals of $\mathscr{B}(\mathcal{H})$ is denoted by $\mathscr{J}_{p,w}$. Note that these spaces are also often denoted by $\ell_{p,\infty}$ and $\mathscr{L}_{p,\infty}$, and one has

$$\mathscr{L}_{p,\infty} = \big\{A \in \mathscr{K}(\mathcal{H}) \mid \mu_n(A) \in O\big(n^{-1/p}\big)\big\}. \tag{3.3}$$

For applications, the space $\mathscr{L}_{1,\infty}$ is the most important one of the above family. Note that one can define a quasi-norm[1] on this space by the formula

$$\|A\|_{1,\infty} := \sup_{n\geq 1} n\,\mu_n(A).$$

For an increasing and concave function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\lim_{x\searrow 0}\psi(x) = 0$ and $\lim_{x\to\infty}\psi(x) = \infty$ we define *the Lorentz sequence space*

$$m_\psi := \Big\{a \in c_0 \mid \|a\|_{m_\psi} := \sup_{n\geq 1} \frac{1}{\psi(n)} \sum_{j=1}^{d} a_j^* < \infty\Big\}. \tag{3.4}$$

Examples of such functions ψ are $x \mapsto x^\alpha$ or $x \mapsto \big(\ln(x+1)\big)^\alpha$ for any $\alpha \in (0,1]$. Again, m_ψ is a Calkin space, and the corresponding two-sided ideal is denoted by $\mathscr{J}_\psi$. Note that in the special case $\psi(x) = \ln(x+1)$ the notations $m_{1,\infty}$ and $\mathscr{M}_{1,\infty}$ are also often used in the literature, and one has

$$\mathscr{M}_{1,\infty} = \Big\{A \in \mathscr{K}(\mathcal{H}) \mid \sup_{n\geq 1} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A) < \infty\Big\}. \tag{3.5}$$

Remark 3.2.1. *The notations in the literature are not fully fixed and one has to pay attention to the definition used in each paper or book. The spaces $\mathscr{L}_{1,\infty}$ and $\mathscr{M}_{1,\infty}$ are often presented with different notations. We refer also to the Example 1.2.9 in [LSZ].*

Exercise 3.2.2. *Show that the following inclusions hold: $\ell_1 \subset \ell_{1,\infty} \subset m_{1,\infty}$. For that purpose one can also look at [LSZ, Lem. 1.2.8].*

3.3 Dixmier's construction

Even if the following proof does not correspond exactly to the content of [Dix] it is very close to it. For the arguments we mainly follow [Les, Sec. 2.3] and [CS1, Sec. 5.1].

Theorem 3.3.1. *Let ω be a state on ℓ_∞ which vanishes on c_0 and which is invariant under D_2. For any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ let us set*

$$\mathrm{Tr}_\omega(A) := \omega\Big(\Big(\frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A)\Big)_{n\in\mathbb{N}}\Big). \tag{3.6}$$

Then Tr_ω extends by linearity to a non-trivial trace on $\mathscr{M}_{1,\infty}$, and by setting $\mathrm{Tr}_\omega(A) = \infty$ for all $A \in \mathscr{B}(\mathcal{H})_+ \setminus \big(\mathscr{M}_{1,\infty}\big)_+$ one extends Tr_ω to a non-normal tracial weight on $\mathscr{B}(\mathcal{H})$. If $A \in \mathscr{J}_1$, then $\mathrm{Tr}_\omega(A) = 0$.

[1] A quasi-norm Φ on a complex vector space V is a map $V \to \mathbb{R}_+$ which satisfies for any V and $\lambda \in \mathbb{C}$ the following properties: i) $\Phi(\lambda a) = |\lambda|\Phi(a)$, ii) $\Phi(a+b) \leq c\big(\Phi(a) + \Phi(b)\big)$ for some $c > 0$, iii) $\Phi(a) = 0$ if and only if $a = 0$.

Before starting with the proof, let us mention that the existence of a state on ℓ_∞ satisfying the condition required by this theorem is already a consequence of Theorem 3.1.4. In fact, the states mentioned in this theorem satisfy an unnecessary condition related to the Cesàro operator H. The larger subset Ω_1 of states mentioned in the proof of Theorem 3.1.4 is also suitable for our purpose. Let us also mention that alternative notations are often used for (3.6) as for example

$$\mathrm{Tr}_\omega(A) \equiv \omega - \lim_{n\to\infty} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A) \equiv \lim_{\omega} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A). \tag{3.7}$$

One reason for these notations is that if the sequence $\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{d}\mu_j(A)\big)_{n\in\mathbb{N}}$ has a limit, then one has $\mathrm{Tr}_w(A) = \lim_{n\to\infty}\frac{1}{\ln(n+1)}\sum_{j=1}^{d}\mu_j(A)$. This property clearly follows from the facts that $\omega(\mathbf{1}) = 1$ and that $\omega(a) = 0$ for any $a \in c_0$.

The proof of the above statement is divided into several lemmas and exercises. For each of them, the assumptions of Theorem 3.3.1 are implicitly taken into account. First of all, recall that the notions of positive homogeneous and additive have been introduced just before Definition 2.6.7.

Lemma 3.3.2. Tr_ω *is positive homogeneous and additive on* $\big(\mathscr{M}_{1,\infty}\big)_+$.

Proof. Homogeneity property directly follows from the property $\mu_n(\lambda A) = \lambda\mu_n(A)$ for any $\lambda \geq 0$. The proof of the additivity is much more difficult and will use the assumptions made on the state ω.

i) For shortness let us set

$$\sigma_n(A) := \sum_{j=1}^{d} \mu_j(A) \quad \text{for any } A \in \mathscr{K}(\mathcal{H})_+, \tag{3.8}$$

and observe that for any $A, B \in \mathscr{K}(\mathcal{H})$ and any $n \in \mathbb{N}$ the following inequalities hold:

$$\sigma_n(A+B) \leq \sigma_n(A) + \sigma_n(B) \leq \sigma_{2n}(A+B). \tag{3.9}$$

Their proof is quite similar to the min-max principle introduced in Theorem 2.2.1. Indeed, one easily observes that

$$\sigma_n(A) = \sup\big\{\mathrm{Tr}(AP) \mid P \in \mathscr{P}(\mathcal{H}) \text{ with } \dim(P\mathcal{H}) = n\big\}.$$

The first inequality follows then directly from this observation and from the linearity of the trace Tr. For the second, fixed any $\varepsilon > 0$ and let P_A, P_B be such that $\dim(P_A\mathcal{H}) = n = \dim(P_B\mathcal{H})$ and $\mathrm{Tr}(AP_A) > \sigma_n(A) - \varepsilon$ and $\mathrm{Tr}(BP_B) > \sigma_n(B) - \varepsilon$. By setting P for the orthogonal projection on $P_A\mathcal{H} + P_B\mathcal{H}$ (often denoted by $P := P_A \vee P_B$) then we infer that

$$\mathrm{Tr}\big((A+B)P\big) = \mathrm{Tr}(AP) + \mathrm{Tr}(BP) \geq \mathrm{Tr}(AP_A) + \mathrm{Tr}(BP_B) > \sigma_n(A) + \sigma_n(B) - 2\varepsilon.$$

Since $\dim(P\mathcal{H}) \leq 2n$ and since ε is arbitrarily small, one gets

$$\sigma_{2n}(A+B) \geq \mathrm{Tr}\big((A+B)P\big) \geq \sigma_n(A) + \sigma_n(B)$$

which corresponds to the second inequality of (3.9).

ii) Let us now define $a, b, c \in \ell_\infty$ by

$$a_n := \frac{1}{\ln(n+1)}\sigma_n(A), \; b_n := \frac{1}{\ln(n+1)}\sigma_n(B), \text{ and } c_n := \frac{1}{\ln(n+1)}\sigma_n(A+B).$$

Then the inequality (3.9) reads

$$c_n \leq a_n + b_n \leq \tfrac{\ln(2n+1)}{\ln(n+1)} c_{2n}. \tag{3.10}$$

The first inequality together with the positivity of the state ω directly leads to the inequality $\mathrm{Tr}_\omega(A+B) \leq \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for any $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$. On the other hand, since $\lim_{n\to\infty} \frac{\ln(2n+1)}{\ln(n+1)} = 1$ and since ω vanishes on c_0 we infer that

$$\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\Big(\big(\tfrac{\ln(2n+1)}{\ln(n+1)} c_{2n}\big)_{n\in\mathbb{N}}\Big).$$

Thus, since we will show below that $\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\big((c_n)_{n\in\mathbb{N}}\big)$, one infers from (3.10) that $\omega(a) + \omega(b) \leq \omega(c)$, or in other words that $\mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B) \leq \mathrm{Tr}_\omega(A+B)$. The two inequalities obtained above prove the additivity of the map Tr_ω.

iii) It remains to show that

$$\omega\big((c_{2n})_{n\in\mathbb{N}}\big) = \omega\big((c_n)_{n\in\mathbb{N}}\big). \tag{3.11}$$

For that purpose, let us simply write the l.h.s. by $\omega\big((c_{2n})\big)$, and observe that by the invariance of ω under D_2 one has

$$\omega\big((c_{2n})\big) = \omega\big(D_2(c_{2n})\big) = \omega\big((c_2, c_2, c_4, c_4, c_6, c_6, \dots)\big).$$

Then, since $\omega(a) = 0$ for any $a \in c_0$, it is sufficient to show that

$$(c_2, c_2, c_4, c_4, c_6, c_6, \dots) - (c_1, c_2, c_3, c_4, c_5, c_6, \dots) \in c_0$$

in order to obtain (3.11). Thus, we are left in proving that $\lim_{n\to\infty}\big(c_{2n} - c_{2n-1}\big) = 0$.

By the definitions of quantities introduced so far one has

$$\begin{aligned} &c_{2n} - c_{2n-1} \\ &= \frac{1}{\ln(2n+1)}\sigma_{2n}(A+B) - \frac{1}{\ln(2n)}\sigma_{2n-1}(A+B) \\ &= \big(\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)}\big)\sigma_{2n-1}(A+B) + \frac{1}{\ln(2n+1)}\mu_{2n}(A+B). \end{aligned}$$

Clearly, the second term on the last line tends to 0 as $n \to \infty$. For the first term of the last line, since $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$, one infers that $\sigma_{2n-1}(A+B) = O\big(\ln(2n)\big)$. Then, since $\big(\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)}\big) = o\big(\frac{1}{\ln(2n+1)}\big)$ one deduces that the first term goes to 0 as $n \to \infty$ as well. This completes the proof of the statement. □

Before extending the map Tr_ω, let us observe that this map is non-trivial.

Exercise 3.3.3. *Show that there exists an element $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ which satisfies $\mathrm{Tr}_\omega(A) = 1$.*

By linearity, the map Tr_ω can then be extended to any element of $\mathscr{M}_{1,\infty}$. More precisely, for any self-adjoint $B \in \mathscr{M}_{1,\infty}$ we set $\mathrm{Tr}_\omega(B) = \mathrm{Tr}_\omega(B_+) - \mathrm{Tr}_\omega(B_-)$, and the Dixmier trace for an arbitrary $B \in \mathscr{M}_{1,\infty}$ is defined by $\mathrm{Tr}_\omega(B) = \mathrm{Tr}_\omega\big(\Re(B)\big) + i\mathrm{Tr}_\omega\big(\Im(B)\big)$. In addition, by setting $\mathrm{Tr}_\omega(A) = \infty$ for all $A \in \mathscr{B}(\mathcal{H})_+ \setminus \big(\mathscr{M}_{1,\infty}\big)_+$ one gets that Tr_ω is a weight on $\mathscr{B}(\mathcal{H})$. Also, since $\mu_j(BB^*) = \mu_j(B^*B)$ for any $B \in \mathscr{K}(\mathcal{H})$ and when these expressions are different from 0 one easily infers that Tr_ω is a tracial weight on $\mathscr{B}(\mathcal{H})$

Exercise 3.3.4. *Show that for any $A \in \mathscr{M}_{1,\infty}$ one has $|\mathrm{Tr}_\omega(A)| \leq \|A\|_{\mathscr{M}_{1,\infty}}$, with*

$$\|A\|_{\mathscr{M}_{1,\infty}} := \sup_{n\geq 1} \frac{1}{\ln(n+1)} \sum_{j=1}^{d} \mu_j(A).$$

Exercise 3.3.5. *Show that for any $A \in \mathscr{J}_1$ one has $\mathrm{Tr}_\omega(A) = 0$.*

As a consequence of the statement contained in the previous exercise, the tracial weight Tr_ω is non-normal, see Definition 2.6.9. Indeed, any approximation of a compact operator by finite rank operators would lead to a trivial trace Tr_ω. It only remains to show that the Tr_ω is a trace on $\mathscr{M}_{1,\infty}$.

Lemma 3.3.6. *For any $A \in \mathscr{M}_{1,\infty}$ and $B \in \mathscr{B}(\mathcal{H})$ one has $\mathrm{Tr}_\omega(AB) = \mathrm{Tr}_\omega(BA)$.*

Proof. Recall that every element of $\mathscr{B}(\mathcal{H})$ can be written has a linear combination of four unitary operators, see for example [Mur, Rem. 2.2.2]. Thus, by linearity it is sufficient to show that $\mathrm{Tr}_\omega(AU) = \mathrm{Tr}_\omega(UA)$ for any unitary $U \in \mathscr{B}(\mathcal{H})$. In addition, since A itself is a linear combination of positive operators, if is sufficient to show the previous equality for positive A. Now, such an equality follows directly from the observation that $\mu_j(AU) = \mu_j(UA) = \mu_j(A)$ for any $j \in \mathbb{N}$. □

Let us finally observe that the trace Tr_ω is a *symmetric* functional in the following sense: If $A, B \in \big(\mathscr{M}_{1,\infty}\big)_+$ satisfy $\mu_n(A) = \mu_n(B)$ for any $n \in \mathbb{N}$, then $\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(B)$.

Remark 3.3.7. *The construction above is based on an invariant states ω and on the use of the function $n \mapsto \ln(n+1)$. It it natural to wonder how much freedom one has for these choices, and how many different Dixmier traces exist ? Deep investigations in that direction have recently been performed and lot's of material has been gathered in [LSZ]. In the next section we present part of this material.*

3.4 Generalizations of the Dixmier trace

In this section we recast the construction of the Dixmier trace in a more general framework, as presented in [SU, SUZ1].

3.4.1 Extended limits

The first step consists in using the more developed theory of extended limits on L^∞ instead of states on ℓ_∞. More precisely, we shall consider $L^\infty(\mathbb{R})$ and $L^\infty(\mathbb{R}_+)$ as the set of essentially bounded Lebesgue measurable functions on $\mathbb{R}$ and $\mathbb{R}_+$ endowed with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$ or $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}_+}|f(x)|$ respectively. One aim for considering more general extended limits is to analysis the dependence on ω of the r.h.s. of (3.6).

In analogy to the operations acting on ℓ_∞ we start by introducing the translation operators, namely for any $y \in \mathbb{R}$ we define the operator $T_y : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by the relation

$$[T_y f](x) := f(x+y), \quad f \in L^\infty(\mathbb{R}).$$

We can now set:

Definition 3.4.1. *A linear functional ω on $L^\infty(\mathbb{R})$ is called* a translation invariant extended limit on $L^\infty(\mathbb{R})$ *if the following conditions are satisfied:*

(i) ω is positive, i.e. $\omega(f) \geq 0$ *whenever $f \in L_\infty$ satisfies $f \geq 0$,*

(ii) $\omega(\mathbf{1}) = 1$ where $\mathbf{1}$ is the constant function equal to 1 in $L^\infty(\mathbb{R})$,

(iii) $\omega(\chi_{(-\infty,0)}) = 0$ where $\chi_{(-\infty,0)}$ corresponds to the characteristic function on $\mathbb{R}_-$,

(iv) $\omega(T_y f) = \omega(f)$ for every $y \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R})$.

Let us note that a more appropriate name would be an *extended limit at* $+\infty$ since the behavior near $-\infty$ does not really matter.

Exercise 3.4.2. *Show that if $\lim_{x\to\infty} f(x)$ exists, then one has $\omega(f) = \lim_{x\to\infty} f(x)$, which justifies the name* extended limit. *For that purpose, one can start by showing that if $f \in L^\infty(\mathbb{R})$ has support on $\mathbb{R}_-$, then $\omega(f) = 0$.*

The following functional has been introduced and studied in [SUZ1, Sec. 3]. For any real-valued $f \in L^\infty(\mathbb{R})$ we set

$$p_T(f) := \lim_{x\to\infty} \sup_{h\geq 0} \frac{1}{x}\int_0^x f(y+h)\,\mathrm{d}y \,. \tag{3.12}$$

Note that the index T refers to translation. The main utility of this functional is contained in the following statements, whose proofs are given in [SUZ1, Thms. 13 & 14].

Theorem 3.4.3. *For any uniformly continuous and real-valued function $f \in L^\infty(\mathbb{R})$ the following equality holds:*

$$[-p_T(-f), p_T(f)] = \big\{\omega(f) \mid \omega \text{ is a translation invariant extended limit on } L^\infty(\mathbb{R})\big\}.$$

Note that the assumption about uniform continuity is necessary. As a consequence, one infers a continuous analogue of the classical result on extended limits of [Lor].

Theorem 3.4.4. *Let f be a uniformly continuous and real-valued function $f \in L^\infty(\mathbb{R})$ and let $c \in \mathbb{R}$. The equality $\omega(f) = c$ holds for every translation invariant extended limits on $L^\infty(\mathbb{R})$ if and only if*

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(y+h)\,\mathrm{d}y = c$$

uniformly in $h \geq 0$.

Extension 3.4.5. *Study the previous two theorems and their proof.*

Let us now switch from extended limits on $L^\infty(\mathbb{R})$ to extended limits on $L^\infty(\mathbb{R}_+)$. Again, by analogy with the operation acting on ℓ_∞ we can introduce the dilation operator by $\beta > 0$ by $\sigma_{1/\beta} : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ defined by

$$[\sigma_{1/\beta} f](x) := f(\beta x), \quad f \in L^\infty(\mathbb{R}_+).$$

In this framework, the notion of dilation invariant extended limit is provided by:

Definition 3.4.6. *A linear functional ω on $L^\infty(\mathbb{R}_+)$ is called* a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ *if the following conditions are satisfied:*

(i) ω is positive,

(ii) $\omega(\mathbf{1}) = 1$ where $\mathbf{1}$ is the constant function equal to 1 in $L^\infty(\mathbb{R}_+)$,

(iii) $\omega(\chi_{(0,1)}) = 0$ where $\chi_{(0,1)}$ corresponds to the characteristic function on $(0,1)$,

(iv) $\omega(\sigma_{1/\beta} f) = \omega(f)$ for every $\beta > 0$ and $f \in L^\infty(\mathbb{R}_+)$.

Obviously, Definitions 3.4.1 and 3.4.6 have been chosen such that there is a one-to-one correspondence between them. Indeed if ω is a translation invariant extended limit on $L^\infty(\mathbb{R})$, then the linear functional $\exp^*\omega$ defined on any $f \in L^\infty(\mathbb{R}_+)$ by $[\exp^*\omega](f) := \omega(f\circ\exp)$ is a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$. The converse statement also holds, by using a logarithmic function.

Exercise 3.4.7. *Fix the details of the previous observation.*

By analogy to (3.12) it is now natural to introduce the functional on any real-valued $f \in L^\infty(\mathbb{R}_+)$ by

$$p_D(f) := \lim_{x\to\infty} \sup_{\beta\geq 1} \frac{1}{\ln(x)}\int_1^x f(\beta y)\frac{\mathrm{d}y}{y}\ . \tag{3.13}$$

From the previous correspondence and from Theorems 3.4.3 and 3.4.4 one directly deduces that:

Theorem 3.4.8. *For any real $f \in L^\infty(\mathbb{R}_+)$ such that $f \circ \exp$ is uniformly continuous on $\mathbb{R}$, the following equality holds:*

$$[-p_D(-f), p_D(f)] = \big\{\omega(f) \mid \omega \text{ is a dilation invariant extended limit on } L^\infty(\mathbb{R}_+)\big\}.$$

Theorem 3.4.9. *Let $f \in L^\infty(\mathbb{R})$ be real and such that $f \circ \exp$ is a uniformly continuous function on $\mathbb{R}$, and let $c \in \mathbb{R}$. The equality $\omega(f) = c$ holds for every dilation invariant extended limits on $L^\infty(\mathbb{R}_+)$ if and only if*

$$\lim_{x\to\infty} \frac{1}{\ln(x)} \int_1^x f(\beta y)\frac{\mathrm{d}y}{y} = c$$

uniformly in $\beta \geq 1$.

3.4.2 Additional spaces on $\mathbb{R}_+$

In this subsection we mention the analogue of the sequence spaces introduced in Section 3.2 but in the continuous setting. As a first step and in order to take benefit of $\mathbb{R}_+$ instead of $\mathbb{N}$ let us provide an extension of the function μ giving the singular values of any $A \in \mathscr{K}(\mathcal{H})$. More precisely, for any $A \in \mathscr{K}(\mathcal{H})$ let us set

$$\mu(\cdot, A) := \sum_{j=1}^{\infty} \mu_j(A)\chi_{(j-1,j]}(\cdot) \tag{3.14}$$

Clearly, this function is non-increasing and satisfies the equality $\mu(n, A) = \mu_n(A)$ for any $n \in \mathbb{N}$. It is natural to call $\mu(\cdot, A)$ the *singular values function* of A.

Remark 3.4.10. *A slightly different but more common definition for this function could be given by*

$$\mu(t, A) := \inf\big\{s \geq 0 \mid \mathrm{Tr}\big(\chi_{(s,\infty)}(|A|)\big) \leq t\big\} \tag{3.15}$$

where $\chi_{(s,\infty)}(|A|)\big)$ denotes the spectral projection associated with $|A|$ on the interval (s,∞). Clearly, $\mathrm{Tr}\big(\chi_{(s,\infty)}(|A|)\big)$ gives the number of eigenvalues of $|A|$ inside the interval (s,∞) multiplicity counted. Thus, for a given $t > 0$ the r.h.s. of (3.14) *provides the minimal value s such that $|A|$ has t eigenvalues in the interval (s,∞). With the notation of* (3.14) *this function is equal to $\sum_{j=1}^{\infty} \mu_j(A)\chi_{[j-1,j)}(\cdot)$, and thus $\mu(n, A)$ would not be equal to $\mu_n(A)$ but to $\mu_{n+1}(A)$. By changing our convention on the index of the singular values (and starting with $\mu_0(A)$ instead of $\mu_1(A)$), one could have used* (3.15). *Note that the interest in* (3.15) *is that it extends quite straightforwardly to the more general context of semi-finite von Neumann algebra endowed with a semi-finite normal trace, see [LSZ] for this general framework.*

Let us now denote by $\Psi^{\mathrm{inc}}_{\mathrm{con}}(\mathbb{R}_+)$ the set of increasing and concave functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{x\to 0} \psi(x) = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$. In the present context and for any $\psi \in \Psi^{\mathrm{inc}}_{\mathrm{con}}(\mathbb{R}_+)$ it is natural to define the Lorentz ideal $\mathscr{M}_\psi$ by

$$\mathscr{M}_\psi = \Big\{A \in \mathscr{K}(\mathcal{H}) \mid \|A\|_\psi := \sup_{x>0} \frac{1}{\psi(x)} \int_0^x \mu(y, A)\,\mathrm{d}y < \infty\Big\}. \tag{3.16}$$

Also, when $\psi(x) = \ln(1 + x)$ the Lorentz ideal will be denoted by $\mathscr{M}_{1,\infty}$. This ideal is sometimes called *the Dixmier ideal.* The spaces $\mathscr{L}_{p,\infty}$ are then defined for any $p \geq 1$ by

$$\mathscr{L}_{p,\infty} = \big\{A \in \mathscr{K}(\mathcal{H}) \mid \sup_{x>0} x^{1/p}\mu(x, A) < \infty\big\}. \tag{3.17}$$

3.4.3 Dixmier traces

In this subsection we generalize the construction of Dixmier by considering dilation invariant extended limits on $\mathbb{R}_+$. Recall that the notion of weight has been introduced in Definition 2.6.7 and corresponds to a positive homogeneous and additive functional.

Definition 3.4.11. *Let ω be a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ and let $\psi \in \Psi^{\rm inc}_{\rm con}(\mathbb{R}_+)$. If the functional $\mathrm{Tr}_\omega : (\mathscr{M}_\psi)_+ \to [0,\infty)$ defined on $A \in (\mathscr{M}_\psi)_+$ by*

$$\mathrm{Tr}_\omega(A) := \omega\Big(x \mapsto \frac{1}{\psi(x)} \int_0^x \mu(y,A)\,\mathrm{d}y\Big) \tag{3.18}$$

is a weight on $\mathscr{M}_\psi$, then its extension by linearity on $\mathscr{M}_\psi$ is called a Dixmier trace on $\mathscr{M}_\psi$.

Based on a rather deep analysis, the following result has been proved in [DPSS, Thm. 3.4] and in [LSZ, Thm.6.3.3]. Note that the result is in fact proved in a slightly more general context, namely without referring to compact operators and to the specific functions $\mu(\cdot, A)$. In addition, more precise information on the functional Tr_ω are provided in [DPSS].

Theorem 3.4.12. *The Lorentz ideal $\mathscr{M}_\psi$ admits non-trivial Dixmier traces if and only if the function $\psi \in \Psi^{\rm inc}_{\rm con}(\mathbb{R}_+)$ satisfies the additional condition*

$$\liminf_{x\to\infty} \frac{\psi(2x)}{\psi(x)} = 1. \tag{3.19}$$

Before going on, let us compare this result with the result obtained in the previous section. Here we consider arbitrary $\psi \in \Psi^{\rm inc}_{\rm con}(\mathbb{R}_+)$ while in Section 3.3 only the special case $\psi(x) = \ln(x+1)$ was considered. In addition, the properties

$$\lim_{n\to\infty} \frac{\ln(2n+1)}{\ln(n+1)} = 1$$
$$\frac{1}{\ln(2n+1)} - \frac{1}{\ln(2n)} = o\Big(\frac{1}{\ln(2n+1)}\Big)$$

have been explicitly used in the previous proof. In the result mentioned above, only the condition (3.19) is necessary. In addition, since the above result corresponds to a necessary and sufficient condition it can be considered as a rather deep extension of the construction of Dixmier.

Our next aim is to characterize the dilation invariant extended limits which generate a Dixmier trace on $\mathscr{M}_\psi$. For that purpose, the following definition is useful.

Definition 3.4.13. *For any $\psi \in \Psi^{\rm inc}_{\rm con}(\mathbb{R}_+)$, a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ is* ψ-compatible *or* compatible with ψ *if*

$$\omega\Big(x \mapsto \frac{\psi(2x)}{\psi(x)}\Big) = 1.$$

With this definition at hand, the following result has been proved in [KSS, Thm. 10] or in [SU, Thm. 2.15].

Theorem 3.4.14. *Let* $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ *satisfying condition* (3.19)*. Let* ω *be a dilation invariant extended limit on* $L^\infty(\mathbb{R}_+)$ *which is compatible with* ψ*. Then the functional* Tr_ω *defined by* (3.18) *on* $(\mathscr{M}_\psi)_+$ *defines a non-normal Dixmier trace.*

In fact, a stronger statement has been proved in these references. First of all, if the functional defined by (3.18) defines a Dixmier trace, then the corresponding state ω is ψ-compatible. In addition, to any normalized fully symmetric functional φ on $\mathscr{M}_\psi$ one can associate a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ such that $\mathrm{Tr}_\omega = \varphi$. Since the notion of fully symmetric has not been introduced here (but corresponds to the property appearing in Theorem 2.3.11.(b) in the restricted setting of Section 2) we shall not go further in this direction.

It is now time to show that the continuous approach considered in this section coincides with the discrete approach of Section 3.3.

Exercise 3.4.15. *Show that if* $\psi(x) = \ln(x + 1)$*, then Theorem 3.3.1 and the results presented in this section are equivalent.*

Up to now, one question has not been discussed: how many values can one generate by $\mathrm{Tr}_\omega(A)$ for different dilation invariant extended limits ω ? In order to answer this question, let us first introduce the following definition:

Definition 3.4.16. *Let* $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ *satisfying condition* (3.19)*. An operator* $A \in \mathscr{M}_\psi$ *is called* Dixmier measurable *if all values of the Dixmier traces* $\mathrm{Tr}_\omega(A)$ *coincide.*

Let us also recall that for any $A \in \mathscr{M}_\psi$ one has the unique decomposition $A = A_1 - A_2 + iA_3 - iA_4$ with each $A_j \in (\mathscr{M}_\psi)_+$. It then follows that

$$\begin{aligned}
\mathrm{Tr}_\omega(A) &= \mathrm{Tr}_\omega(A_1) - \mathrm{Tr}_\omega(A_2) + i\mathrm{Tr}_\omega(A_3) - i\mathrm{Tr}_\omega(A_4)\\
&= \omega\Big(x \mapsto \frac{1}{\psi(x)}\int_0^x \big(\mu(y, A_1) - \mu(y, A_2) + i\mu(y, A_3) - i\mu(y, A_4)\big)\,\mathrm{d}y\Big)\\
&= \omega\Big(x \mapsto \frac{1}{\psi(x)}\int_0^x \tilde{\mu}(y, A)\,\mathrm{d}y\Big)
\end{aligned}$$

with $\tilde{\mu}(y, A) := \mu(y, A_1) - \mu(y, A_2) + i\mu(y, A_3) - i\mu(y, A_4)$. Since the function

$$x \mapsto \frac{1}{\psi(x)}\int_0^x \tilde{\mu}(y, A)\,\mathrm{d}y$$

is absolutely continuous, we can then use the criterion introduced in Theorem 3.4.9 and infer (see also [SUZ2, Cor. 3.2]) :

Theorem 3.4.17. *Let* $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ *satisfying condition* (3.19)*. An element* $A \in \mathscr{M}_\psi$ *is Dixmier measurable if and only if the limit*

$$\lim_{x\to\infty} \frac{1}{\ln(x)}\int_1^x \Big(\frac{1}{\psi(\beta y)}\int_0^{\beta y} \tilde{\mu}(z, A)\,\mathrm{d}z\Big)\frac{\mathrm{d}y}{y}$$

exists uniformly in $\beta \geq 1$*. If so,* $\mathrm{Tr}_\omega(A)$ *is equal to this limit for all Dixmier traces.*

Let us finally mention that for positive operators, the above condition can be simplified, but the condition on ψ is slightly more restrictive.

Theorem 3.4.18. *Let $\psi \in \Psi_{\rm con}^{\rm inc}(\mathbb{R}_+)$ satisfying the condition $\lim_{x\to\infty} \frac{\psi(2x)}{\psi(x)} = 1$. An element $A \in (\mathscr{M}_\psi)_+$ is Dixmier measurable if and only if the limit*

$$\lim_{x\to\infty} \frac{1}{\psi(x)} \int_0^x \mu(y, A)\,\mathrm{d}y$$

exists. If so, $\mathrm{Tr}_\omega(A)$ is equal to this limit for all Dixmier traces.

As a final remark, let us recall that this theory can be applied to a large class of von Neumann algebra instead of $\mathscr{B}(\mathcal{H})$. However, one has to be cautious with the hypotheses in all the statements since counterexamples have been constructed for checking the optimality of several results. Some of them are recalled in the reference [SU] which has been the main source of inspiration for this section.

Chapter 4

Heat kernel and ζ-function

In this chapter we present the links between the Dixmier traces and two other functions which are also quite well-known. Some additional definitions or results skipped in Chapter 3 will be introduced on the way. First of all, the notion of *symmetric* or *fully symmetric* linear functional can not be avoided any further. Recall that for any $A \in \mathscr{K}(\mathcal{H})$ the function $\mu(\cdot, A)$ has been introduced in (3.14).

Definition 4.0.19. *Let $\mathscr{M}_\psi$ be a Lorentz ideal introduced in* (3.16)*, and let φ be a linear functional on $\mathscr{M}_\psi$.*

(i) φ is symmetric *if for any $A, B \in \mathscr{M}_\psi$ with $A \geq 0$, $B \geq 0$ and satisfying $\mu(\cdot, B) = \mu(\cdot, A)$ one has $\varphi(B) = \varphi(A)$,*

(ii) φ is fully symmetric *if for any $A, B \in \mathscr{M}_\psi$ with $A \geq 0$, $B \geq 0$ and satisfying $\int_0^x \mu(y, B)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ for any $x > 0$ one has $\varphi(B) \leq \varphi(A)$.*

Note that for the notion of a symmetric norm on ℓ_∞ had already been introduced in Definition 2.3.8 and coincide with the previous one in the discrete setting. On the other hand, the notion of fully symmetric functional was only mentioned in Section 3.4.3 but was not further developed at this place. However, the inequality $\int_0^x \mu(y, B)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ corresponds to the notation $B \lll A$ in the discrete setting of Section 2.3. Note finally that a fully symmetric linear functional φ is automatically positive since $0 \leq A$ implies that $0 = \int_0^x \mu(y, 0)\,\mathrm{d}y \leq \int_0^x \mu(y, A)\,\mathrm{d}y$ for any $x > 0$, from which one infers that $0 = \varphi(0) \leq \varphi(A)$.

4.1 ζ-function residue

For a positive operator A the corresponding ζ-function is defined by the map

$$s \mapsto \zeta(s) := \mathrm{Tr}(A^s)$$

whenever this expression is meaningful. For example if there exists $s_0 > 1$ such that A^{s_0} belongs to the trace class ideal $\mathscr{J}_1$, then the previous expression is well-defined for any

$s \geq s_0$. A rather common assumption on A is to assume that $A^s \in \mathscr{J}_1$ for any $s > 1$ and to study the asymptotic behavior of $(s-1)\zeta(s)$ as $s \searrow 1$. For example if $A \in \mathscr{J}_1$, then the limit clearly exists and is equal to 0. The aim of this section is to consider more general positive operator A and to relate the limits (suitably defined) at $s = 1$ with some Dixmier traces. Here suitable means that we shall consider the limits in a broad sense, namely with the notion of extended limits already used in the previous chapter. Note that for convenience and in order to stay closer to the notations introduced so far, we shall replace the parameter s by $1 + 1/x$ and consider the limit $x \to \infty$.

First of all, recall that an extended limit γ on $L^\infty(\mathbb{R}_+)$ is a positive element of $L^\infty(\mathbb{R}_+)^*$ satisfying $\gamma(\mathbf{1}) = 1$ and such that $\gamma(f) = 0$ whenever $f \in L^\infty(\mathbb{R}_+)$ has compact support. Then, for any extended limit γ on $L^\infty(\mathbb{R}_+)$ one can define the function $\zeta_\gamma : \big(\mathscr{M}_{1,\infty}\big)_+ \to \mathbb{R}_+$ by

$$\zeta_\gamma(A) := \gamma\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(A^{1+1/x}\big)\Big). \tag{4.1}$$

Our first duty is to check that this expression is well-defined. For that purpose, we shall need a result of which can be useful in other context. Its proof can be found in [Fac, Lem. 4.1].

Lemma 4.1.1. *Let $\mu_1, \mu_2 : \mathbb{R}_+ \to \mathbb{R}$ be two decreasing and upper-bounded functions satisfying for any $x > 0$*

$$\int_0^x \mu_1(y)\,\mathrm{d}y \leq \int_0^x \mu_2(y)\,\mathrm{d}y.$$

Then, for any convex and increasing function $f : \mathbb{R} \to \mathbb{R}$ and for any $x > 0$ one has

$$\int_0^x f\big(\mu_1(y)\big)\,\mathrm{d}y \leq \int_0^x f\big(\mu_2(y)\big)\,\mathrm{d}y.$$

Note that if μ_1 and μ_2 takes values in $\mathbb{R}_+$ an important example for the function f consists in the map $\mathbb{R}_+ \ni x \mapsto x^t$ for any $t \geq 1$.

Lemma 4.1.2. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ and for any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ one has $\zeta_\gamma(A) < \infty$.*

Proof. Observe first that if C is trace class and positive, then

$$\mathrm{Tr}(C) = \sum_j \lambda_j(C) = \sum_j \mu_j(C) = \int_0^\infty \mu(y, C)\,\mathrm{d}y$$

where the function $\mu(\cdot, C)$ was introduced in (3.14). Thus, for any $C \geq 0$ such that $C^{1+1/x} \in \mathscr{J}_1$ for some $x > 0$, one deduces by functional calculus that

$$\mathrm{Tr}(C^{1+1/x}) = \int_0^\infty \mu(y, C)^{1+1/x}\,\mathrm{d}y. \tag{4.2}$$

On the other hand, for any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ one has

$$\sup_{z>0} \frac{1}{\ln(z+1)} \int_0^z \mu(y,A)\,\mathrm{d}y =: \|A\|_{1,\infty}$$

which implies that for any $z > 0$

$$\int_0^z \mu(y,A)\,\mathrm{d}y \leq \|A\|_{1,\infty} \ln(1+z) = \int_0^z \frac{\|A\|_{1,\infty}}{1+y}\,\mathrm{d}y. \tag{4.3}$$

Now, by taking these results into account as well as the content of the previous lemma one infers that

$$\begin{aligned}\operatorname{Tr}(A^{1+1/x}) = \int_0^\infty \mu(y,A)^{1+1/x}\,\mathrm{d}y &\leq \int_0^\infty \Big(\frac{\|A\|_{1,\infty}}{1+y}\Big)^{1+1/x}\mathrm{d}y \\ &= \|A\|_{1,\infty}^{1+1/x} \int_0^\infty \frac{1}{(1+y)^{1+1/x}}\,\mathrm{d}y = x\,\|A\|_{1,\infty}^{1+1/x}.\end{aligned}$$

As a consequence of this inequality and since γ is an extended limit one directly gets that $\zeta_\gamma(A) \leq \|A\|_{1,\infty}$. □

The main property of the map ζ_γ is summarized in the following statement whose proof can be find either in [LSZ, Thm. 8.6.4] or in [SZ, Thm. 8].

Theorem 4.1.3. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ the map ζ_γ extends by linearity to a fully symmetric linear functional on $\mathscr{M}_{1,\infty}$.*

Let us just mention that for the linearity it is sufficient to show that ζ_γ is a weight on $\big(\mathscr{M}_{1,\infty}\big)_+$, namely that it is positive homogeneous and additive, see Definition 2.6.7. The map ζ_γ is sometimes called *a ζ-function residue*.

As already mentioned at the end of Chapter 3, the set of all normalized (*i.e.* of norm 1) fully symmetric linear functionals on $\mathscr{M}_{1,\infty}$ is in bijective correspondence with the set of all Dixmier traces, as defined in Definition 3.4.11. This statement corresponds to the main result of [KSS]. We are naturally led to the following result.

Corollary 4.1.4. *For any extended limit γ on $L^\infty(\mathbb{R}_+)$ there exists a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ such that*

$$\zeta_\gamma = \operatorname{Tr}_\omega.$$

It is then natural to wonder about the relation between γ and ω. In fact, a simple relation has been exhibited only in a restricted setting, see [SZ, Thm.] or [LSZ, Thm. 8.6.8]. For stating the result, let us recall from Section 3.4.1 that starting from a translated invariant extended limit ω on $L^\infty(\mathbb{R})$ we have defined a dilatation invariant extended limit $\exp^* \omega$ on $L^\infty(\mathbb{R}_+)$. Conversely, starting from a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ one easily observes that defining $\ln^* \omega$ by $[\ln^* \omega](f) = \omega(f \circ \ln)$ we get a translation invariant extended limit on $L^\infty(\mathbb{R})$. Note that this extended limit is often denoted by $\omega \circ \ln$ but we shall avoid this ambiguous notation.

Theorem 4.1.5. *Let ω be a dilatation invariant extended limit on $L^\infty(\mathbb{R}_+)$ and assume that the extended limit $\ln^*\omega$ is also dilation invariant on $\mathbb{R}_+$. Then one has*

$$\zeta_{\ln^*\omega} = \mathrm{Tr}_\omega.$$

Remark 4.1.6. *In Corollary 4.1.4 it is claimed that one can associate to every ζ-function residue constructed with an extended limit γ on $L^\infty(\mathbb{R}_+)$ a Dixmier trace Tr_ω. However, let us mention that the converse is not true, namely the set of Dixmier traces on $\mathscr{M}_{1,\infty}$ is strictly larger than the set of ζ-function residues. We refer to [LSZ, Sec. 8.7] for more explanations and for a concrete counterexample.*

Let us close this section with a result about the uniqueness of the values taken by the ζ-function residues. This result will complement the one already mentioned in Theorem 3.4.18. Its proof can be found in [CS2, Thm. 7]. Recall that the notion of Dixmier measurable has been introduced in Definition 3.4.16 and means that all values obtained by $\mathrm{Tr}_\omega(A)$ are the same, for all dilation invariant extended limits ω.

Theorem 4.1.7. *For any $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ the following conditions are equivalent:*

(i) A is Dixmier measurable,

(ii) The limit $\lim_{x\to\infty} \frac{1}{\ln(x+1)} \int_0^x \mu(y,A)\,\mathrm{d}y$ exits,

(iii) The limit $\lim_{x\to\infty} \frac{1}{x}\mathrm{Tr}\big(A^{1+1/x}\big)$ exits,

(iv) The limit $\lim_{s\searrow 1}(s-1)\mathrm{Tr}\big(A^s\big)$ exits,

Furthermore, if any of these conditions is satisfied, all limiting values exist and coincide with $\mathrm{Tr}_\omega(A)$ for any dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$. These values also coincide with the limit $\lim_{s\searrow 1}(s-1)\zeta_\gamma(s)$ for any extended limit γ on $L^\infty(\mathbb{R}_+)$.

4.2 The heat kernel functional

The ζ-function mentioned in the previous section shares many properties with the heat kernel functional that we shall briefly introduce here. For a positive operator A the corresponding heat kernel function is defined by the map

$$s \mapsto \mathrm{Tr}\big(\exp(-sA^{-1})\big)$$

whenever such an expression is meaningful. Since the behavior of this function is usually studied around 0, we shall replace the parameter s by $1/x$ and consider the map $x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)$.

In order to study this function, we introduce the logarithmic mean $M : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ defined for $f \in L^\infty(\mathbb{R}_+)$ and any $x > 1$ by

$$[Mf](x) := \frac{1}{\ln(x)} \int_1^x f(y)\frac{\mathrm{d}y}{y}. \tag{4.4}$$

With this definition at hand, we define for any extended limit ω on $L^\infty(\mathbb{R}_+)$ the functional $\xi_\omega : \big(\mathscr{M}_{1,\infty}\big)_+ \to \mathbb{R}_+$ by

$$\xi_\omega(A) := (\omega \circ M)\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big). \tag{4.5}$$

In the next statement we ensure that the above expression is well defined.

Lemma 4.2.1. *Let $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ and consider ω an extended limit on $L^\infty(\mathbb{R}_+)$*

(i) The image by M of the function $x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)$ belongs to $L^\infty(\mathbb{R}_+)$,

(ii) The following equality holds

$$\xi_\omega(A) = \omega\Big(x \mapsto \frac{1}{\ln(x+1)}\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big)\Big)$$

where $\xi_\omega(A)$ is defined by (4.5).

Proof. 1) Let us first consider $A \in \big(\mathscr{M}_{1,\infty}\big)_+$ and $\mu(y) := \|A\|_{1,\infty}\frac{1}{1+y}$ for any $y > 0$. Then, by the inequality (4.3) one has for any $z > 0$

$$\int_0^z \mu(y, A)\,\mathrm{d}y \le \int_0^z \frac{\|A\|_{1,\infty}}{1+y}\,\mathrm{d}y = \int_0^z \mu(y)\,\mathrm{d}y. \tag{4.6}$$

For any fixed $x > 0$, since the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f_x(z) := z\,\mathrm{e}^{-(xz)^{-1}}$ is convex and increasing on $\mathbb{R}_+$ we infer from Lemma 4.1.1 and from the functional calculus of self-adjoint operators that for any $x > 0$ one has

$$\begin{aligned}
\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big) &= \int_0^\infty f_x\big(\mu(y,A)\big)\,\mathrm{d}y \\
&\le \int_0^\infty f_x\big(\mu(y)\big)\,\mathrm{d}y \\
&= \int_0^\infty \mu(y)\,\mathrm{e}^{-(x\mu(y))^{-1}}\,\mathrm{d}y \\
&= \int_0^\infty \|A\|_{1,\infty}\frac{1}{1+y}\,\mathrm{e}^{-(x\|A\|_{1,\infty})^{-1}(1+y)}\,\mathrm{d}y \\
&= \|A\|_{1,\infty}\int_{(x\|A\|_{1,\infty})^{-1}}^\infty \frac{1}{z}\,\mathrm{e}^{-z}\,\mathrm{d}z \\
&< \infty.
\end{aligned}$$

One thus deduces that $A\exp(-(xA)^{-1}) \in \mathscr{J}_1$ and that

$$\mathrm{Tr}\big(A\exp(-(xA)^{-1})\big) \in \mathcal{O}\big(\ln(x+1)\big) \qquad \text{for } x \to \infty. \tag{4.7}$$

2) By definition we have

$$M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big) = \Big(x \mapsto \frac{1}{\ln(x)}\int_1^x \mathrm{Tr}\big(\mathrm{e}^{-(yA)^{-1}}\big)\frac{\mathrm{d}y}{y^2}\Big).$$

However, since

$$\int_1^x \mathrm{e}^{-(yA)^{-1}}\frac{\mathrm{d}y}{y^2} = \int_{1/x}^1 \mathrm{e}^{-uA^{-1}}\,\mathrm{d}u = A\,\mathrm{e}^{-(xA)^{-1}} - A\,\mathrm{e}^{-A^{-1}},$$

it follows that

$$M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big) = \Big(x \mapsto \frac{1}{\ln(x)}\big(\mathrm{Tr}(A\,\mathrm{e}^{-(xA)^{-1}}) - \mathrm{Tr}(A\,\mathrm{e}^{-A^{-1}})\big)\Big). \quad (4.8)$$

3) By taking the previous expression into account as well as the estimate obtained in (4.7), one infers that $M\Big(x \mapsto \frac{1}{x}\mathrm{Tr}\big(\exp(-(xA)^{-1})\big)\Big)$ is bounded for x large. In addition, since the r.h.s. of (4.8) is continuous and vanishes when $x \searrow 0$, one deduces the statement (i). Since ω is an extended limit and thus vanishes on $C_0(\mathbb{R}_+)$, the statement (ii) easily follows from the expression obtained in (4.8). □

The next statement is the analogue of Theorem 4.1.3 but for the heat kernel. Its proof can be found in [LSZ, Thm. 8.2.4].

Theorem 4.2.2. *For any dilation invariant extended limit γ on $L^\infty(\mathbb{R}_+)$ the map ξ_γ extends by linearity to a fully symmetric linear functional on $\mathscr{M}_{1,\infty}$.*

By the previous result one directly infers that a statement similar to the content of Corollary 4.1.4 holds for the functional ξ_γ. However, an even stronger result holds in this case.

Theorem 4.2.3. *(i) If ω is a dilation invariant extended limit on $L^\infty(\mathbb{R}_+)$ satisfying $\omega \circ M = \omega$, then $\xi_\omega = \mathrm{Tr}_\omega$,*

(ii) For any normalized fully symmetric linear functional φ on $\mathscr{M}_{1,\infty}$ there exists a dilation invariant extended limit ω on $L^\infty(\mathbb{R}_+)$ such that $\varphi = \xi_\omega$.

These two results can be found in [LSZ, Thm. 8.2.9 & Thm. 8.3.6] to which we refer for the proofs and for more information.

Chapter 5

Traces of pseudo-differential operators

In this chapter we look at applications of the Dixmier traces in the context of pseudo-differential operators. Again, our main source of inspirations will be [SU] and [LSZ] but also the book [RT].

5.1 Pseudo-differential operators on $\mathbb{R}^d$

In this first section we recall a few classical definitions and results related to pseudo-differential operators. Our setting is clearly not the most general one and many extensions are possible. We shall use the usual multi-index notation, namely $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j \in \{0, 1, 2, \dots\}$. For shortness, we shall write $\alpha \in \mathbb{N}_0^d$ with $\mathbb{N}_0 = \{0, 1, 2 \dots\}$ (recall that the convention of Chapter 2 is that $\mathbb{N} = \{1, 2, 3 \dots\}$). We shall also use $|\alpha| := \sum_{j=1}^d \alpha_j$ and $\alpha! = \alpha_1! \dots \alpha_d!$. The other standard notations which are going to be used are $\nabla := (\partial_1, \dots, \partial_d)$ with $\partial_j := \partial_{x_j}$, $-\Delta := -\sum_{j=1}^d \partial_j^2$ which is a positive operator, and $\langle x \rangle := \big(1 + \sum_{j=1}^d x_j^2\big)^{1/2}$ for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. In the Hilbert space $L^2(\mathbb{R}^d)$, we shall also use the notation $X = (X_1, \dots, X_d)$ with X_j the self-adjoint operator of multiplication by the variable x_j, and $D = (D_1, \dots, D_d)$ with D_j the self-adjoint operator corresponding to the operator $-i\partial_j$.

Definition 5.1.1. *1) For any $m \in \mathbb{R}$, $\rho \in [0, 1]$, and $\delta \in [0, 1)$, a function $a \in C^\infty(\mathbb{R}^{3d})$ is called* an amplitude of order m *if it satisfies*

$$\big|[\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha a](x, y, \xi)\big| \le C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta(|\beta|+|\gamma|)} \tag{5.1}$$

for any $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ and all $x, y, \xi \in \mathbb{R}^d$. The set of all amplitudes satisfying (5.1) *is denoted by $\mathcal{A}_{\rho,\delta}^m(\mathbb{R}^d)$. Note that the constants $C_{\alpha,\beta,\gamma}$ depend also on the function a but not on x, y and ξ.*

2) For any amplitude $a \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$ the corresponding amplitude operator of order *m is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by*

$$[a(X,Y,D)f](x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i(x-y)\cdot\xi}\, a(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{5.2}$$

The corresponding set of operators is denoted by $\mathfrak{Op}\big(\mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)\big)$.

Remark 5.1.2. *It can be shown that the operator defined in (5.2) is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$, when this space is endowed with its usual Fréchet topology. Note also that the presence of 2π is irrelevant and mainly depends on several conventions about the Fourier transform. In fact, the above operator should be denoted by $a\big(X,Y,\frac{1}{2\pi}D\big)$ according to the convention taken in [RT]. In these notes, we mainly follow the convention of [LSZ] but warn the reader(s) that some constants have not been double-checked. It is possible that sometimes the equality $2\pi = 1$ holds !*

The main interest for dealing with amplitudes is that the expression for the adjoint operator is simple. Indeed, by using the usual scalar product $\langle\cdot,\cdot\rangle$ of $L^2(\mathbb{R}^d)$ one defines the adjoint of $a(X,Y,D)$ by the relation

$$\big\langle a(X,Y,D)^* f, g\big\rangle = \big\langle f, a(X,Y,D)g\big\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{5.3}$$

It then follows that $a(X,Y,D)^*$ is also an amplitude operator of order m with symbol a^* given by

$$a^*(x,y,\xi) = \overline{a(y,x,\xi)}. \tag{5.4}$$

The adjoint operator plays an important role for the extension by duality to operators acting on tempered distributions. Indeed, if $\mathcal{S}'(\mathbb{R}^d)$ denotes the set of tempered distributions on $\mathbb{R}^d$ and if $\Psi \in \mathcal{S}'(\mathbb{R}^d)$, then we can define $a(X,Y,D) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$[a(X,Y,D)\Psi](f) := \Psi\big(a(X,Y,D)^* f\big) \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Let us now explain the link between amplitudes and more usual symbols of the class $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$. For that purpose, we define the Fourier transform for any $f \in L^1(\mathbb{R}^d)$ by

$$[\mathscr{F} f](\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x. \tag{5.5}$$

The inverse Fourier transform is then provided by $[\mathscr{F}^{-1} f](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} f(\xi)\,\mathrm{d}\xi$

Definition 5.1.3. *1) For any $m \in \mathbb{R}$, $\rho \in [0,1]$, and $\delta \in [0,1)$, a function $a \in C^\infty(\mathbb{R}^{2d})$ is called* a symbol of order m *if it satisfies*

$$\big|[\partial_x^\beta \partial_\xi^\alpha a](x,\xi)\big| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{5.6}$$

for any $\alpha,\beta \in \mathbb{N}_0^d$ and all $x,\xi \in \mathbb{R}^d$. The set of all symbols satisfying (5.6) is denoted by $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$. Note that the constants $C_{\alpha,\beta}$ depend also on the function a but not on x and ξ.

2) For any symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ the corresponding pseudo-differential operator of order *m is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by*

$$[a(X,D)f](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \tag{5.7}$$

and $a(X,D)f \in \mathcal{S}(\mathbb{R}^d)$. The corresponding set of operators is denoted by $\mathfrak{Op}\big(\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)\big)$.

3) We set $\mathcal{S}^{-\infty}(\mathbb{R}^d) := \bigcap_{m\in\mathbb{R}} \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ (which is independent of ρ and δ) and call a smoothing operator *a pseudo-differential operator $a(X,D)$ with $a \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$.*

Before going on, let us look at the Fourier transform of a symbol. More precisely, observe that

$$\begin{aligned}[a(X,D)f](x) &= \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i (x-y)\cdot\xi}\, a(x,\xi)f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)\,\mathrm{d}y\end{aligned}$$

with

$$k(x,y) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i (x-y)\cdot\xi}\, a(x,\xi)\,\mathrm{d}\xi. \tag{5.8}$$

Remark 5.1.4. *The integral in* (5.8) *does not converge absolutely in general. This integral is usually understood as an oscillatory integral. We shall not develop this any further in these notes. However, if the function $(x,\xi) \mapsto a(x,\xi)$ decreases fast enough in ξ, then the integral can be understood in the usual sense.*

The map $(x,y) \mapsto k(x,y)$ is sometimes called the kernel (or Schwartz kernel[1]) of the operator $a(X,D)$. One of its important property is given in the following statement, see [RT, Thm. 2.3.1].

Theorem 5.1.5. *For any $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$, the corresponding kernel $k(x,y)$ defined by* (5.8) *satisfies*

$$\big|\big[\partial^\beta_{x,y} K\big](x,y)\big| \leq C_{N,\beta}|x-y|^{-N}$$

for any $N > m+n+|\beta|$ and $x \neq y$. In other words, for $x \neq y$ the map $(x,y) \mapsto k(x,y)$ is a smooth function which decays at infinity, together with all its derivatives, faster that any power of $|x-y|^{-1}$.

Additional results for pseudo-differential operators are summarized in the following statements, see [RT, Thm. 2.4.2 and Thm. 2.5.1].

[1] In the context of operators K defined by $[Kf](x) = \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y$ for $f \in C_c(\mathbb{R}^d)$ with kernel $k \in L^1_{loc}(\mathbb{R}^{2d})$ one can not prevent from recalling the important Schur's lemma which says that if the two conditions $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|k(x,y)|\mathrm{d}y < \infty$ and $\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|k(x,y)|\mathrm{d}x < \infty$ hold, then K defines a bounded operator on $L^2(\mathbb{R}^d)$.

Theorem 5.1.6 (L^2-boundedness)**.** *Let $a \in \mathcal{S}^0_{1,0}(\mathbb{R}^d)$, then the operator $a(X,D)$ extends continuously to an element of $\mathscr{B}\big(L^2(\mathbb{R}^d)\big)$.*

Theorem 5.1.7 (Composition formula)**.** *Let $a \in \mathcal{S}^{m_a}_{1,0}(\mathbb{R}^d)$ and $b \in \mathcal{S}^{m_b}_{1,0}(\mathbb{R}^d)$, then there exists $c \in \mathcal{S}^{m_a+m_b}_{1,0}(\mathbb{R}^d)$ such that the equality*

$$a(X,D)\,b(X,D) = c(X,D)$$

holds, where the product of operators in considered on the l.h.s. Moreover, one has the asymptotic formula

$$c \sim \sum_{\alpha\in\mathbb{N}_0^d} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha b) \tag{5.9}$$

where the meaning of (5.9) *is*

$$c - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha b) \;\in\; \mathcal{S}^{m_a+m_b-N}(\mathbb{R}^d) \tag{5.10}$$

for any $N > 0$.

Exercise 5.1.8. *Provide a proof of the previous statements, and check what happens for symbols in $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta < \rho \leq 1$. In particular for Theorem 5.1.7 show that if $a \in \mathcal{S}^{m_a}_{\rho,\delta}(\mathbb{R}^d)$ and $b \in \mathcal{S}^{m_b}_{\rho,\delta}(\mathbb{R}^d)$ then $c \in \mathcal{S}^{m_a+m_b}_{\rho,\delta}(\mathbb{R}^d)$.*

The link between amplitudes and symbols can now be established. Clearly, any symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ defines the amplitude $a \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$. Conversely one has:

Theorem 5.1.9. *For any amplitude $c \in \mathcal{A}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta < \rho \leq 1$, there exits a symbol $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ such that $a(X,D) = c(X,Y,D)$. Moreover, the symbol a admits the asymptotic expansion given by*

$$(x,\xi) \mapsto a(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}[\partial_\xi^\alpha \partial_y^\alpha c](x,x,\xi) \;\in\; \mathcal{S}^{m-(\rho-\delta)N}(\mathbb{R}^d)$$

for any $N > 0$.

The proof of the previous statement for $(\rho,\delta) = (1,0)$ can be found in [RT, Thm. 2.5.8]. Its extension to amplitudes with $(\rho,\delta) \neq (1,0)$ can be performed as an exercise.

For any operator $a(X,D)$, we define its L^2-adjoint by the formula

$$\big\langle a(X,D)^* f, g\big\rangle = \big\langle f, a(X,D)g\big\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d)$$

which corresponds to the relation (5.3) for amplitudes. Then, by the content of Theorem 5.1.9 together with the formula (5.4) for the amplitude of an adjoint one directly infers:

Corollary 5.1.10. *For any $a \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ with $0 \le \delta < \rho \le 1$ there exists a symbol $a^* \in \mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ such that $a(X,D)^* = a^*(X,D)$. Moreover a^* admits the asymptotic expansion*

$$(x,\xi) \mapsto a^*(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!}[\partial_\xi^\alpha \partial_y^\alpha \bar{a}](x,\xi) \;\in\; \mathcal{S}^{m-(\rho-\delta)N}(\mathbb{R}^d)$$

for any $N \ge 0$. Here $\bar{a}$ means the complex conjugate function.

The previous result implies that any pseudo-differential operator $a(X,D)$ extends to a continuous linear map from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Let us then note that a rather simple criterion allows us to know if a continuous linear operator from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is of the previous form. More precisely, if for any $\xi \in \mathbb{R}^d$ one sets $\mathrm{e}_\xi : \mathbb{R}^d \to \mathbb{C}$ by $\mathrm{e}_\xi(x) := \mathrm{e}^{2\pi i x\cdot\xi}$ then one has:

Theorem 5.1.11. *A continuous linear operator T from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is a pseudo-differential (with symbol a) if and only if the symbol a defined by*

$$a(x,\xi) := \mathrm{e}_{-\xi}(x)[T\,\mathrm{e}_\xi](x)$$

belong to $\mathcal{S}^\infty(\mathbb{R}^d) := \bigcup_{m\in\mathbb{R}} \mathcal{S}^m_{1,0}(\mathbb{R}^d)$.

Among the set of pseudo-differential operators let us still introduce those which have a *classical symbol.* For that purpose, we say that a function $a \in C^\infty(\mathbb{R}^{2d})$ is *homogeneous of order k* for some $k \in \mathbb{R}$ if for all $x \in \mathbb{R}^d$

$$a(x,\lambda\xi) = \lambda^k a(x,\xi), \quad \forall\lambda > 1,\ \forall\xi \in \mathbb{R}^d \text{ with } |\xi| \ge 1. \tag{5.11}$$

Definition 5.1.12. *1) A symbol $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ is called* classical *if there exists an asymptotic expansion $a \sim \sum_{k=0}^\infty a_{m-k}$ where each function a_{m-k} is homogeneous of order $m-k$, and if $a - \sum_{k=0}^N a_{m-k} \in \mathcal{S}^{m-N-1}_{1,0}(\mathbb{R}^d)$ for all $N \ge 0$. The set of all classical symbols of order m is denoted by $\mathcal{S}^m_{cl}(\mathbb{R}^d)$.*

2) For a classical symbol $a \in \mathcal{S}^m_{cl}(\mathbb{R}^d)$, its principal symbol *corresponds to the term a_m in the mentioned expansion.*

Note that the notion of principal symbol can be defined for more general pseudo-differential operators. For a symbol in $\mathcal{S}^m_{1,0}(\mathbb{R}^d)$ its principal symbol corresponds to the equivalent class of this symbol modulo the subclass $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$. More precisely, we set:

Definition 5.1.13. *For $a, b \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ we write $a \sim b$ if the difference $a - b$ belongs to $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$. For $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ we denote by $[a]$ the equivalent class defined by the previous equivalence relation and call it* the principal symbol *of $a(X,D)$.*

Examples 5.1.14. *1) The simplest and main example of a pseudo-differential operator is provided by the relation*

$$\big[(1-\Delta)^{m/2} f\big](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} \langle 2\pi\xi\rangle^m \hat{f}(\xi)\,\mathrm{d}\xi \qquad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{5.12}$$

In other terms the symbol corresponding to the operator $(1-\Delta)^{m/2}$ *is the map* $\xi \mapsto \langle 2\pi\xi\rangle^m$*. In addition, since the equality* $\langle x\rangle^m = |x|^m(1+|x|^{-2})^{m/2}$ *holds, by using a binomial expansion for* $|x|>1$ *one observes that any symbol which agrees with* $(2\pi)^m|x|^m$ *for* $|x|\geq 1$*, up to a symbol in* $\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)$*, share the same principal symbol as the one of* $(1-\Delta)^{m/2}$*.*

2) Let ϕ *be an element of* $C^\infty_b(\mathbb{R}^d)$*, which corresponds to the set of smooth functions with all derivatives bounded. Then the multiplication operator* $\phi(X)$ *defined by* $[\phi(X)f](x) = \phi(x)f(x)$ *is a pseudo-differential operator belonging to* $\mathcal{S}^0_{1,0}(\mathbb{R}^d)$*. In addition, the operator* $\phi(X)(1-\Delta)^{m/2}$ *belongs to* $\mathfrak{Op}\big(\mathcal{S}^m_{1,0}(\mathbb{R}^d)\big)$ *and the corresponding symbol is the map* $(x,\xi)\mapsto \phi(x)\langle 2\pi\xi\rangle^m$*.*

3) For any $\phi\in C^\infty_b(\mathbb{R}^d)$*, one infers from Theorem 5.1.7 that*

$$[\phi(X),(1-\Delta)^{m/2}] \in \mathfrak{Op}\big(\mathcal{S}^{m-1}_{1,0}(\mathbb{R}^d)\big).$$

More generally, if $A\in\mathfrak{Op}\big(\mathcal{S}^{m_a}_{1,0}(\mathbb{R}^d)\big)$ *and* $B\in\mathfrak{Op}\big(\mathcal{S}^{m_b}_{1,0}(\mathbb{R}^d)\big)$ *then one has* $[A,B]\in\mathfrak{Op}\big(\mathcal{S}^{m_a+m_b-1}_{1,0}(\mathbb{R}^d)\big)$*. This information means also that* AB *and* BA *share the same principal symbol.*

Let us briefly mention the link between pseudo-differential operators and Sobolev spaces. First of all recall that for any $s\geq 0$ the Sobolev space $\mathcal{H}^s(\mathbb{R}^d)$ is defined by

$$\mathcal{H}^s(\mathbb{R}^d) := \big\{f\in L^2(\mathbb{R}^d)\mid \|f\|_{\mathcal{H}^s} := \|\langle X\rangle^s\mathscr{F}f\|<\infty\big\}. \tag{5.13}$$

Note that this space coincide with the completion of $\mathcal{S}(\mathbb{R}^d)$ with the norm $\|\cdot\|_{\mathcal{H}^s}$. For $s>0$, the spaces $\mathcal{H}^{-s}(\mathbb{R}^d)$ can either be defined by duality, namely $\mathcal{H}^{-s}(\mathbb{R}^d) = \mathcal{H}^s(\mathbb{R}^d)^*$, or by the completion of $\mathcal{S}(\mathbb{R}^d)$ with the norm $\|f\|_{\mathcal{H}^{-s}} := \|\langle X\rangle^{-s}\mathscr{F}f\|$. Then, the main link between these spaces and pseudo-differential operators is summarized in the following statement. Recall that the definition of closed operators has been provided in Definition 1.4.6.

Theorem 5.1.15. *Let* $A := a(X,Y,D)$ *be the operator defined on* $\mathcal{S}(\mathbb{R}^d)$ *by an amplitude* $a\in\mathcal{A}^m_{1,0}(\mathbb{R}^d)$ *with* $m\geq 0$*.*

(i) *A extends continuously to a bounded linear operator from* $\mathcal{H}^s(\mathbb{R}^d)$ *to* $\mathcal{H}^{s-m}(\mathbb{R}^d)$ *for any* $s\in\mathbb{R}$*,*

(ii) *If* $m>0$ *then the extension of* $A:\mathcal{H}^m(\mathbb{R}^d)\to\mathcal{H}^0(\mathbb{R}^d)\equiv L^2(\mathbb{R}^d)$ *defines a closed operator,*

(iii) *If* $m=0$ *then the extension of* $A: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ *defines an element of* $\mathscr{B}\big(L^2(\mathbb{R}^d)\big)$*.*

Clearly, the point (iii) in the previous statement is a slight extension of the result already mentioned in Theorem 5.1.6 for symbols instead of for amplitudes Let us now close this section with the notion of compactly supported and compactly based pseudo-differential operators. Such operators have nice extension properties.

Definition 5.1.16. *Let $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be a pseudo-differential operator.*

(i) A is compactly supported *if the exists $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A\psi(X)$,*

(ii) A is compactly based *if the exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A$.*

Based on these definitions one easily infers the following lemma. Recall that the set $\mathcal{S}^{-\infty}(\mathbb{R}^d)$ has been introduced in Definition 5.1.3.

Lemma 5.1.17. *Let A, B be pseudo-differential operators. Then*

(i) If A, B are compactly supported, so are A^*, AB and BA,

(ii) A is compactly supported if and only if A and A^* are compactly based,

(iii) If A is compactly based, so is AB,

(iv) If A is compactly based, then there exists a compactly supported pseudo-differential operator A' such that $A - A' \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$.

Exercise 5.1.18. *Provide a proof of the last statement of the previous lemma.*

In relation with this last statement, let us mention a useful result about the difference $A - A'$. Unfortunately, we can not prove it here because it would require the definition of the so-called Shubin pseudo-differential operators. These operators are defined with slightly different classes of symbols. We refer to the book [Shu] for a different approach to pseudo-differential operators, and especially to Section 27 of this reference for a proof of the subsequent statement.

Lemma 5.1.19. *For any compactly based pseudo-differential operator A of order m there exists a compactly supported pseudo-differential operator A' of order m such that the difference $A - A'$ is trace class,* i.e. $A - A' \in \mathscr{J}_1$.

The following statement will play an essential role subsequently. We shall comment about its generality and its proof after the statement.

Theorem 5.1.20. *Let A be a compactly based pseudo-differential operator of order m. If $m < 0$ then the extension of $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defines a compact operator. If $m < -d$ then this extension defines a trace class operator.*

The previous statement is quite well-known for compactly supported pseudo-differential operators. A more general statement for arbitrary Schatten ideals can be found in [Ars], or an approach using operators of the form $f(X)g(D)$ can be borrowed from [Sim, Chap. 4]. The extension to compactly based pseudo-differential operators follows then directly from Lemma 5.1.19.

Remark 5.1.21. *In relation with the paper [Ars] let us mention that there exists several types of quantization of symbols on $\mathbb{R}^{2d}$. The one introduced so far corresponds to the so-called Kohn-Nirenberg quantization. Each of these quantizations has some properties of special interest: for example the Weyl quantization of real-valued functions provides self-adjoint operators, the Berezin quantization of positive functions provides positive operators, etc. Some of these quantizations can be recast in a single quantization (τ-quantization) which depends on an additional parameter τ. We refer the interested reader to Chapter 2 of the reference [Del] which presents the similarities and the differences between some of these quantizations.*

Extension 5.1.22. *Study some alternative quantization, as presented for example in [Del].*

5.2 Noncommutative residue

In this section we introduce the concept of noncommutative residue on the set of classical and compactly based pseudo-differential operators of order $-d$, where n is the space dimension. This concept is also called *Wodzicki residue* after the seminal papers [Wod1, Wod2]. In these papers the general theory is presented in the framework of global analysis on manifolds, and the special case presented here corresponds to the Remark 7.13 of [Wod1].

Before introducing the definition of noncommutative residue let us observe that if a is a classical symbol of order m with $a(X, D)$ compactly based, then one can impose that each term the expansion $a \sim \sum_{k=0}^{\infty} a_{m-k}$ has a compact support for the first variable (the variable x). Note that such symbols will simply be called classical and compactly based symbols. In addition, if m is an integer, then a_{-d} is well-defined and is equal to 0 if $-d > m$, while if m is not an integer, then we set $a_{-d} := 0$.

Definition 5.2.1. *Let $a \in \mathcal{S}^m_{cl}(\mathbb{R}^d)$ be a classical and compactly based symbols of order m. The* noncommutative residue *of $a(X, D)$ is defined by*

$$\mathsf{Res}_W\big(a(X, D)\big) := \frac{1}{d} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} a_{-d}(x, \theta)\, \mathrm{d}x\, \mathrm{d}\theta. \tag{5.14}$$

Before stating and proving some of the properties of this noncommutative residue, let us come back to the Examples 5.1.14. For ϕ in $C_c^\infty(\mathbb{R}^d)$ let us consider the operator $\phi(X)(1 - \Delta)^{-d/2}$ which is associated with a classical and compactly based symbol of order $-d$. Its principal symbol is given for $|\xi| \geq 1$ by the map $(x, \xi) \mapsto \phi(x)(2\pi|\xi|)^{-d}$. Thus, we easily get

$$\begin{aligned}\mathsf{Res}_W\big(\phi(X)(1 - \Delta)^{-d/2}\big) &= \frac{1}{d} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \phi(x)(2\pi)^{-d}\, \mathrm{d}x\, \mathrm{d}\theta \\ &= \frac{\mathsf{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d} \int_{\mathbb{R}^d} \phi(x)\, \mathrm{d}x,\end{aligned}$$

where $\mathsf{Vol}(\mathbb{S}^{d-1})$ denotes the volume of the sphere $\mathbb{S}^{d-1}$.

Proposition 5.2.2. *The noncommutative residue* Res_W

(i) is a linear functional on the set of all classical and compactly based pseudo-differential operators with symbol of order $-d$*,*

(ii) vanishes on compactly based pseudo-differential operators with symbol of order m *with* $m < -d$ *(and in particular on trace class operators),*

(iii) is a trace in the sense that if A *is a classical and compactly based pseudo-differential operator of order* m_a *and if* B *is a classical and compactly based pseudo-differential operator of order* m_b *with* $m_a + m_b = -d$*, then*

$$\mathsf{Res}_W([A, B]) = 0.$$

Proof. For (i), the linearity of Res_W is a direct consequence of the linearity of the action of taking the principal symbol on $\mathcal{S}_{cl}^{-d}(\mathbb{R}^d)$. For symbols of order $m < -d$ the noncommutative residue is trivial by its definition, which implies the statement (ii). Finally, since AB and BA share the same principal symbol, as mentioned in Examples 5.1.14, it follows that the principal symbol of $[A, B]$ is of order $-d - 1$. The statement (iii) follows then from (ii). □

It was A. Connes who realized in [Con] that this noncommutative residue can be linked to the Dixmier trace, with an equality of the form $\mathsf{Res}_W(A) = \mathrm{Tr}_\omega(A)$ for some states ω. Such an equality is often called *Connes' trace theorem*. Again this was proved in the context of global analysis on manifolds. In order to understand such a result in our context of pseudo-differential operators on $\mathbb{R}^d$ and in the framework developed in Chapter 3, additional information are necessary. In particular, since there exist several different Dixmier traces on operator which are not Dixmier measurable (see Definition 3.4.16) it is important to understand when an equality with the noncommutative residue is possible ?

5.3 Modulated operators

In this section we introduce the concept of modulated operators and study their properties. Most of this material is borrowed from [KLPS] and [LSZ]. In the sequel $\mathcal{H}$ denotes the Hilbert space $L^2(\mathbb{R}^d)$, and we recall that the Hilbert-Schmidt norm is denoted by $\|\cdot\|_2$. Part of the theory can be built with an abstract bounded and positive operator V in $\mathcal{H}$, but for simplicity and for our purpose, we shall only consider the operator $V := (1 - \Delta)^{-d/2}$.

Definition 5.3.1. *An operator* $T \in \mathscr{B}(\mathcal{H})$ *is* Laplacian-modulated *if the operator* $T\big(1 + t(1 - \Delta)^{-d/2}\big)^{-1}$ *is a Hilbert-Schmidt operator for any* $t > 0$*, and*

$$\|T\|_{mod} := \sup_{t>0} t^{1/2}\big\|T\big(1 + t(1 - \Delta)^{-d/2}\big)^{-1}\big\|_2 < \infty.$$

Note that a Laplacian-modulated operator T is automatically Hilbert-Schmidt since one has

$$\begin{aligned}\|T\|_2 &= \big\|T\big(1+(1-\Delta)^{-d/2}\big)^{-1}\big(1+(1-\Delta)^{-d/2}\big)\big\|_2 \\ &\le \big\|1+(1-\Delta)^{-d/2}\big\|\big\|T\big(1+(1-\Delta)^{-d/2}\big)^{-1}\big\|_2 \\ &\le \big(1+\big\|(1-\Delta)^{-d/2}\big\|\big)\|T\|_{mod}.\end{aligned}$$

The following statement can also easily be proved by taking into account the completeness of $\mathscr{J}_2$, see also [LSZ, Prop. 11.2.2].

Proposition 5.3.2. *The set of all Laplacian-modulated operator is a Banach space with the norm $\|\cdot\|_{mod}$. In addition, if B is Laplacian-modulated and $A \in \mathscr{B}(\mathcal{H})$ one has $\|AB\|_{mod} \le \|A\|\|B\|_{mod}$.*

In order to further study this Banach space, let us come back to some algebras of functions.

Definition 5.3.3. *A function $f \in L^1(\mathbb{R}^d)$ is a* modulated function*, written $f \in L^1_{mod}(\mathbb{R}^d)$, if*

$$\|f\|_{L^1_{mod}} := \sup_{t>0}\,(1+t)^d \int_{|x|>t} |f(x)|\,\mathrm{d}x \; < \infty. \tag{5.15}$$

Clearly, the inequality $\|f\|_{L^1} \le \|f\|_{L^1_{mod}}$ holds. Observe also that the natural operation on such functions is the convolution, as shown in the next statement.

Lemma 5.3.4. *If $f, g \in L^1_{mod}(\mathbb{R}^d)$ then the convolution $f * g$ belongs to $L^1_{mod}(\mathbb{R}^d)$.*

Proof. For any $t > 0$ observe that for $|y| > |x|/2$ one has

$$\begin{aligned}\int_{|x|>t}\int_{|y|>|x|/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x &\le \int_{\mathbb{R}^d}\int_{|y|>t/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\ &= \|f\|_{L^1}\int_{|y|>t/2}|g(y)|\,\mathrm{d}y.\end{aligned}$$

On the other hand, if $|y| \le |x|/2$ and $|x| > t$ it follows that $|x-y| \ge |x|/2 \ge t/2$, and then

$$\begin{aligned}\int_{|x|>t}\int_{|y|<|x|/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x &\le \iint_{|x-y|>t/2} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint_{|x|>t/2} |g(y)||f(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &= \|g\|_{L^1}\int_{|x|>t/2}|f(x)|\,\mathrm{d}x.\end{aligned}$$

By splitting the following integral into two parts and by using the previous estimates one gets

$$\begin{aligned}\|f * g\|_{L^1_{mod}} &= \sup_{t>0}\,(1+t)^d \int_{|x|>t} |[f*g](x)|\,\mathrm{d}x \\ &= \sup_{t>0}\,(1+t)^d \int_{|x|>t}\int_{\mathbb{R}^d} |g(y)||f(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \|f\|_{L^1}\|g\|_{L^1_{mod}} + \|g\|_{L^1}\|f\|_{L^1_{mod}} \\ &\leq 2\|f\|_{L^1_{mod}}\|g\|_{L^1_{mod}} \end{aligned} \tag{5.16}$$

which leads directly to the result. □

Based on the previous result, one gets:

Lemma 5.3.5. *$L^1_{mod}(\mathbb{R}^d)$ endowed with the convolution product is a Banach algebra.*

Proof. 1) With the definition of $\|f\|_{L^1_{mod}}$ provided in (5.15) the space $L^1_{mod}(\mathbb{R}^d)$ is clearly a normed space. We first show that this space is complete. Since the inequality $\|f\| \leq \|f\|_{L^1_{mod}}$ holds, if $\{f_p\}$ is a Cauchy sequence in the L^1_{mod}-norm it is also a Cauchy sequence in the L^1-norm. Let $f \in L^1(\mathbb{R}^d)$ denote the limit of this Cauchy sequence. Then, for any fixed $\varepsilon > 0$ let us choose $N \in \mathbb{N}$ such that

$$(1+t)^d \int_{|x|>t} |f_n(x) - f_m(x)|\,\mathrm{d}x \leq \varepsilon$$

for any $n, m \geq N$ and every $t > 0$. Then one infers by the dominated convergence theorem that for arbitrary $t > 0$ and $n \geq N$ one has

$$(1+t)^d \int_{|x|>t} |f_n(x) - f(x)|\,\mathrm{d}x = \lim_{q\to\infty}(1+t)^d \int_{|x|>t} |f_n(x) - f_m(x)|\,\mathrm{d}x \leq \varepsilon.$$

Since ε is arbitrary, one concludes that $L^1_{mod}(\mathbb{R}^d)$ is a complete vector space.

2) It has already been proved in the previous lemma that $L^1_{mod}(\mathbb{R}^d)$ is an algebra with the convolution product. In addition, it has been proved in (5.16) that $\|f*g\|_{L^1_{mod}} \leq 2\|f\|_{L^1_{mod}}\|g\|_{L^1_{mod}}$ which proves the continuity of the product, and hence makes $L^1_{mod}(\mathbb{R}^d)$ a Banach algebra. □

Additional properties of this Banach algebra are presented in [LSZ, Sec. 11.3]. For example, it is proved that the set of compactly supported L^1-functions is not dense in $L^1_{mod}(\mathbb{R}^d)$. A similar space with L^2-functions is also introduced and studied, namely

$$L^2_{mod}(\mathbb{R}^d) := \big\{ f \in L^2(\mathbb{R}^d) \mid |f|^2 \in L^1_{mod}(\mathbb{R}^d) \big\}$$

endowed with the norm $\|f\|_{L^2_{mod}} := \||f|^2\|^{1/2}_{L^1_{mod}}$. This space is again a Banach space, but despite the fact that it is made of L^2-functions, this space has not good properties with respect to the Fourier transform.

Extension 5.3.6. *Study the previous statements.*

Our next aim is to connect this Banach algebra $L^1_{mod}(\mathbb{R}^d)$ with the concept of Laplacian-modulated operators. For that purpose, let us recall that there exists a bijective relation between the set of Hilbert-Schmidt operators in $\mathcal{H} = L^2(\mathbb{R}^d)$ and the set of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$-functions, see Theorem 2.5.1. We present below a slightly modified version of this correspondence, which is based on the mentioned theorem and on Plancherel theorem.

Lemma 5.3.7. *For any Hilbert-Schmidt operator $T \in \mathscr{B}(\mathcal{H})$ there exists a unique function $p_T \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that the following relation holds:*

$$[Tf](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi}\, p_T(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi, \qquad \forall f \in L^2(\mathbb{R}^d). \tag{5.17}$$

Definition 5.3.8. *For any Hilbert-Schmidt operator T, the unique function $p_T \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying* (5.17) *is called* the symbol *of the operator T.*

Clearly, the previous definition is slightly ambiguous since it does not require the regularity conditions of the symbols of a pseudo-differential operators. However, the context together with the index $_T$ should prevent any confusion. On the other hand, the very good point of this definition is that if T is a pseudo-differential operator and a Hilbert-Schmidt operator, its symbol as a pseudo-differential operator and its symbol as a Hilbert-Schmidt operator coincide.

The main result linking all these notions is:

Proposition 5.3.9. *A Hilbert-Schmidt operator $T \in \mathscr{B}(\mathcal{H})$ is Laplacian-modulated if and only if its symbol p_T satisfies*

$$\int_{\mathbb{R}^d} |p_T(x,\cdot)|^2\,\mathrm{d}x \ \in L^1_{mod}(\mathbb{R}^d).$$

We provide below a proof of this statement. However, it involves an equivalent definition for Laplacian-modulated operator which is only provided in Lemma 5.4.9 in a slightly more general context.

Proof. It follows from Lemma 5.4.9 that T is Laplacian-modulated if and only if

$$\big\|TE_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)\big\|_2 = O(t^{-1/2}) \qquad \forall t > 0, \tag{5.18}$$

where $E_{(1-\Delta)^{-d/2}}$ denotes the spectral measure associated with the operator $(1-\Delta)^{-d/2}$. The key point is that the spectral projection $E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)$ is explicitly known, namely for suitable f and any $x \in \mathbb{R}^d$

$$\big[E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big)f\big](x) = \int_{(1+4\pi^2|\xi|^2)^{-d/2}\le t^{-1}} \mathrm{e}^{2\pi i x\cdot\xi}[\mathscr{F}f](\xi)\,\mathrm{d}\xi.$$

Now, let us define a family of projections P_t by the formula

$$[P_t f](x) := \int_{|\xi|>t} \mathrm{e}^{2\pi i x\cdot\xi}[\mathscr{F} f](\xi)\,\mathrm{d}\xi.$$

By a simple computation we then find that for any $t \geq 1$

$$P_{(c_{min}t)^{1/d}} \leq E_{(1-\Delta)^{-d/2}}\big([0,t^{-1}]\big) \leq P_{(c_{max}t)^{1/d}}$$

with $c_{min} := (4\pi^2+1)^{-d/2}$ and $c_{max} := (4\pi^2)^{-d/2}$. It follows from (5.18) that T is Laplacian-modulated if and only if $\|TP_t\|_2 = O(t^{-d/2})$. The statement can finally easily be obtained by observing that

$$\|TP_t\|_2^2 = \int_{|\xi|>t}\int_{\mathbb{R}^d} |p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi.$$

□

Remark 5.3.10. *By endowing the set of symbols of Hilbert-Schmidt Laplacian-modulated operators with the norm*

$$\|p_T\|_{mod} := \Big(\sup_{t>0}\,(1+t)^d \int_{|\xi|>t}\int_{\mathbb{R}^d} |p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2}, \tag{5.19}$$

it follows from the previous proposition and its proof that there is an isometry between the Banach space of Laplacian-modulated symbols and the Banach space of Laplacian-modulated operators mentioned in Proposition 5.3.2. Both norms have been denoted by $\|\cdot\|_{mod}$ *for that purpose.*

We shall soon show that the set of Laplacian-modulated operators is an extension of the set of compactly based pseudo-differential operators of order $-d$. For that purpose, observe first that the definition of compactly supported or compactly based operators can also be used in the context of bounded operators, namely an operator $A \in \mathscr{B}\big(L^2(\mathbb{R}^d)\big)$ is compactly supported if the exists $\phi,\psi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A\psi(X)$, while A is compactly based if the exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $A = \phi(X)A$. Then, one easgets that a Laplacian-modulated operator T is compactly supported if and only if its Schwartz kernel is compactly supported. On the other hand, this operator is compactly based if and only if its symbol p_T is compactly supported in the first variable. Note that the notion of a compactly supported operator does not really fit well with the notion of the symbol of a pseudo-differential operator or of a Laplacian-modulated operator. On the other hand, this notion can be used for the Schwartz kernel or for the kernel of an amplitude operator.

In the next statement we show that the concept of Laplacian-modulated operator extends the notion of compactly based pseudo-differential operator of degree $-d$.

Theorem 5.3.11. *Let $A = a(X,D)$ be a compactly based pseudo-differential operator with symbol in $a \in \mathcal{S}^{-d}_{1,0}(\mathbb{R}^d)$, Then A and A^* extends continuously to Laplacian-modulated operators.*

We provide below the proof of the statement for the operator A. The proof for A^* is slightly more complicated and involves Shubin pseudo-differential operators already mentioned in Section 5.1. We refer to[LSZ, Thm. 11.3.17] for the details.

Proof. By assumption one has $|a(x,\xi)| \le C\langle\xi\rangle^{-d}$ for all $x \in \mathbb{R}^d$ and a constant C independent of x and ξ. In addition, since the operator A is compactly based, its symbol a is compactly supported in the first variable. Thus, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $a(x,\xi) = 0$ for any $x \notin \Omega$. We then infer that

$$\begin{aligned}\sup_{t>0}(1+t)^d \int_{|\xi|>t}\int_{\mathbb{R}^d} |a(x,\xi)|^2 \,\mathrm{d}x\,\mathrm{d}\xi &= \sup_{t>0}(1+t)^d \int_{|\xi|>t}\int_{\Omega} |a(x,\xi)|^2 \,\mathrm{d}x\,\mathrm{d}\xi \\ &\le C^2\,|\Omega| \sup_{t>0}(1+t)^d \int_{|\xi|>t} \langle\xi\rangle^{-2d}\,\mathrm{d}\xi \\ &\le C'|\Omega| \sup_{t>0}(1+t)^d \int_t^\infty r^{-2d} r^{d-1}\,\mathrm{d}r \\ &= \frac{C'}{d}|\Omega| \sup_{t>0}(1+t)^d t^{-d} \\ &< \infty,\end{aligned}$$

where $|\Omega|$ means the Lebesgue measure of the set Ω, and C' is a constant. It follows from Proposition 5.3.9 that a corresponds to the symbol of a Laplacian-modulated operator. As a consequence, the operator A extends continuously to a Hilbert-Schmidt operator which is Laplacian-modulated. $\square$

In order to extend the noncommutative residue to all compactly based Laplacian-modulated operators, the following rather technical lemma is necessary. For that purpose we recall that any Laplacian-modulated operator T is itself a Hilbert-Schmidt operator.

Lemma 5.3.12. *Let T be a compactly based Laplacian-modulated operator, and let p_T denotes its symbol. Then the map*

$$\mathbb{N} \ni\ n \mapsto \frac{1}{\ln(n+1)} \int_{|\xi|\le n^{1/d}} \int_{\mathbb{R}^d} p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\ \in \mathbb{C}$$

is bounded

Proof. Recall first that since the operator T is compactly based, its symbol p_T is compactly supported in the first variable. Thus, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $p_T(x,\xi) = 0$ for any $x \notin \Omega$. Observe in addition that there exists a constant C (depending only on the space dimension d) such that for any $k \ge 0$

$$\big|\Omega \times \big\{\xi \in \mathbb{R}^d \mid \mathrm{e}^k \le |\xi| \le \mathrm{e}^{k+1}\big\}\big| = C|\Omega|\,\mathrm{e}^{kd}\,.$$

It then follows by an application of Cauchy-Schwartz inequality that

$$\begin{aligned}
&\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\mathbb{R}^d}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\Omega}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq C^{1/2}|\Omega|^{1/2}\Big((e^k)^d\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\Omega}|p_T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2} \\
&\leq C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod},
\end{aligned}$$

where the definition (5.19) has been used in the last step.

Based on this estimate one infers that for $t>1$

$$\begin{aligned}
&\Big|\int_{|\xi|\leq t}\int_{\mathbb{R}^d}p_T(x;\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big| \\
&\leq \Big|\int_{|\xi|\leq 1}\int_{\mathbb{R}^d}p_T(x;\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big| + \sum_{k=0}^{\lfloor\ln(t)\rfloor}\int_{e^k\leq|\xi|\leq e^{k+1}}\int_{\mathbb{R}^d}|p_T(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq \big(\ln(t)+1\big)C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+D
\end{aligned}$$

with D independent of t. By setting then $t=n^{1/d}$ for $n>1$ one gets

$$\begin{aligned}
&\frac{1}{\ln(n+1)}\Big|\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d}p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big| \\
&\leq \frac{\ln(n^{1/d})+1}{\ln(n+1)}C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+o(n) \\
&= \frac{\frac{1}{d}\ln(n)+1}{\ln(n+1)}C^{1/2}|\Omega|^{1/2}\|p_T\|_{mod}+o(n)
\end{aligned}$$

which clearly defines a bounded function of $n\in\mathbb{N}$. □

Based on this result, it is now natural to set:

Definition 5.3.13. *The map* Res, *from the set of compactly based Laplacian-modulated operator to the quotient* ℓ_∞/c_0, *is defined for any compactly based Laplacian-modulated operator* T *by*

$$\mathsf{Res}(T) := \Big[\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d}p_T(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}\Big] \tag{5.20}$$

where p_T *denotes the symbol associated with* T *and* $[\,\cdot\,]$ *denotes the equivalence class in* ℓ_∞/c_0. *This map is called the* generalized residue.

Let us directly check that this notion extends the noncommutative residue introduced in Section 5.2. First of all we need a preliminary lemma, which uses the fact proved in Theorem 5.3.11 that any compactly based pseudodifferential operator of order $-d$ extends to a compactly based Laplacian-modulated operator.

Lemma 5.3.14. *The generalized residue of a compactly based pseudo-differential operator of order $-d$ depends only on its principal symbol.*

Proof. Let A_1 and A_2 be two compactly based pseudo-differential operators of order $-d$ sharing the same principal symbol. Then the difference $A_1 - A_2$ is a compactly based pseudo-differential operator of order $-d-1$, which means that its symbol a satisfies $a|(x,\xi)| \leq C\langle\xi\rangle^{-d-1}$ for all $x, \xi \in \mathbb{R}^d$ and a constant C independent of x and ξ. Since a has compact support in the first variable, there exists a compact set $\Omega \subset \mathbb{R}^d$ such that $a(x,\xi) = 0$ if $x \notin \Omega$. Then one has

$$\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big| \leq C|\Omega| \int_{\mathbb{R}^d} \langle\xi\rangle^{-d-1}\,\mathrm{d}\xi < \infty.$$

As a consequence of this estimate, it follows from the definition of Res provided in (5.20) that $\mathsf{Res}(A_1 - A_2) = 0$, and therefore that $\mathsf{Res}(A_1) = \mathsf{Res}(A_2)$. □

In the next statement we clearly identify $\mathbb{C}$ with the set of constant elements of ℓ_∞.

Proposition 5.3.15. *For any $a \in \mathcal{S}^{-d}_{cl}(\mathbb{R}^d)$ with compact support in the first variable of all elements of its asymptotic expansion, or equivalently for any classical and compactly based pseudo-differential operator A of order $-d$ (with $A = a(X,D)$) one has*

$$\mathsf{Res}_W(A) = \mathsf{Res}(A).$$

Proof. Let us denote by a_{-d} the principal symbol of the operator a. By the previous lemma $\mathsf{Res}(A)$ depends only on the symbol a_{-d}, and is determined by the equivalence class in ℓ_∞/c_0 of the sequence

$$\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}.$$

Since a_{-d} is homogeneous of order $-d$ and is compact in its first variable one has

$$\begin{aligned}
\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi &= \int_{\mathbb{R}^d}\int_{1<|\xi|\leq n^{1/d}} |\xi|^{-d} a_{-d}\Big(x,\frac{\xi}{|\xi|}\Big)\,\mathrm{d}\xi\,\mathrm{d}x + C \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \int_1^{n^{1/d}} r^{-d} r^{d-1}\,\mathrm{d}r + C \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \ln\big(n^{1/d}\big) + C \\
&= \frac{\ln(n)}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x \ln(n) + C
\end{aligned}$$

with C a constant independent of n. As a consequence one infers that

$$\begin{aligned}
\mathsf{Res}(A) &= \Big[\Big(\frac{1}{\ln(n+1)}\int_{|\xi|\leq n^{1/d}}\int_{\mathbb{R}^d} a_{-d}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\Big)_{n\in\mathbb{N}}\Big] \\
&= \Big[\Big(\frac{\ln(n)}{\ln(n+1)}\frac{1}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x + \frac{C}{\ln(n+1)}\Big)_{n\in\mathbb{N}}\Big] \\
&= \Big[\Big(\frac{1}{d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} a_{-d}(x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x\Big)_{n\in\mathbb{N}}\Big] \\
&= \mathsf{Res}_W(A)
\end{aligned}$$

with the identification mentioned before the statement of the proposition. □

In reference [LSZ], it is shown that there exist some symbols for which the generalized residue is not a constant sequence. We provide a counterexample in the following exercise, and refer to Example 10.2.10 and Proposition 11.3.22 of that reference for more information.

Exercise 5.3.16. *Consider the smooth function $a_\star : \{\xi \in \mathbb{R}^d \mid |\xi| > 4\} \to \mathbb{R}$ given by*

$$a_\star(\xi) := |\xi|^m \Big(\sin\big(\ln(\ln(|\xi|))\big) + \cos\big(\ln(\ln(|\xi|))\big)\Big) \qquad \forall |\xi| > 4.$$

1) Based on this function, show that there exists a symbol $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^d)$ such that its principal symbol can not be a homogeneous function. For that purpose one can show that the map

$$\xi \mapsto a_\star(2|\xi|) - 2^m a_\star(|\xi|)$$

does not belong to $S^{m-1}_{1,0}(\mathbb{R}^d)$.

2) In the special case $m = -d$, let $\phi \in C^\infty_c(\mathbb{R}^d)$ and let $a \in \mathcal{S}^{-d}_{1,0}(\mathbb{R}^d)$ satisfying $a(x,\xi) := \phi(x)a_\star(\xi)$ for any $|\xi| > 4$ and any $x \in \mathbb{R}^d$. Show that

$$\mathsf{Res}\big(a(X,D)\big) = \big[(b_n)_{n\in\mathbb{N}}\big]$$

with $b_n = \frac{1}{d}\sin\big(\ln(\ln(n^{1/d}))\big)$ for n large enough. The sequence (b_n) is clearly not a convergent sequence.

5.4 Connes' trace theorem

In this section we state a generalized version of Connes' trace theorem and sketch the main arguments of its proof. Again, our framework are operators acting $\mathbb{R}^d$ while the original setting was for operators acting on compact manifolds.

Recall that the space $\mathscr{L}_{1,\infty}$ has been introduced in (3.3) and corresponds to

$$\big\{A \in \mathscr{K}(\mathcal{H}) \mid \mu_n(A) \in O\big(n^{-1}\big)\big\}. \tag{5.21}$$

Theorem 5.4.1. *Let T be a compactly supported Laplacian modulated operator with symbol p_T, and let ω be any dilation invariant extended limit on ℓ_∞. Then:*

(i) T belongs to $\mathscr{L}_{1,\infty}$ and

$$\mathrm{Tr}_\omega(T) = \omega\big(\mathsf{Res}(T)\big),$$

(ii) T is Dixmier measurable if and only if $\mathsf{Res}(T)$ is a constant sequence, and then $\mathrm{Tr}_\omega(T) = \mathsf{Res}(T)$.

Remark 5.4.2. *In the corresponding statement [LSZ, Thm. 11.5.1] the dilation invariance of the extended limit ω is not required. Indeed, it is shown in [LSZ, Sec. 9.7] that once applied to operators in $\mathscr{L}_{1,\infty}$ the dilation invariance of ω holds automatically. However, since we have not introduced this material and since our Dixmier traces were introduced on the more general space $\mathscr{M}_{1,\infty}$ we shall not consider this refinement here.*

As mentioned before, the sketch of the proof will be given subsequently. Our aim is to mention some corollaries of the previous statement.

Theorem 5.4.3. *Let A be a compactly based pseudo-differential operator of order $-d$. Then A extends continuously to an element of $\mathscr{L}_{1,\infty}$ and satisfies $\mathrm{Tr}_\omega(A) = \omega\big(\mathsf{Res}(A)\big)$ for any dilation invariant extended limit ω on ℓ_∞.*

Proof. First of all, it follows from Theorem 5.3.11 that the operator A extends continuously to a Laplacian modulated operator. In addition, there exists a function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(X)A = A$. The operator $A' := A\phi(X)$ is then compactly supported and the difference $A - A'$ is a compactly based operator and a pseudodifferential operator of order $-\infty$. Note that the operator A' corresponds to the one already mentioned in the statement (iv) of Lemma 5.1.17 and in Lemma 5.1.19. It then follows from Lemma 5.3.14 that $\mathsf{Res}(A) = \mathsf{Res}(A')$, and from Lemma 5.1.19 that $A - A' \in \mathscr{J}_1$. Thus, one infers from Theorem 5.4.1 that $A' \in \mathscr{L}_{1,\infty}$, and since $\mathscr{J}_1 \subset \mathscr{L}_{1,\infty}$ one also gets that $A \in \mathscr{L}_{1,\infty}$. Finally, again from Theorem 5.4.1 one deduces that

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A') = \omega\big(\mathsf{Res}(A')\big) = \omega\big(\mathsf{Res}(A)\big) \tag{5.22}$$

which corresponds to the statement. □

Note that this result makes the Dixmier trace of any compactly based pseudo-differential operator easily computable. Indeed, for a classical symbol the residue $\mathsf{Res}(A)$ of the corresponding pseudo-differential operator A can be computed by its Wodzicki residue, see Proposition 5.3.15, and the expression $\omega\big(\mathsf{Res}(A)\big)$ does not depend on ω. On the other hand, if the symbol is not classical, then the generalized residue $\mathsf{Res}(A)$ of the corresponding operator can be computed by (5.20) in Definition 5.3.13. Then, if this sequence is not constant, the r.h.s. of (5.22) does depend on ω, but nevertheless it makes the Dixmier trace $\mathrm{Tr}_\omega(A)$ computable. For example, the pseudo-differential operator $a(X, D)$ exhibited in Exercise 5.3.16 is compactly based and possesses a generalized residue $\mathsf{Res}\big(a(X,D)\big)$ which is not a constant sequence. It then follows that the r.h.s. of (5.22) depends on the choice of ω.

By collecting the information obtained so far one directly deduces the following statement:

Corollary 5.4.4. *For any $\phi \in C_c^\infty(\mathbb{R}^d)$ and for any dilation invariant extended limit ω on ℓ_∞ one has*

$$\mathrm{Tr}_\omega\big(\phi(X)(1-\Delta)^{-d/2}\big) = \frac{\mathsf{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d}\phi(x)\,\mathrm{d}x.$$

Let us mention that the above equality still holds if ϕ belongs to $L^2(\mathbb{R}^d)$ and has compact support. We refer to [LSZ, Thm. 11.7.5] for the proof of this extension.

We now come to the proof of Theorem 5.4.1. In fact, its content is a simple consequence of the following two major statements.

Theorem 5.4.5. *Let T be a compactly supported Laplacian-modulated operator with symbol p_T. Then $T \in \mathscr{L}_{1,\infty}$ and the map*

$$\mathbb{N} \ni\; n \mapsto \sum_{j=1}^{n}\lambda_j(T) - \int_{\mathbb{R}^d}\int_{|\xi|<n^{1/d}} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \;\in \mathbb{C} \tag{5.23}$$

is bounded, where $\lambda_j(T)$ denote the eigenvalues of T and these eigenvalues are ordered such that their modulus decrease.

Note that this result should be read with the content of Theorem 2.6.6 in mind. Indeed, in that result and for a trace class operator A its trace was expressed as a integral over its Schwartz symbol. Here, the operator T is not trace class, and p_T is not a Schwartz kernel, but anyway the difference between the partial sum of eigenvalues and a partial integral over the kernel p_T remains bounded, as a function of n.

Theorem 5.4.6 (Lidskii's type formula for the Dixmier trace)**.** *For any $A \in \mathscr{M}_{1,\infty}$ and for any dilation invariant extended limit on ℓ_∞ the following formula holds:*

$$\mathrm{Tr}_\omega(A) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=0}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big)$$

where again $\lambda_j(T)$ denote the eigenvalues of T and these eigenvalues are ordered such that their modulus decrease.

Based on the previous two statements one has:

Proof of Theorem 5.4.1. i) By Theorem 5.4.5 and the definition of the residue $\mathsf{Res}(T)$ one has $T \in \mathscr{L}_{1,\infty}$ and

$$\mathsf{Res}(T) = \Big[\Big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(T)\Big)_{n\in\mathbb{N}}\Big].$$

Since $\mathscr{L}_{1,\infty} \subset \mathscr{M}_{1,\infty}$ we can then apply Theorem 5.4.6 and infer that

$$\mathrm{Tr}_\omega(T) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(T)\big)_{n\in\mathbb{N}}\Big) = \omega\big(\mathsf{Res}(T)\big).$$

ii) It is clear that if $\mathsf{Res}(T)$ is a constant sequence, then $\omega\big(\mathsf{Res}(T)\big) = \mathsf{Res}(T)$ for any dilation invariant extended limit ω on ℓ_∞. For the reverse implication, we refer to [LSZ, Thm. 10.1.3.(f)] since the statement is based on the notion of Tauberian operator (see Definition 9.7.1 of that reference) which has not been introduced in these notes. □

In the rest of this section we provide some information about the proofs of Theorems 5.4.5 and 5.4.6. These results are rather deep statements and we shall not be able to prove them in detail. We start with Theorem 5.4.6 which also provides the necessary tools for the proof of the initial Lidskii's theorem. We first prove a necessary estimate.

Lemma 5.4.7. *Let ω be a dilation invariant extended limit on ℓ_∞, and let $A \in \mathscr{M}_{1,\infty}$. Then one has*

$$\omega\Big(\big(\frac{n}{\ln(n+1)}\mu_n(A)\big)_{n\in\mathbb{N}}\Big) = 0$$

Proof. Since $\omega = \omega \circ D_2$ with D_2 the dilation operator introduced in Section 3.1 one infers that

$$\begin{aligned}\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\mu_j(A)\big)_{n\in\mathbb{N}}\Big) &= \omega\Big(\big(\frac{1}{\ln(\lfloor n/2\rfloor+1)}\sum_{j=1}^{\lfloor n/2\rfloor}\mu_j(A)+o(n)\big)_{n\in\mathbb{N}}\Big)\\ &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{\lfloor n/2\rfloor}\mu_j(A)+o(n)\big)_{n\in\mathbb{N}}\Big)\end{aligned}$$

where we have used that $\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(\lfloor n/2\rfloor+1)} = 1$. As a consequence, one has

$$\begin{aligned}0 &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\mu_j(A)\big)_{n\in\mathbb{N}}\Big) - \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{\lfloor n/2\rfloor}\mu_j(A)+o(n)\big)_{n\in\mathbb{N}}\Big)\\ &= \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=\lfloor n/2\rfloor+1}^{n}\mu_j(A)+o(n)\big)_{n\in\mathbb{N}}\Big)\\ &\geq \omega\Big(\big(\frac{n}{2\ln(n+1)}\mu_n(A)\big)_{n\in\mathbb{N}}\Big)\end{aligned}$$

from which one deduces the statement. □

Proof of Theorem 5.4.6. 1) First of all, let $A \in \mathscr{M}_{1,\infty}$ be self-adjoint, and recall that $A = A_+ - A_-$ with $A_\pm \geq 0$. By the linearity of the Dixmier trace one has

$$\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A_+) - \mathrm{Tr}_\omega(A_-) = \omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\big\{\lambda_j(A_+) - \lambda_j(A_-)\big\}\big)_{n\in\mathbb{N}}\Big).$$

In the point 2) below we shall prove that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j(A_+)+\lambda_j(A_-)\big\}\Big|\leq n\mu_n(A). \tag{5.24}$$

It then follows from Lemma 5.4.7 that $\omega\Big(\big(\frac{1}{\ln(n+1)}n\mu_n(A)\big)_{n\in\mathbb{N}}\Big)=0$, and therefore

$$\mathrm{Tr}_\omega(A)=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big).$$

If $A\in\mathscr{M}_{1,\infty}$ is a normal operator, it follows from the previous paragraph that

$$\begin{aligned}\mathrm{Tr}_\omega(A)&=\mathrm{Tr}_\omega\big(\Re(A)\big)+i\mathrm{Tr}_\omega\big(\Im(A)\big)\\&=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\big\{\lambda_j\big(\Re(A)\big)+i\lambda_j\big(\Im(A)\big)\big\}\big)_{n\in\mathbb{N}}\Big).\end{aligned}$$

Again in the point 2) below we shall prove that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j\big(\Re(A)\big)-i\lambda_j\big(\Im(A)\big)\big\}\Big|\leq 5n\mu_n(A), \tag{5.25}$$

from which one infers with Lemma 5.4.7 that

$$\mathrm{Tr}_\omega(A)=\omega\Big(\big(\frac{1}{\ln(n+1)}\sum_{j=1}^{n}\lambda_j(A)\big)_{n\in\mathbb{N}}\Big).$$

For the general case $A\in\mathscr{M}_{1,\infty}$ one has to rely on a rather deep decomposition of A, namely $A=N+Q$ with $N,Q\in\mathscr{M}_{1,\infty}$, N normal, Q satisfying $\mathrm{Tr}_\omega(Q)=0$, and $\lambda_j(A)=\lambda_j(N)$. This decomposition is provided for example in [LSZ, Thm. 5.5.1] in a more general framework. Note also that this decomposition can be used for proving the usual Lidskii's theorem, see (2.33). With this information at hand, the proof of the statement follows directly.

2) For (5.24) one first observes that for any $n\in\mathbb{N}$

$$\big\{\lambda_j(A)\big\}_{j=1}^{n}\subset\Big\{\big\{\lambda_j(A_+)\big\}_{j=1}^{n}\cup\big\{-\lambda_j(A_-)\big\}_{j=1}^{n}\Big\}.$$

Indeed, this easily follows from the functional calculus of the self-adjoint operator A. In addition, one also observes that

$$\Big\{\big\{\lambda_j(A_+)\big\}_{j=1}^{n}\cup\big\{-\lambda_j(A_-)\big\}_{j=1}^{n}\Big\}\setminus\big\{\lambda_j(A)\big\}_{j=1}^{n}\subset\big\{\lambda\in\mathbb{C}\mid|\lambda|\leq|\lambda_n(A)|\big\}$$

and that the cardinality of the set on the l.h.s. contains at most n elements. It then follows that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j(A_+)+\lambda_j(A_-)\big\}\Big|\leq n|\lambda_n(A)|=n\mu_n(A). \tag{5.26}$$

For (5.25) recall first that since A is a normal compact operator it has the canonical form $A = \sum_j \lambda_j(A)|f_j\rangle\langle f_j|$ for $\lambda_j(A) \in \mathbb{C}$ ordered with a decrease of their modulus. It then follows that for any $n \in \mathbb{N}$

$$\big\{\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\lambda| > \mu_n(A)\}\big\} \subset \{\lambda_j(A)\}_{j=1}^n .$$

One also observes that

$$\{\lambda_j(A)\}_{j=1}^n \setminus \big\{\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\lambda| > \mu_n(A)\}\big\} \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \mu_n(A)\}$$

and that the cardinality of the set on the l.h.s. contains at most n elements. As a consequence one has

$$\Big|\sum_{j=1}^n \lambda_j(A) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \lambda\Big| \leq n\mu_n(A).$$

By a similar argument one also gets that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(\Re(A)),|\lambda|>\mu_n(A)} \lambda\Big| \leq n\mu_n(A).$$

Since $\Re\big(\sigma(A)\big) = \sigma\big(\Re(A)\big)$, by the normality of A, this is equivalent to

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(A),|\Re(\lambda)|>\mu_n(A)} \Re(\lambda)\Big| \leq n\mu_n(A). \tag{5.27}$$

On the other hand one infers that

$$\begin{aligned}
&\Big|\sum_{\lambda\in\sigma(A),|\Re(\lambda)|>\mu_n(A)} \Re(\lambda) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Re(\lambda)\Big| \\
&\leq \sum_{\lambda\in\sigma(A),|\Re(\lambda)|\leq\mu_n(A)| \text{ and } |\lambda|\geq\mu_n(A)} |\Re(\lambda)| \\
&\leq \sum_{\lambda\in\sigma(A),|\lambda|\geq\mu_n(A)} \mu_n(A) \\
&= n\mu_n(A).
\end{aligned}$$

By this estimate together with (5.27) we finally infer that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Re(A)\big) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Re(\lambda)\Big| \leq 2n\mu_n(A).$$

Similarly, one can also deduce that

$$\Big|\sum_{j=1}^n \lambda_j\big(\Im(A)\big) - \sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)} \Im(\lambda)\Big| \leq 2n\mu_n(A).$$

By combining the previous two estimates one infers that

$$\Big|\sum_{j=1}^{n}\lambda_j\big(\Re(A)\big)+i\lambda_j\big(\Im(A)\big)-\sum_{\lambda\in\sigma(A),|\lambda|>\mu_n(A)}\lambda\Big|\le 4n\mu_n(A), \tag{5.28}$$

It finally follows from (5.26) and (5.28) that

$$\Big|\sum_{j=1}^{n}\big\{\lambda_j(A)-\lambda_j\big(\Re(A)\big)-i\lambda_j\big(\Im(A)\big)\big\}\Big|\le 5n\mu_n(A),$$

as announced. □

Let us now come to the proof of Theorem 5.4.5 which is at the heart of Connes' trace theorem. A first step in the proof consists in studying more deeply the notion of V-modulated operator. As already mentioned at the beginning of Section 5.3 this notion is more general than Laplacian-modulated and has some advantages. For the record:

Definition 5.4.8. *Let $V\in\mathscr{B}(\mathcal{H})$ be positive. An operator $T\in\mathscr{B}(\mathcal{H})$ is* V-modulated *if the operator $T\big(1+tV\big)^{-1}$ is a Hilbert-Schmidt operator for any $t>0$, and*

$$\|T\|_{mod}:=\sup_{t>0}\ t^{1/2}\big\|T\big(1+tV\big)^{-1}\big\|_2\ <\infty. \tag{5.29}$$

Before going on with the main result related to V-modulated operator, let us provide an equivalent definition. It proof involves the functional calculus of the self-adjoint operator V.

Lemma 5.4.9. *Let $V\in\mathscr{J}_2$ be positive. An operator $T\in\mathscr{B}(\mathcal{H})$ is* V-modulated *if and only if*

$$\big\|TE_V\big([0,t^{-1}]\big)\big\|_2=O(t^{-1/2})\qquad\forall t>0, \tag{5.30}$$

where E_V denotes the spectral measure associated with the operator V.

Proof. For fixed $t>0$, observe first that for any $x\in\mathbb{R}_+$ one has $1\le 2(1+tx)^{-1}$ if and only if $x\le 1/t$. Since in addition $2(1+tx)^{-1}>0$ one infers that the following inequality holds for functions: $\chi_{[0,1/t]}\le 2(1+t\,\cdot\,)^{-1}$. By functional calculus for V it follows that

$$E_V\big([0,t^{-1}]\big)\equiv\chi_{[0,1/t]}(V)\le 2(1+tV)^{-1}.$$

Thus, if we assume that T satisfies (5.29) one infers that

$$\begin{aligned}\big\|TE_V\big([0,t^{-1}]\big)\big\|_2&=\big\|T\big(2(1+tV)^{-1}\big)E_V\big([0,t^{-1}]\big)\big(2(1+tV)^{-1}\big)^{-1}\big\|_2\\&\le 2\big\|T(1+tV)^{-1}\big\|_2\ \big\|E_V\big([0,t^{-1}]\big)\big(2(1+tV)^{-1}\big)^{-1}\big\|\\&\le\big\|T(1+tV)^{-1}\big\|_2\\&\le\|T\|_{mod}\,t^{-1/2}.\end{aligned}$$

For the converse assertion, let us first assume that $\|V\| < 1$. Since the inequality (5.29) is always satisfied for $t \in (0, 1)$ we can consider without restriction that $t \geq 1$. Let $k \in \mathbb{N}_0$ such that $t \in [2^k, 2^{k+1})$. Then by assuming (5.30) one has

$$\begin{aligned}
\|T(1+tV)^{-1}\| &\leq \|TE_V([0,2^{-k}])\|_2 + \sum_{j=0}^{k-1} \|TE_V((2^{-j-1},2^{-j}])(1+tV)^{-1}\|_2 \\
&\leq O(t^{-1/2}) + \sum_{j=0}^{k-1} (1+t2^{-j-1})^{-1} \|TE_V((2^{-j-1},2^{-j}])\|_2 \\
&\leq O(t^{-1/2}) + C\sum_{j=0}^{k-1} (1+2^{k-j-1})^{-1}\, 2^{-j/2} \\
&= O(t^{-1/2}) + C\sum_{j=0}^{k-1} \frac{\sqrt{2}\, 2^{-k/2}}{2^{(j-k+1)/2} + 2^{-(j-k+1)/2}} \\
&\leq O(t^{-1/2}).
\end{aligned}$$

Note that for the summation in the last term one can use an argument involving the estimate $\int_{\mathbb{R}} \frac{1}{\cosh(x)}\,\mathrm{d}x < 0$.

For arbitrary $V > 0$ one can consider $(1 + tV) = (1 + \{t\|V\|\}\hat{V})$ with $\hat{V} = \frac{V}{\|V\|}$ which is of norm 1. The adaptation of the proof is then straightforward. □

The main result in the present context is provided in [LSZ, Thm. 11.2.3]. We can not provide a proof of this statement without additional efforts, but let us state it and see its role in the proof of Theorem 5.4.5. By a strictly positive operator we denote a positive operator with empty kernel.

Theorem 5.4.10. *Let $V \in \mathcal{L}_{1,\infty}$ be a strictly positive operator, and let $T \in \mathscr{B}(\mathcal{H})$ be a V-modulated operator. Let $\{f_n\}$ be an orthonormal basis of $\mathcal{H}$ ordered such that $Vf_n = \mu_n(V)f_n$ for any $n \in \mathbb{N}$. Then we have:*

(i) $T \in \mathcal{L}_{1,\infty}$ and the sequence $\big(\langle f_n, Tf_n\rangle\big)_{n\in\mathbb{N}}$ belongs to $\ell_{1,\infty}$,

(ii) The map

$$\mathbb{N} \ni \sum_{j=1}^{n} \lambda_j(T) - \sum_{j=1}^{n} \langle f_j, Tf_j\rangle \in \mathbb{C} \tag{5.31}$$

is bounded.

Note that equation (5.31) should be read with the results of Chapter 2 on the usual trace in mind. Indeed, for a trace class operator A, the sum $\sum_n \langle f_n, Af_n\rangle$ gives the same value for an arbitrary orthonormal basis of $\mathcal{H}$, and by Lidskii's theorem this sum is equal to $\sum_j \lambda_j(A)$. In the present situation, the operator T is not trace class, and therefore neither $\sum_j \lambda_j(T)$ nor $\sum_n \langle f_n, Tf_n\rangle$ are well-defined. However, equation (5.31) states that a suitable difference (depending on a parameter n) of these expressions remains bounded for all n. One additional difference with the content of Chapter 2 is that the basis of $\mathcal{H}$ is not arbitrary but is adapted to the operator V to which the operator T is modulated. In a vague sense it means that the chosen basis of $\mathcal{H}$ is made of elements which have a certain regularity with respect to T.

Clearly, the above result can not be applied for any Laplacian-modulated operator since the operator $(1-\Delta)^{-d/2}$ is never a compact operator. However, the trick is to replace the Laplacian operator by the Laplacian on a bounded domain. This change will be possible thanks to the assumption on the support of the operator T. So, for any $m \in \mathbb{Z}^d$ let us set $\mathrm{e}_m \in L^2([0,1]^d)$ by $\mathrm{e}_m(x) := \mathrm{e}^{2\pi i m\cdot x}$ and let us denote by $-\Delta_0$ the Laplacian in $L^2([0,1]^d)$ with domain $\mathsf{D}(-\Delta) := \mathsf{Span}\big(\{\mathrm{e}_m\}_{m\in\mathbb{Z}^d}\big)$. Clearly, $-\Delta_0\,\mathrm{e}_m = 4\pi^2 m^2\,\mathrm{e}_m$. One major interest in this operator is that its resolvent has very good spectral properties, more precisely one has $(1-\Delta_0)^{-d/2} \in \mathcal{L}_{1,\infty}$ as a consequence of Weyl law. In addition, this operator is strictly positive, and therefore satisfies the assumptions of Theorem 5.4.10

Exercise 5.4.11. *By using the Weyl asymptotic provided in the theorem on page 30 of [Cha] show that $(1-\Delta_0)^{-d/2} \in \mathcal{L}_{1,\infty}$. Show also that such an inclusion holds for the Laplacian Δ_0 for any bounded rectangular domain in $\mathbb{R}^d$.*

In the sequel, we shall consider the functions e_m as periodic functions on $\mathbb{R}^2$. Clearly, these functions are not in $L^2(\mathbb{R}^d)$, but nevertheless they are going to play an important role.

For the next statement, recall that if T is a Hilbert-Schmidt operator and if $\{f_n\}$ is an orthogonal basis of $\mathcal{H}$, then the summation $\sum_n \|Tf_n\|^2$ is finite, see Proposition 2.5.4. Clearly, the family of functions $\{\mathrm{e}_m\}$ is not suitable for such an estimate, but once multiplied by a nice function one gets:

Lemma 5.4.12. *Let $T \in \mathscr{B}(\mathcal{H})$ be a Laplacian-modulated operator, and let ϕ be an arbitrary element of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Then one has*

$$\sum_{|m|>t} \|T(\phi\,\mathrm{e}_m)\|^2 = O(t^{-d}), \qquad \forall t>0.$$

The rather lengthy proof of this lemma is provided in [LSZ, Lem. 11.4.2]. It is only based on the properties of the Schwartz functions and makes an extensive used of the algebra $L^1_{mod}(\mathbb{R}^d)$.

With the previous result we can show that any compactly supported Laplacian-modulated operator is also Δ_0-modulated operator. For the compactly supported operator, we shall assume from now on that the support is inside $[0,1]^d$. Obviously, this is not a loss of generality since other arbitrary cubes could have been chosen, see also Exercise 5.4.11.

Theorem 5.4.13. *Let $T \in \mathscr{B}(\mathcal{H})$ be a compactly supported Laplacian-modulated operator with support in $[0,1]^d$. Then the operator T, considered from $L^2([0,1]^d)$ to $L^2([0,1]^d)$ is Δ_0-modulated.*

Proof. For any $m \in \mathbb{Z}^d$ let e_m be the functions introduced above, seen either as element of $L^2([0,1]^d)$ or as continuous and periodic functions on $\mathbb{R}^d$. Let also $\phi \in \mathcal{S}(\mathbb{R}^d)$ be positive and such that $\phi(x) = 1$ for any $x \in [0,1]^d$. Since T is compactly supported, one has $T\,\mathrm{e}_m = T(\phi\,\mathrm{e}_m)$.

Now, for any $t > 0$ and in the Hilbert space $L^2([0,1]^d)$ one has

$$\begin{aligned}\big\|TE_{(1-\Delta_0)^{-d/2}}([0,t^{-1}])\big\|_2^2 &= \sum_{1+4\pi^2|m|^2\geq t^{2/d}} \|T\,\mathrm{e}_m\|^2 \\ &\leq \sum_{|m|\geq t^{1/d}/2\pi} \|T(\phi\,\mathrm{e}_m)\|^2 \\ &= O(t^{-1})\end{aligned}$$

where the last estimate is provided by Lemma 5.4.12. The statement follows now directly from Lemma 5.4.9. □

Based on Theorem 5.4.10 let us finally provide a sketch of the proof of Theorem 5.4.5.

Proof of Theorem 5.4.5. 1) We shall assume without loss of generality that the compactly supported operator T has support in $[0,1]^d$. As already observed, the operator $V := (1-\Delta_0)^{-d/2}$ belongs to $\mathscr{L}_{1,\infty}$ and is strictly positive. In addition, one has shown in Theorem 5.4.13 that T is Δ_0-modulated, or more precisely that $T : L^2([0,1]^d) \to L^2([0,1]^d)$ is V-modulated. As a consequence of Theorem 5.4.10 one infers that $T \in \mathscr{L}_{1,\infty}\big(L^2([0,1]^d)\big)$, and then by the inclusion of $L^2([0,1]^d)$ into $L^2(\mathbb{R}^d)$ that $T \in \mathscr{L}_{1,\infty}\big(L^2(\mathbb{R}^d)\big)$ as well. In that respect it is worth noting that the eigenvalues of $T : L^2([0,1]^d) \to L^2([0,1]^d)$ and of $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ coincide since the subspace $L^2([0,1]^d)$ is left invariant by T.

Now, let $\{f_n\}$ be a rearrangement of the eigenfunctions $\{\mathrm{e}_m\}$ according to an increase of $|m|$. More precisely for any given $n \in \mathbb{N}$ we have $f_n = \mathrm{e}_{m_n}$ with $|m_n| \geq |m_{n'}|$ whenever $n > n'$. One can also observe that $|m_n| \cong n^{1/d}$. Then Theorem 5.4.10 implies that

$$\sum_{j=1}^{n} \lambda_j(T) = \sum_{j=1}^{n} \langle f_j, Tf_j\rangle + O(1) = \sum_{|m|\leq n^{1/d}} \langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle + O(1). \tag{5.32}$$

2) For the initial statement, it remains to show that for any $t > 0$

$$\int_{|\xi|<t}\int_{\mathbb{R}^d} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x - \sum_{|m|\leq t} \langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle = O(1). \tag{5.33}$$

For that purpose, let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be positive and such that $\phi(x) = 1$ for any $x \in [0,1]^d$. We then have $T\,\mathrm{e}_m = T(\phi\,\mathrm{e}_m)$ and $[\mathscr{F}(\phi\,\mathrm{e}_m)](x) = [\mathscr{F}\phi](x-m)$. It then follows from the explicit formula (5.17) that

$$\langle \mathrm{e}_m, T\,\mathrm{e}_m\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot(\xi-m)}\, p_T(x,\xi)[\mathscr{F}\phi](\xi-m)\,\mathrm{d}\xi\,\mathrm{d}x.$$

By taking into account that $p_T(x,\xi) = 0$ if $x \notin [0,1]^d$, one gets

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \int_{|\xi|<t} p_T(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x - \sum_{|m|\leq t} \langle \mathrm{e}_m, T\,\mathrm{e}_m \rangle \Big| \\
&= \Big| \int_{\mathbb{R}^d} \int_{[0,1]^d} p_T(x,\xi) \Big(\sum_{|m|\leq t} \mathrm{e}^{2\pi i x\cdot(\xi - m)} [\mathscr{F}\phi](\xi - m) - \xi_{[0,t]}(|\xi|) \Big)\,\mathrm{d}x\,\mathrm{d}\xi \Big|
\end{aligned}$$

Now, it has been shown in [LSZ, Lem. 11.4.4] that the term inside the big parenthesis can be further estimated and one gets

$$\sum_{|m|\leq t} \mathrm{e}^{2\pi i x\cdot(\xi - m)} [\mathscr{F}\phi](\xi - m) - \xi_{[0,t]}(|\xi|) = O\big(\langle t - |\xi| \rangle^{-d}\big)$$

for any $t > 0$ and $\xi \in \mathbb{R}^d$, and uniformly in $x \in [0,1]^d$. It only remains then to estimate the term

$$\int_{\mathbb{R}^d} \int_{[0,1]^d} |p_T(x,\xi)| \langle t - |\xi| \rangle^{-d}\,\mathrm{d}x\,\mathrm{d}\xi.$$

It is again shown in the technical statement [LSZ, Lem. 11.4.5] that this term is uniformly bounded for $t > 0$. By setting $t = n^{1/d}$ in (5.33) and by using (5.32) one directly obtains the statement contained in (5.23). □

Bibliography

[Amr] W.O. Amrein, *Hilbert space methods in quantum mechanics*, Fundamental Sciences. EPFL Press, Lausanne; distributed by CRC Press, Boca Raton, FL, 2009.

[Ars] G. Arsu, *On Schatten-von Neumann class properties of pseudodifferential operators, The Cordes-Kato method*, J. Operator Theory 59, no. 1, 81–114, 2008.

[Ban] S. Banach, *Theory of linear operations*, North-Holland Mathematical Library, 38. North-Holland Publishing Co., Amsterdam, 1987.

[Bri] C. Brislawn, *Kernels of trace class operators*, Proc. Amer. Math. Soc. 104, no. 4, 1181–1190, 1988.

[Cal] J.W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. (2) 42, 839–873, 1941.

[CRSS] A.L. Carey, A. Rennie, A. Sedaev, F. Sukochev, *The Dixmier trace and asymptotics of zeta functions*, J. Funct. Anal. 249, no. 2, 253–283, 2007.

[CS1] A.L. Carey, F.A. Sukochev, *Dixmier traces and some applications in noncommutative geometry*, Uspekhi Mat. Nauk 61, no. 6(372), 45–110, 2006; translation in Russian Math. Surveys 61, no. 6, 1039–1099, 2006.

[CS2] A.L. Carey, F.A. Sukochev, *Measurable operators and the asymptotics of heat kernels and zeta functions*, J. Funct. Anal. 262, no. 10, 4582–4599, 2012.

[Cha] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics 115. Academic Press, Inc., Orlando, FL, 1984.

[Con] A. Connes, *The action functional in noncommutative geometry*, Comm. Math. Phys. 117, no. 4, 673–683, 1988.

[Del] G. Dell'Antonio, *Lectures on the mathematics of quantum mechanics II, Selected topics*, Atlantis Studies in Mathematical Physics: Theory and Applications 2. Atlantis Press, Paris, 2016.

[Dix] J. Dixmier, *Existence de traces non normales* (in French), C. R. Acad. Sci. Paris Sér. A-B 262, A1107–A1108, 1966.

[DPSS] P.G. Dodds, B. de Pagter, E.M. Semenov, F.A. Sukochev, *Symmetric functionals and singular traces*, Positivity 2, no. 1, 47–75, 1998.

[Fac] T. Fack, *Sur la notion de valeur caractéristique*, (in French), J. Operator Theory 7, no. 2, 307–333, 1982.

[Fan] K. Fan, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci., U.S.A. 37, 760–766, 1951.

[GK] I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs Vol. 18, American Mathematical Society, Providence, R.I. 1969.

[HLP] G. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Reprint of the 1952 edition, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[Hor] A. Horn, *On the singular values of a product of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. 36, 374–375, 1950.

[KLPS] N. Kalton, S. Lord, D. Potapov, F. Sukochev, *Traces of compact operators and the noncommutative residue*, Adv. Math. 235, 1–55, 2013.

[KSS] N.J. Kalton, A.A. Sedaev, F.A. Sukochev, *Fully symmetric functionals on a Marcinkiewicz space are Dixmier traces*, Adv. Math. 226, no. 4, 3540–3549, 2011.

[Kat] T. Kato, *Perturbation theory for linear operators* (second edition), Classics in mathematics, Springer, 1995.

[Les] M. Lesch, *Pseudodifferential operators and regularized traces*, in Motives, quantum field theory, and pseudodifferential operators, 37–72, Clay Math. Proc. 12, Amer. Math. Soc., Providence, RI, 2010.

[LSZ] S. Lord, F. Sukochev, D. Zanin, *Singular traces, theory and applications*, Studies in mathematics 46, De Gruyter, 2013.

[Lor] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80, 167–190, 1948.

[Ma1] A.S. Markus, *Eigenvalues and singular values of the sum and product of linear operators*, Soviet math. Dokl. 3, 1238–1241, 1962.

[Ma2] A.S. Markus, *Eigenvalues and singular values of the sum and product of linear operators*, Russian Mathematical Surveys 19, No 4, 91–120, 1964.

[MOB] A. Marshall, I. Olkin, A. Barry, *Inequalities: theory of majorization and its applications* (second edition), Springer Series in Statistics, Springer, New York, 2011.

[Mur] G.J. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[Ped] G.K. Pedersen, *Analysis now*, Graduate Texts in Mathematics 118. Springer-Verlag, New York, 1989.

[RS1] M. Reed, B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Academic Press, Inc., 1972.

[RS3] M. Reed, B. Simon, *Methods of modern mathematical physics III: Scattering theory*, Academic Press, Inc., 1979.

[RS4] M. Reed, B. Simon, *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, Inc., 1978.

[RT] M. Ruzhansky, V. Turunen, *Pseudo-differential operators and symmetries; background analysis and advanced topics*, Pseudo-Differential Operators, Theory and Applications 2. Birkhauser Verlag, Basel, 2010.

[Sak] S. Sakai, *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. Springer-Verlag, New York-Heidelberg, 1971.

[Sch] R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 27 Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

[Shu] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Second edition, Springer-Verlag, Berlin, 2001.

[Sim] B. Simon, *Trace ideals and their applications* (second edition), Mathematical surveys and Monographs 120, AMS, 2005.

[SU] F. Sukochev, A. Usachev, *Dixmier traces and non-commutative analysis*, J. Geom. Phys. 105, 102–122, 2016.

[SUZ1] F. Sukochev, A. Usachev, D. Zanin, *Generalized limits with additional invariance properties and their applications to noncommutative geometry*, Adv. Math. 239, 164–189, 2013.

[SUZ2] F. Sukochev, A. Usachev, D. Zanin, *On the distinction between the classes of Dixmier and Connes-Dixmier traces*, Proc. Amer. Math. Soc. 141, no. 6, 2169–2179, 2013.

[SZ] F. Sukochev, D. Zanin, *ζ-function and heat kernel formulae*, J. Funct. Anal. 260, no. 8, 2451–2482, 2011.

[Wey] H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. U.S.A. 35, 408–411, 1949.

[Wod1] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. 75, no. 1, 143–177, 1984.

[Wod2] M. Wodzicki, *Noncommutative residue I, Fundamentals*, in K-theory, arithmetic and geometry (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, Berlin, 1987.