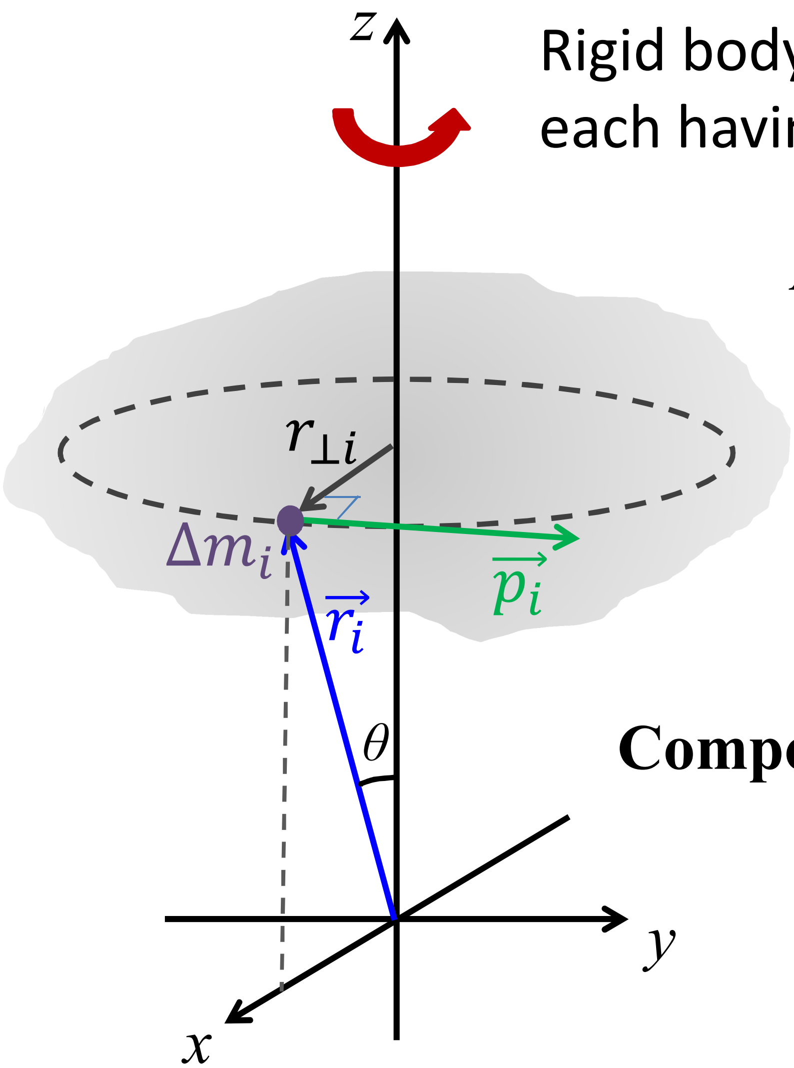
CHAPTER 11 (TORQUE, ANGULAR MOMENTUM)

1. Angular Momentum $\vec{L}$
   a. Rigid body
   b. Conservation of $\vec{L}$
2. Applications
   a. Gyroscope

ANGULAR MOMENTUM OF A RIGID BODY
(ARBITRARY SHAPE AND AXIS POSITION)



Rigid body taken as a system of many elements, each having mass Δm.

About point O: $\overrightarrow{L_O} = \sum_i \overrightarrow{l_i}$

$$\overrightarrow{l_i} = \overrightarrow{r_i} \times \overrightarrow{p_i}$$

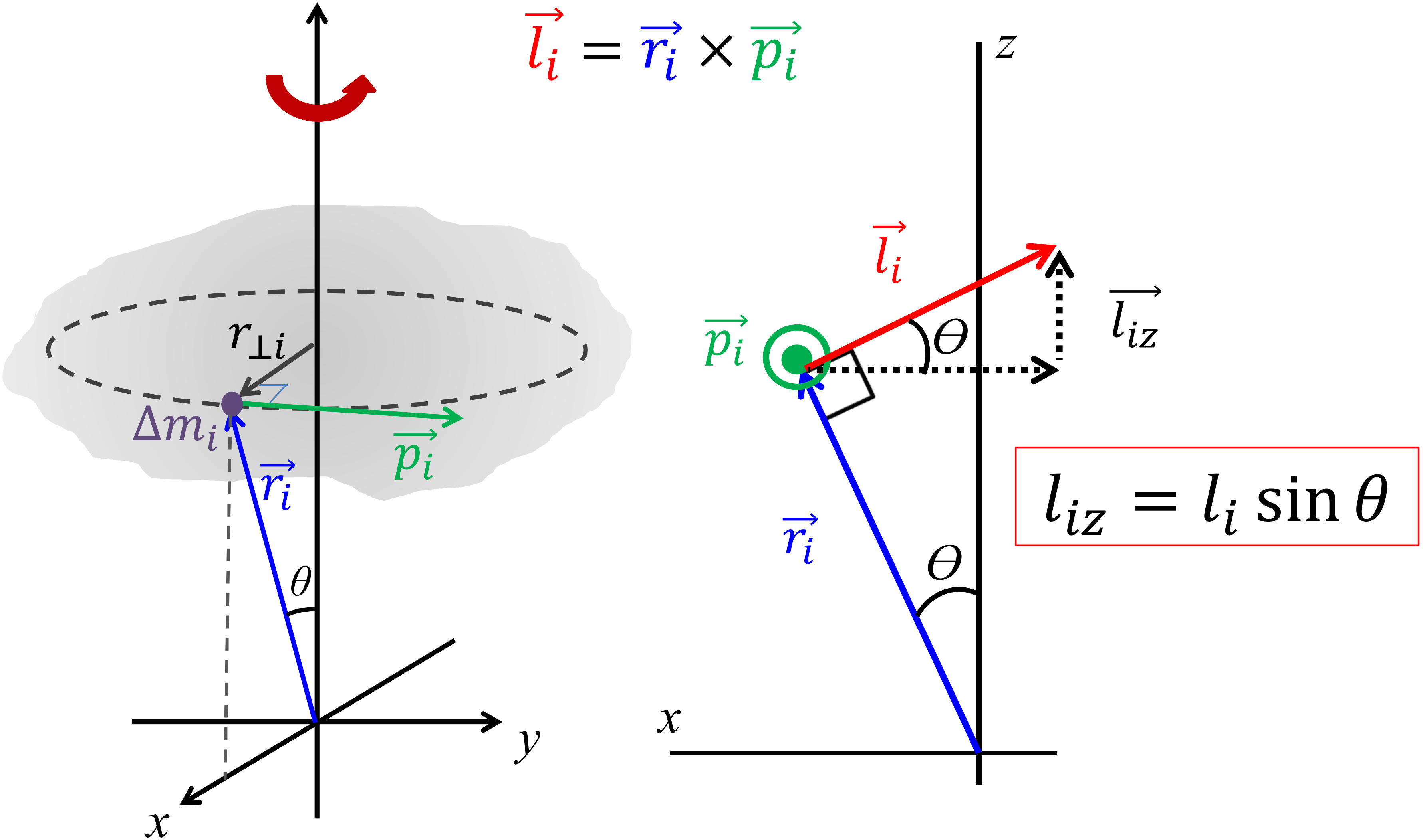
$$\overrightarrow{r_i} \perp \overrightarrow{p_i} \Rightarrow \left|\overrightarrow{l_i}\right| = r_i \Delta m v_i$$

Component L_{Oz} of $\overrightarrow{L_O}$ along rotation (z) axis?

$$L_{Oz} = \sum_i l_{iz}$$

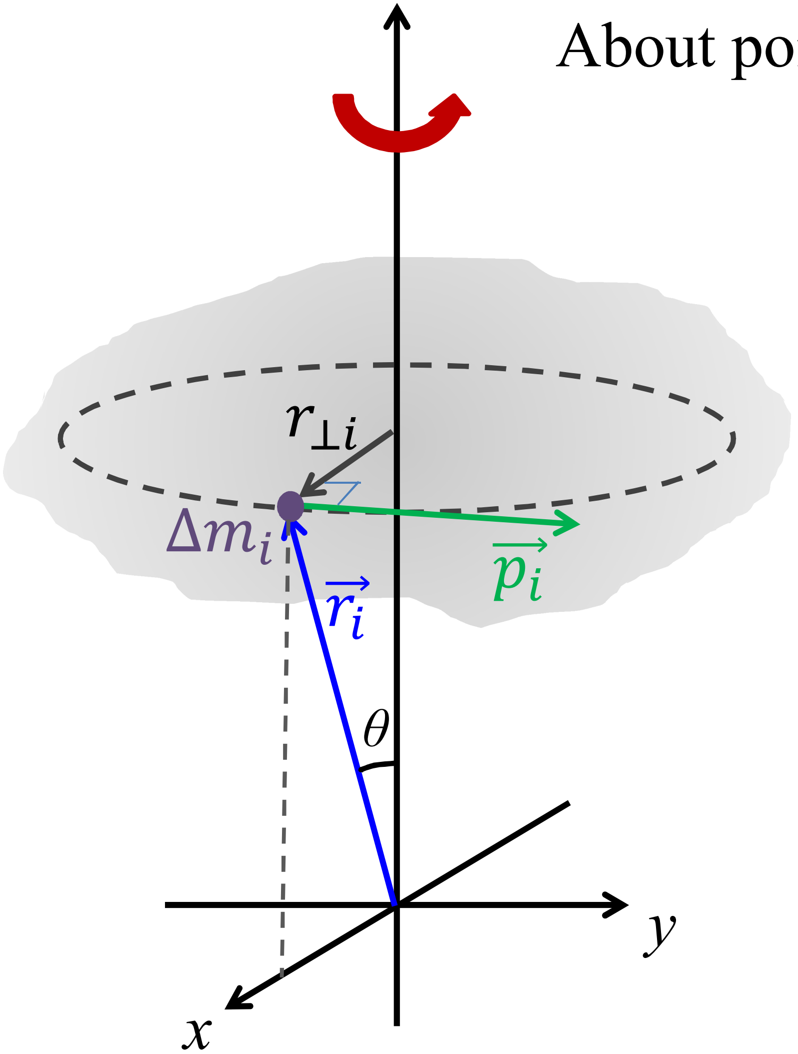
COMPONENT OF l_i ALONG ROTATION AXIS Z

$$\vec{l_i} = \vec{r_i} \times \vec{p_i}$$



$$l_{iz} = l_i \sin\theta$$

ANGULAR MOMENTUM OF A RIGID BODY - COMPONENT ALONG ROTATION AXIS -



About point O: $\overrightarrow{L_O} = \sum_i \vec{l_i}$ $\quad |\vec{l_i}| = r_i \Delta m v_i$

$$l_{iz} = l_i \sin\theta$$
$$= r_i \sin\theta\, \Delta m v_i = r_{\perp i} \Delta m v_i$$

But $v_i = r_{\perp i}\omega$

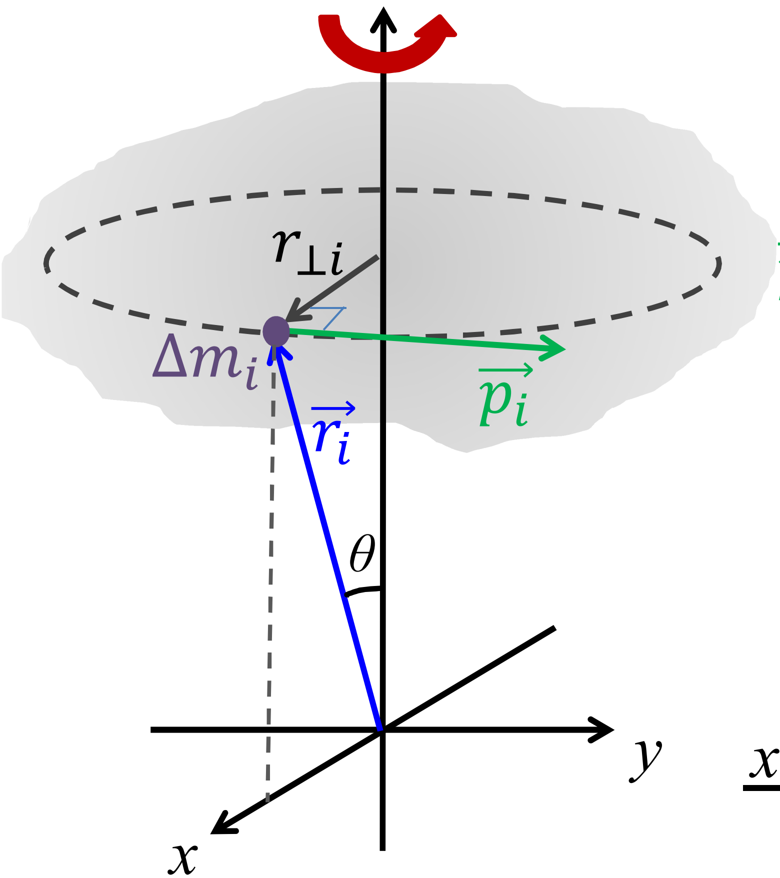
Then, $l_{iz} = r_{\perp i} \Delta m r_{\perp i} \omega = \Delta m {r_{\perp i}}^2 \omega$

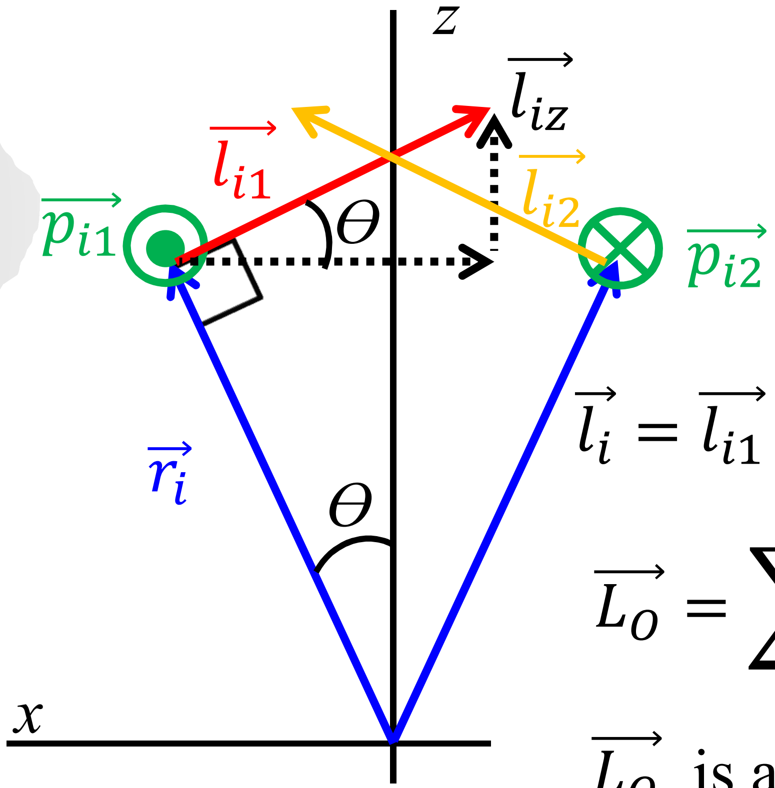
$$L_{Oz} = \sum_i l_{iz} = \omega \sum_i \Delta m {r_{\perp i}}^2$$

$$\Rightarrow L_z = I_z \omega$$

Characteristic of axis *z*, not of point *O*

ANGULAR MOMENTUM OF A SYMMETRIC RIGID BODY ROTATING ABOUT AN AXIS OF SYMMETRY





$$\vec{l_i} = \overrightarrow{l_{i1}} + \overrightarrow{l_{i2}} = 2\overrightarrow{l_{iz}}$$

$$\overrightarrow{L_O} = \sum_i \vec{l_i}$$

$\overrightarrow{L_O}$ is along z axis

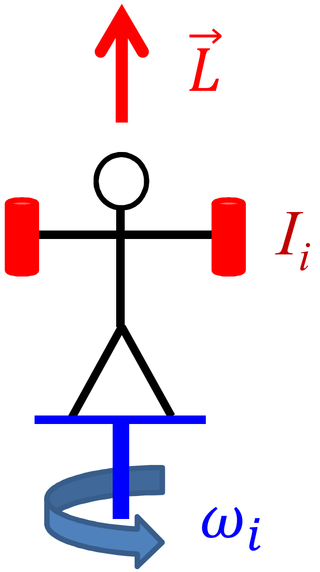
If the axis of rotation is also **an axis of symmetry** of the body:

$$\vec{L} = I_z\vec{\omega}$$

$\vec{L}$ is along *z* axis; characteristic of the axis

CONSERVATION OF ANGULAR MOMENTUM

If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

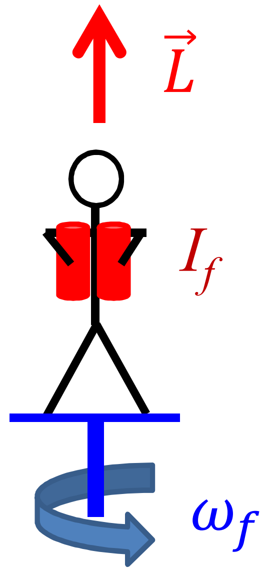


$$I_i\omega_i = I_f\omega_f$$

$$I_\Delta = \sum_i m_i {r_i}^2$$

$$I_i > I_f$$

$$\Rightarrow \omega_i < \omega_f$$



Initial State (i): The man has a relatively large rotational inertia about the rotation axis and a relatively small angular speed.
Final State (f): By decreasing his rotational inertia, the man increases his angular speed. The angular momentum of the rotating system remains unchanged.

FORMULA SUMMARY

Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

Angular momentum: $\vec{l} = \vec{r} \times \vec{p}$

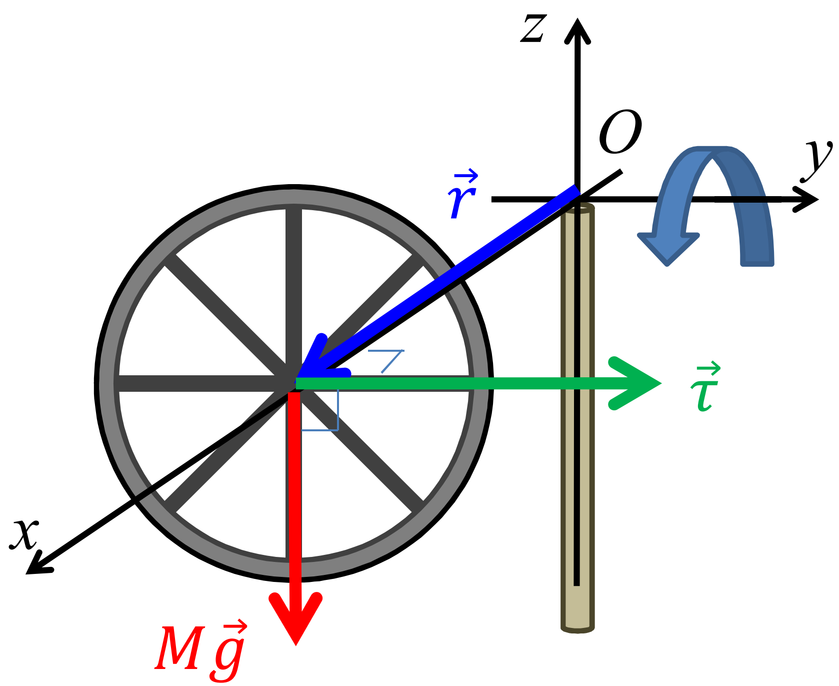
Newton's 2nd law: $\overrightarrow{\tau_{net}} = \frac{d\vec{L}}{dt}$ Pure rotation: $\tau_{net} = I\frac{d\omega}{dt} = I\alpha$

Rigid body, rotation about fixed axis *z*: $L_z = I_z\omega$

Symmetric body rotation, about symmetry axis z: $\vec{L} = I_z\vec{\omega}$

General motion (translation + rotation)
➔ use Newton's 2nd law in **both** linear and angular forms

GYROSCOPE NOT SPINNING



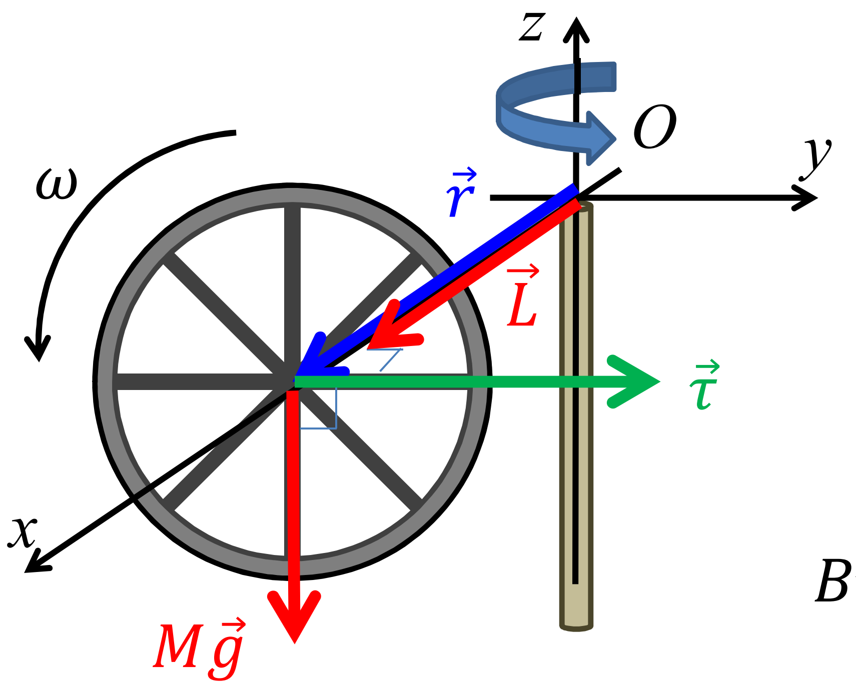
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$|\vec{\tau}| = MGr\sin{}^{\pi}/_{2} = Mgr$$

A non-spinning gyroscope falls by rotating in an *xz* plane because of torque $\boldsymbol{\tau}$.

PRECESSION OF A GYROSCOPE

A rapidly spinning gyroscope, with angular momentum **L**, precesses around the z axis. Its precessional motion is in the xy plane.



Symmetric body rotating about symmetry axis Ox: $\vec{L}(t) = I\vec{\omega}(t)$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = \overrightarrow{\tau_{gravity}}$$

$$But: \left|\vec{L}(t)\right| = I|\vec{\omega}(t)| = Constant!$$

The only way the vector $\vec{L}(t)$ can change with time is by changing its orientation! How does it change?

PRECESSION RATE OF A GYROSCOPE

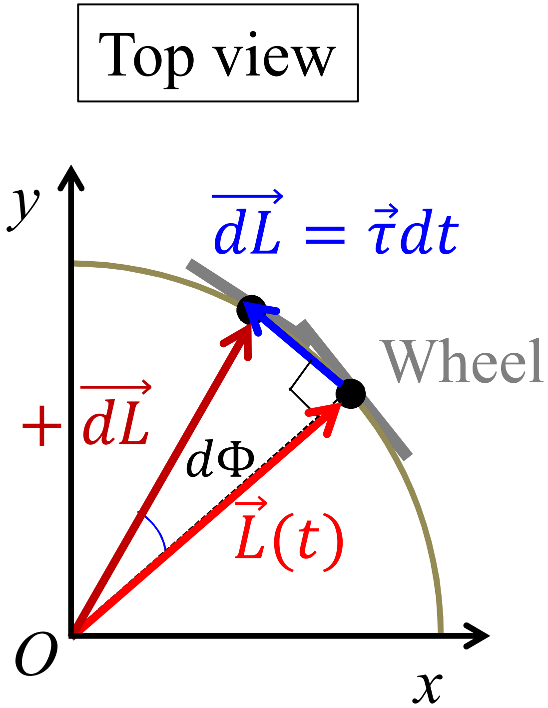
$d\vec{L} = \vec{\tau} dt$ ➔ $\vec{L}$ Changes in the direction of $\vec{\tau}$

$$|d\vec{L}| = |\vec{L}| d\phi$$

$$\Rightarrow \frac{d\phi}{dt} = \frac{1}{|\vec{L}|}\frac{|d\vec{L}|}{dt} = \frac{|\vec{\tau}|}{|\vec{L}|} = \frac{Mgr}{I\omega}$$

$$\boxed{\Omega = \frac{Mgr}{I\omega}}$$

$$\vec{L}(t + \Delta t) = \vec{L}(t) + \overrightarrow{dL}$$



The change of $\vec{L}$, which is induced by the torque $\vec{\tau}$ leads to a rotation of $\vec{L}$ about O, with angular velocity Ω (Precession rate).

PRECESSION OF A GYROSCOPE

Note: precession induces a new component of the angular momentum: $\overrightarrow{l_{prec}} = I_z\vec{\Omega}$ along the z axis! But negligible if ω is large (rapid spinning) : $|\overrightarrow{l_{prec}}| \ll |\vec{L}|$.