CHAPTER 10 (TORQUE, 2ND LAW FOR ROTATION)
CHAPTER 11 (TORQUE, ANGULAR MOMENTUM)

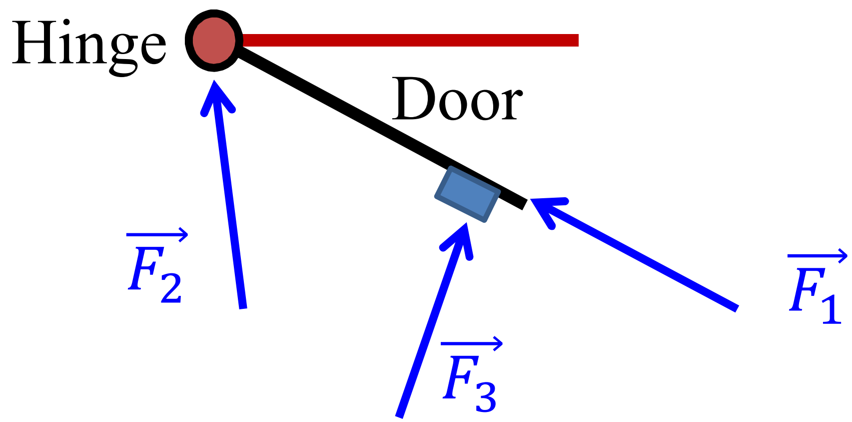
Torque and Angular Momentum

1. Torque
   a. Definition
   b. Work, Power, W-K Theorem
2. Angular Momentum $\vec{L}$
   a. Newton's 2nd Law in angular form
   b. Systems
   c. Conservation of $\vec{L}$

PRODUCING ROTATION ABOUT AN AXIS USING A FORCE

<u>Opening a door</u>

<u>Top view</u>

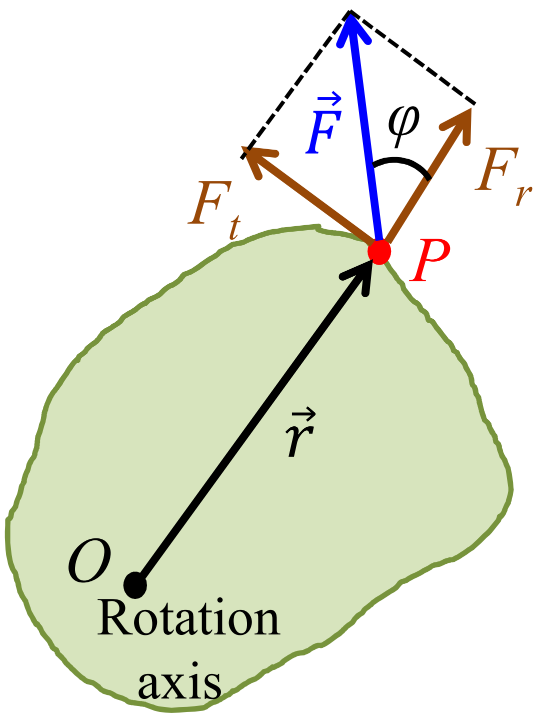


F_1 and F_2 will not produce rotation

<u>Producing rotation around an axis is much easier when:</u>

- the force is greater ➔ **magnitude of $\vec{F}$**
- the force is applied farther from the axis ➔ **distance *r* from axis**
- the direction of $\vec{F}$ is closer to 90˚ with respect to the arm

ROTATION CAUSED BY TANGENTIAL COMPONENT OF F



$\vec{F}$ causes rotation around the axis

But actually only the tangential component F_t of the force causes rotation

TORQUE

we must define a new quantity that describes the effectiveness of an interaction at producing rotation about a specific axis.

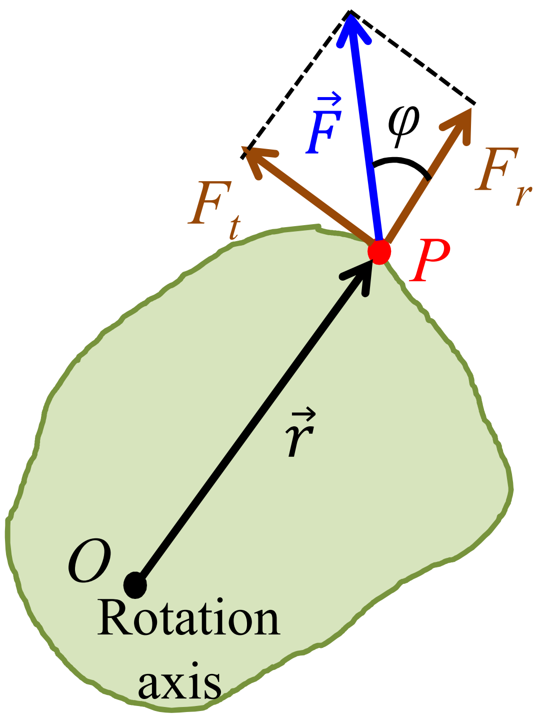
This quantity is called **torque $\boldsymbol{\tau}$** and is a **vector** whose:

- magnitude is **Distance from axis X $|\overrightarrow{F_{tangential}}|$**

  or **(Moment arm* of $\vec{F}$) X $|\vec{F}|$**
- direction is related to the **direction of rotation** (remember $\vec{\omega}$).

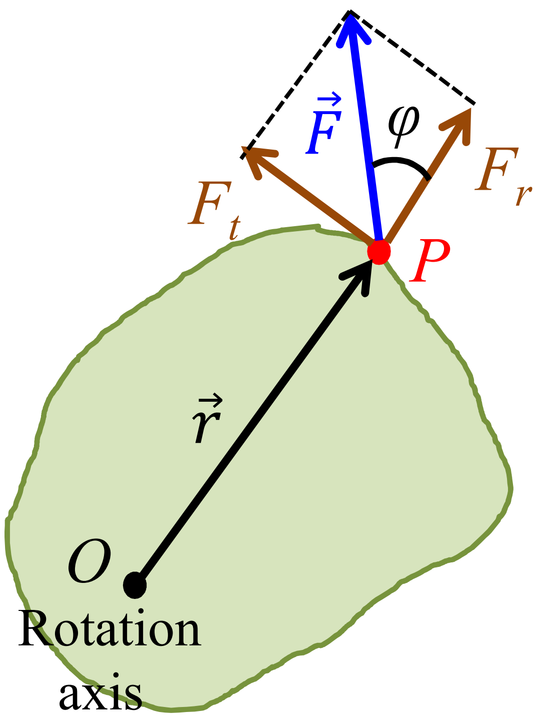
*Also called "lever arm"

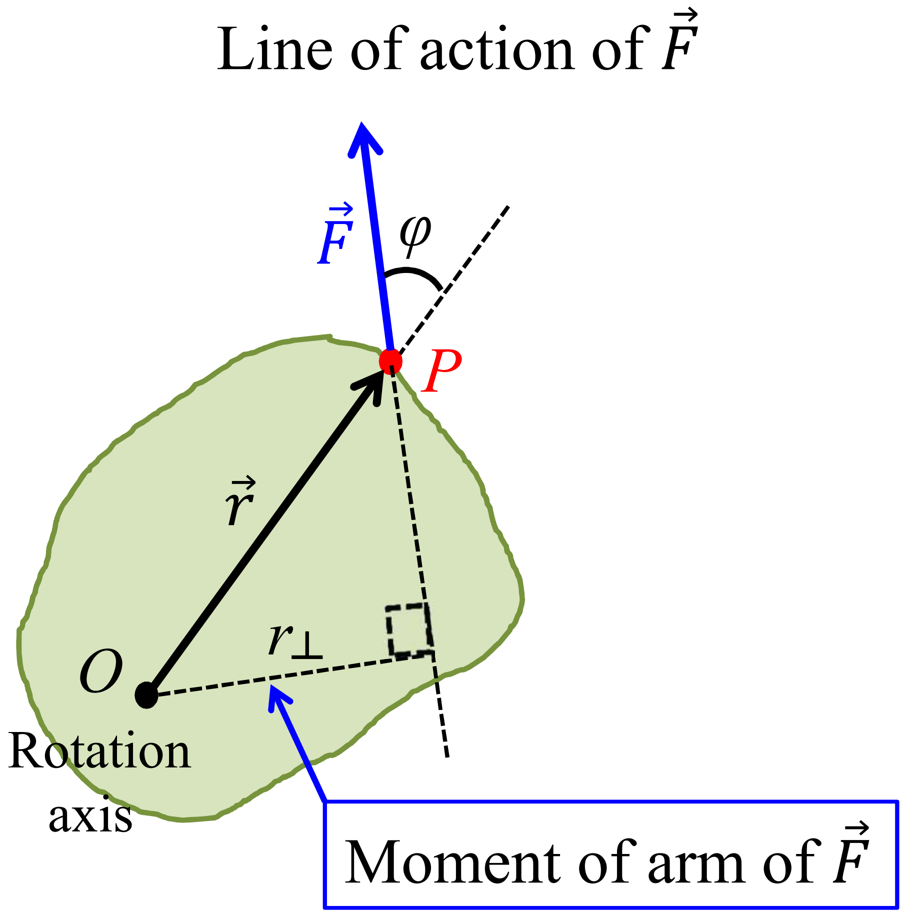
MAGNITUDE OF THE TORQUE



$$\tau = rF_t = rF\sin\varphi$$

2 WAYS TO EXPRESS THE TORQUE MAGNITUDE

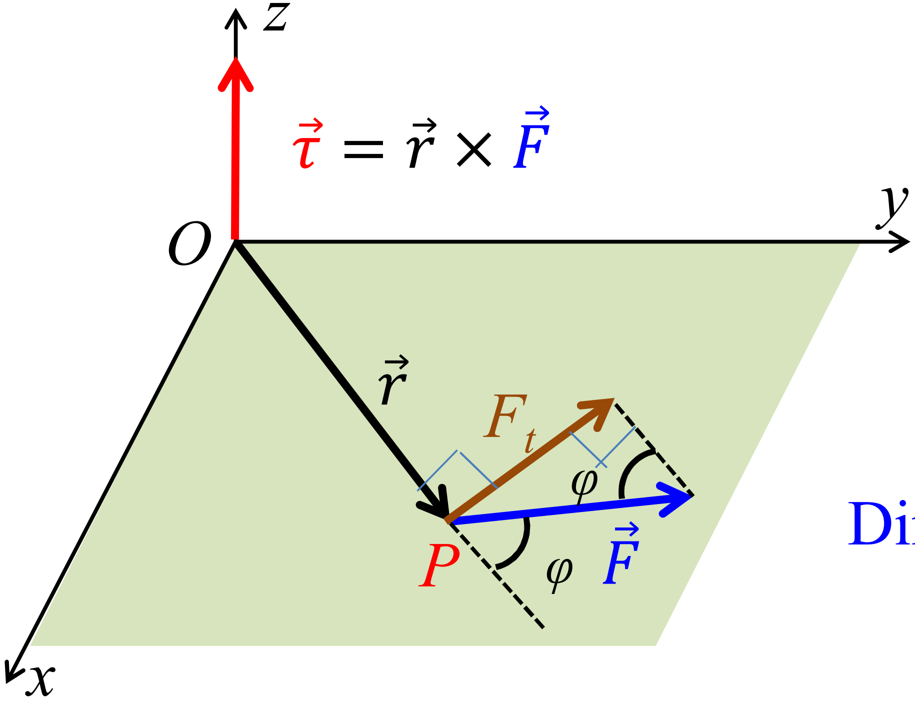




$$\tau = rF_t = r(F \sin \varphi)$$

$$\tau = (r \sin \varphi)F = r_\perp F$$

TORQUE AS A VECTOR



$$\tau = rF\sin\varphi$$

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

To get the cross-product right, use a xyz reference obeying the right-hand rule:

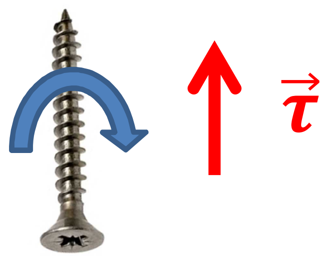
$$\vec{\boldsymbol{z}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$$

Direction of $\vec{\boldsymbol{\tau}}$:

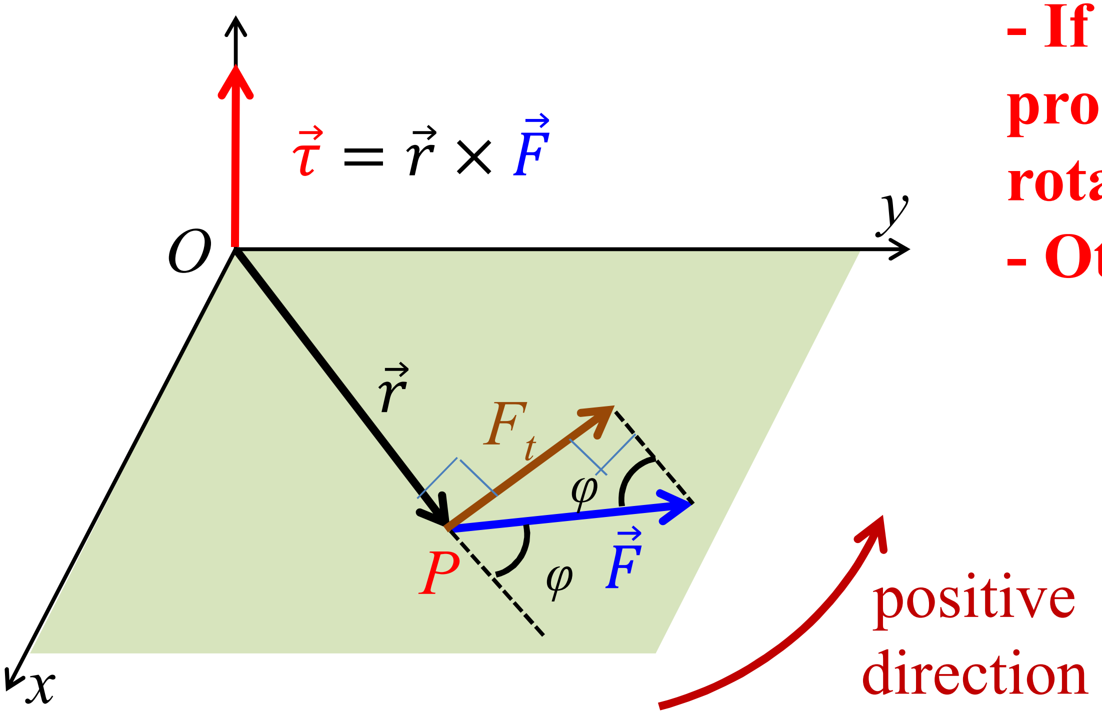
(1) Right-hand rule

(2) Screw rule

Rotation direction that $\vec{\boldsymbol{F}}$ tends to induce



SIGN OF A TORQUE



- If the torque tends to produce a counterclockwise rotation, it is Positive

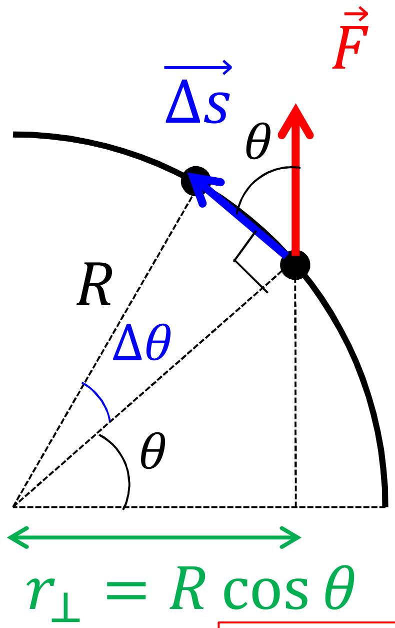
- Otherwise, it is negative.

$\vec{\tau} = \vec{r} \times \vec{F}$ Sign of τ given by direction along z axis

$\tau = rF \sin \varphi$ Sign of τ is in $\sin \varphi$

φ is taken from $\vec{r}$ to $\vec{F}$, negative when going clockwise

WORK DONE BY A TORQUE (PURE ROTATION)



$$\Delta W = \overrightarrow{F} \cdot \overrightarrow{\Delta s} = F \Delta s \cos\theta$$

$$\Delta s = R\Delta\theta$$

$$\Rightarrow \ \Delta W = R\cos\theta \, F\Delta\theta = r_\perp F \Delta\theta = \tau\Delta\theta$$

$$dW = \lim_{\Delta\theta \to 0} \Delta W = \tau d\theta$$

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

If τ is constant, $W = \tau(\theta_f - \theta_i)$

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

WORK-KINETIC ENERGY THEOREM (PURE ROTATION)

$$\Delta K = K_f - K_i = \frac{1}{2}I{\omega_f}^2 - \frac{1}{2}I{\omega_i}^2 = W_{i\to f}(\overrightarrow{\tau_{net}})$$

When using conservation of energy in general, rotational kinetic energy should also be considered.

NEWTON'S SECOND LAW FOR PURE ROTATION

$$F_t = ma_t$$

⇩

$$\tau_O = F_t r = ma_t r$$

⇩

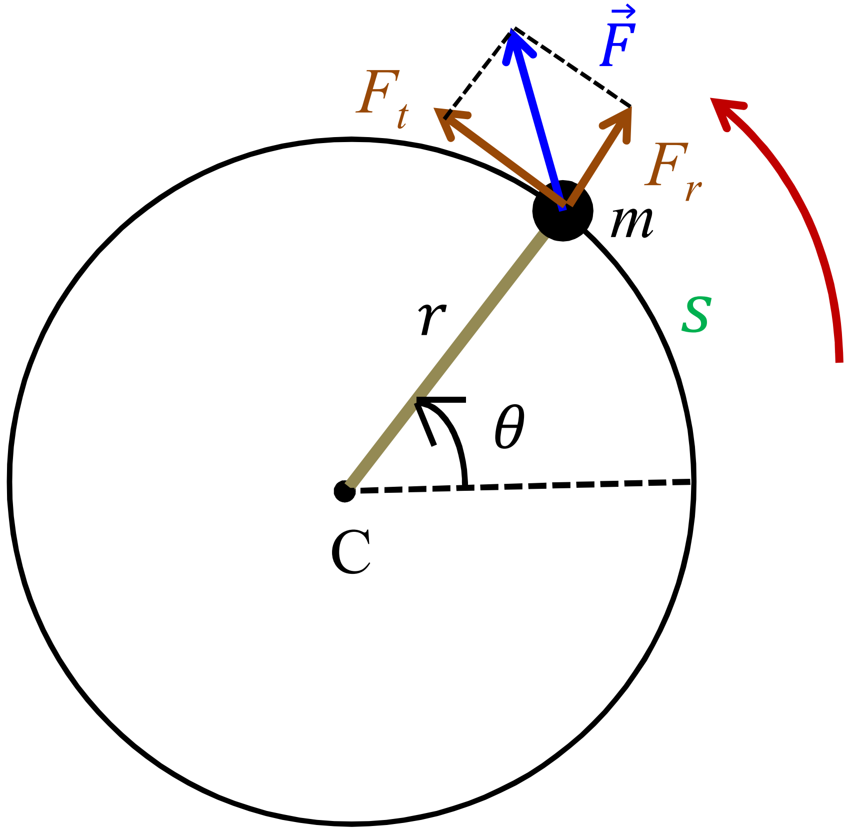
$$\tau_O = m(r\alpha)r = (mr^2)\alpha$$

⇩

$$\tau_O = I_0\alpha$$

For more than one force, we can generalize:

$$\tau_{O,net} = I_0\alpha \quad (radian\ measure)$$



TRANSLATIONAL AND ROTATIONAL MOTION

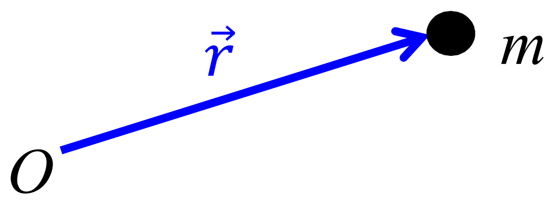
Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

ANGULAR MOMENTUM (SINGLE PARTICLE)

Linear		Angular
$\vec{F}$	$\longrightarrow$	$\vec{\tau} = \vec{r} \times \vec{F}$
$\vec{p} = m\vec{v}$	$\longrightarrow$	$\vec{l} = \vec{r} \times \vec{p}$

Angular momentum defined: $\overrightarrow{l_O} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

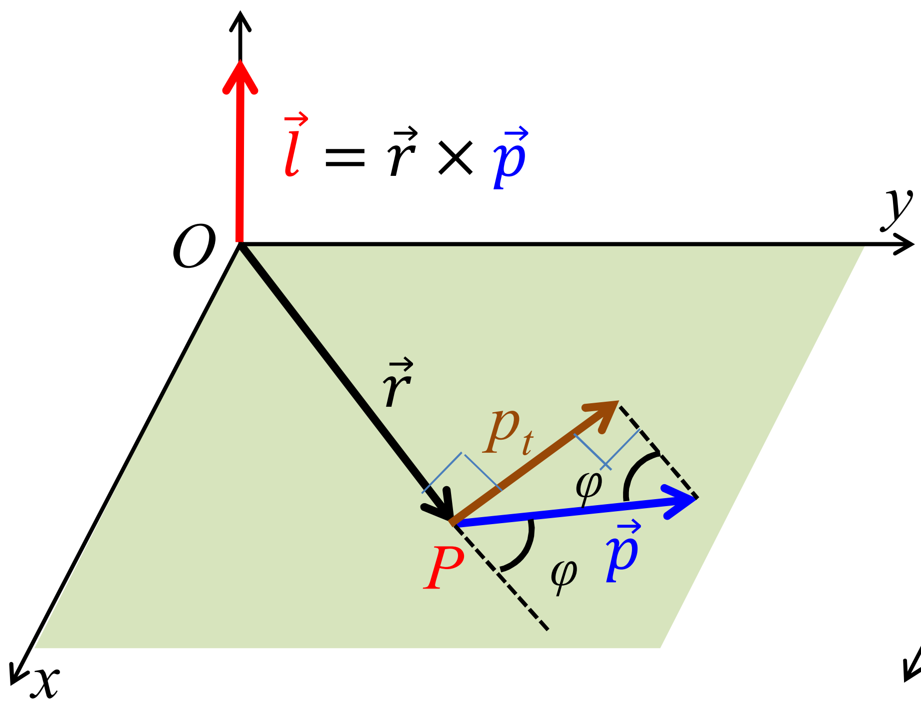
In general, $\vec{l}$ of the particle of mass m is defined **about a point** (O origin of $\vec{r}$)

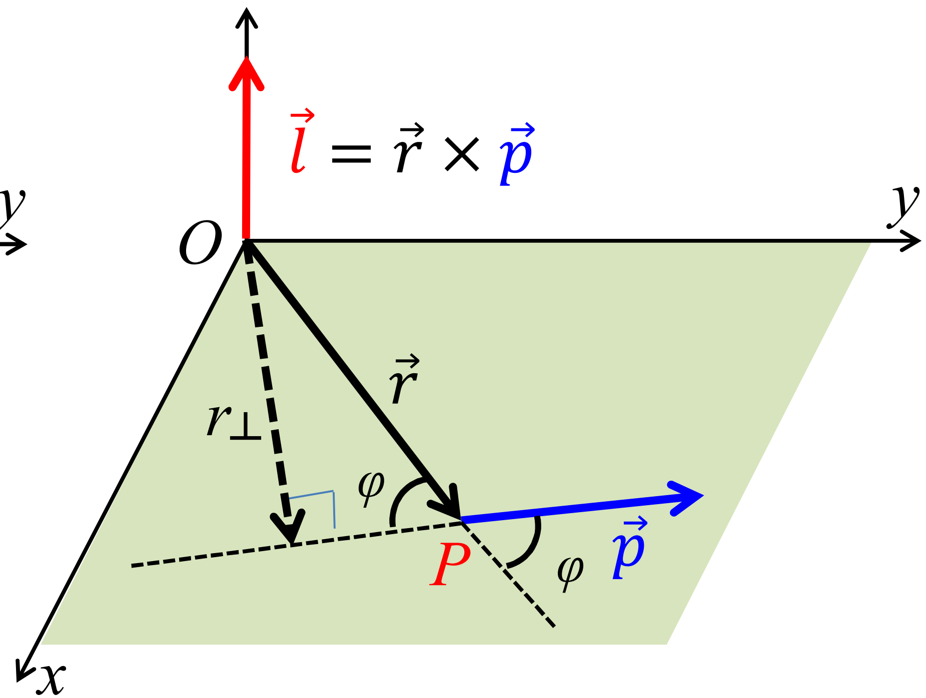


MAGNITUDE OF ANGULAR MOMENTUM

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

$\vec{l}$ of particle A about point O





$$\ell = rmv \sin\phi = rp_\perp = rmv_\perp = r_\perp p = r_\perp mv$$

PURE ROTATION – SINGLE PARTICLE

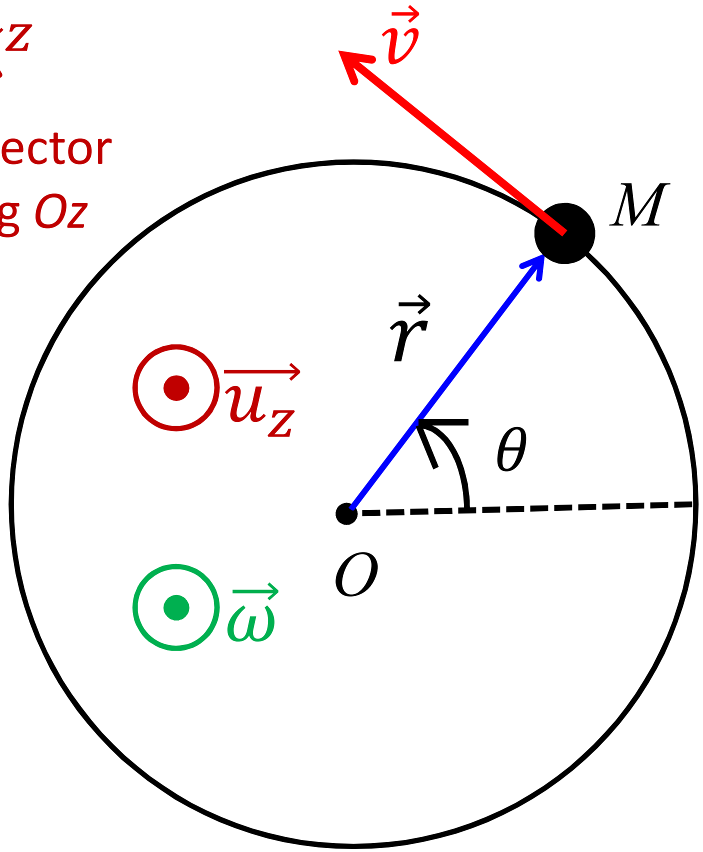
Single particle in pure rotation about axis *Oz* (oriented out of page)

$$\overrightarrow{l_O} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) = mrv\overrightarrow{u_z}$$

unit vector along *Oz*

$$v = r\omega \qquad \Rightarrow \overrightarrow{l_O} = mr^2\vec{\omega}$$

$$I_O = mr^2 \;\Rightarrow\; \overrightarrow{l_O} = I_O\vec{\omega}$$



Another way to write $\vec{l}$ in this case:

$$\boxed{\overrightarrow{l_O} = I_O\vec{\omega}}$$

NEWTON'S SECOND LAW IN ANGULAR FORM FOR A SINGLE PARTICLE

$$\vec{l} = m(\vec{r} \times \vec{v}) \quad \rightarrow \quad \frac{d\vec{l}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right)$$

$$\rightarrow \quad \frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}) \quad \rightarrow \quad \frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a}) = (\vec{r} \times m\vec{a})$$

$$\rightarrow \quad \frac{d\vec{l}}{dt} = \vec{r} \times \overrightarrow{F_{net}} = \sum(\vec{r} \times \vec{F})$$

$$\overrightarrow{\tau_{O,net}} = \frac{d\overrightarrow{l_O}}{dt}$$

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle. All taken with respect to point *O*.

ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

The **total angular momentum** $\vec{\boldsymbol{L}}$ of the system is the (vector) sum of the angular momenta $\vec{\boldsymbol{l}}$ of the individual particles (here with label *i):*

$$\vec{L} = \vec{l_1} + \vec{l_2} + \vec{l_3} + \cdots + \vec{l_n} = \sum_{i=1}^{n} \vec{l_i}$$

With time, the angular momenta of individual particles may change because of interactions between the particles or with the outside.

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l_i}}{dt} = \sum_{i=1}^{n} \overrightarrow{\tau_{net,i}}$$

Therefore, the net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{\boldsymbol{L}}$.

$$\overrightarrow{\tau_{O,net}} = \frac{d\overrightarrow{L_O}}{dt}$$

Newton's 2nd law in angular form for a system of particles

Analogy with Newton's 2nd law in linear form: F=dp/dt

CONSERVATION OF ANGULAR MOMENTUM

If total net torque is zero

If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system is constant, no matter what changes take place within the system.

$$\overrightarrow{\tau_{net}} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = a\ constant \qquad \overrightarrow{L_i} = \overrightarrow{L_f} \quad (isolated\ system)$$

If net torque along axis *z* is zero

If the net external torque acting on a system is zero along a given axis *z*, the angular momentum L_z of the system is constant.

$$\tau_{z,net} = \frac{dL_z}{dt} = 0 \qquad \Rightarrow \quad L_z = a\ constant$$

FORMULA SUMMARY

Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

Work (pure rotation): $W = \int_{\theta_i}^{\theta_f} \tau d\theta$

If τ is constant: $W = \tau(\theta_f - \theta_i)$

Power: $P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$

W-K Theorem for pure rotation: $\Delta K = K_f - K_i = \frac{1}{2} I {\omega_f}^2 - \frac{1}{2} I {\omega_i}^2 = W_{i\to f}(\overrightarrow{\tau_{net}})$

Angular momentum: $\vec{l} = \vec{r} \times \vec{p}$

Newton's 2nd law: $\overrightarrow{\tau_{net}} = \frac{d\vec{L}}{dt}$ Pure rotation: $\tau_{net} = I \frac{d\omega}{dt} = I\alpha$