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Book / WileyPlus: Halliday, Fundamentals of Physics, 10e

1. **Pure rotation (chapter 10)**

2. **Torques, angular momentum and Newton laws for rotation (Chapters 10&11)**

3. **Rolling (Chapter 11)**

ROTATION

- CHAPTER 10 -

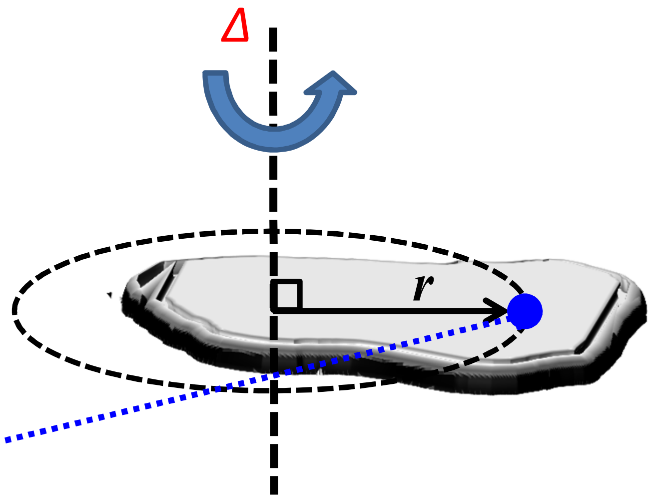
- Rotation defined
- Rotational Variables
- Relating the linear and angular variables
- Kinetic energy of rotation
- Rotational inertia

ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

A **rigid body** is a body that can rotate with all its parts locked together and without any change in its shape.

A **fixed axis** means that the rotation occurs about an axis that does not move.

The blue dot (intersection of the reference line with the body) experiences a **circular motion**



→ **Characterization of circular motion**

ANGULAR POSITION

r is constant ➔ position of point *M* defined by either s or θ

Linear variable (distance; unit: m)

Angular variable (angle; unit: rad)

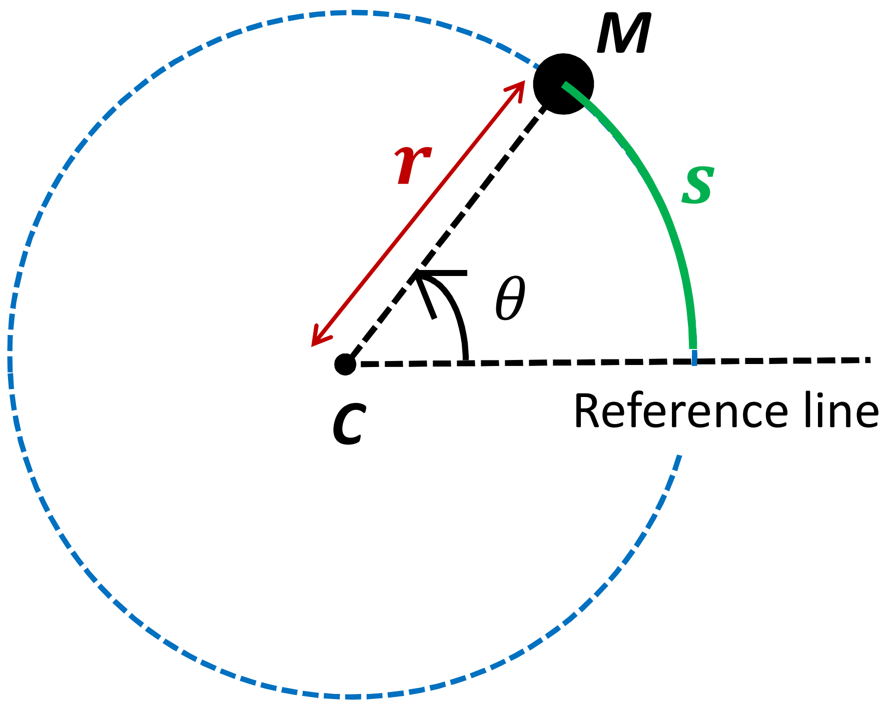
Pure rotation: angular variable θ is sufficient

$$\theta = \frac{s}{r} \quad [rad]$$

s: length of the circular arc

r: radius of the circle

θ measured in **radians (rad)**



ANGULAR POSITION - UNITS

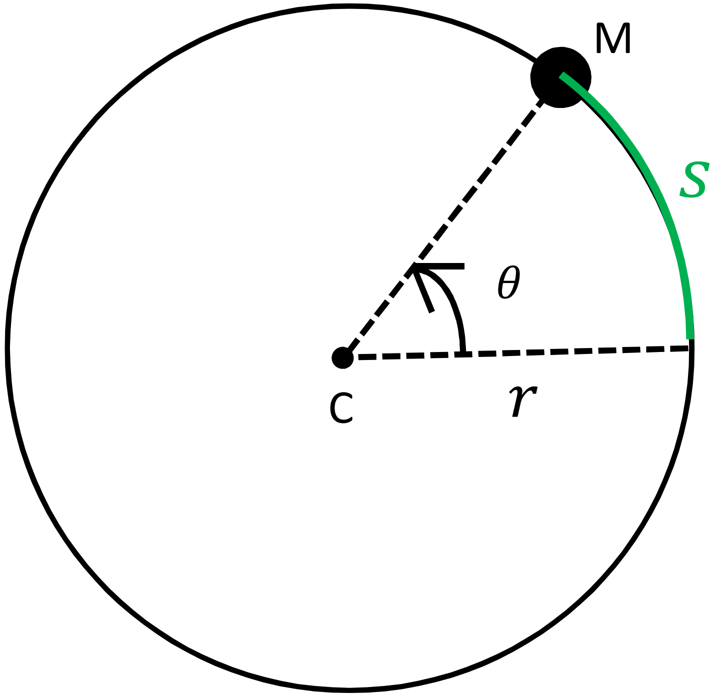
$$\theta = \frac{s}{r} \quad [rad] \qquad \text{SI unit}$$

Definition of radian:
θ= 1 rad when r = 1 and s =1

$$1 \text{ rev} = 360° = \frac{2\pi r}{r} = 2\pi \text{ rad},$$

$$1 \text{ rad} = 57.3° = 0.159 \text{ rev}.$$

1 cycle = 1 revolution



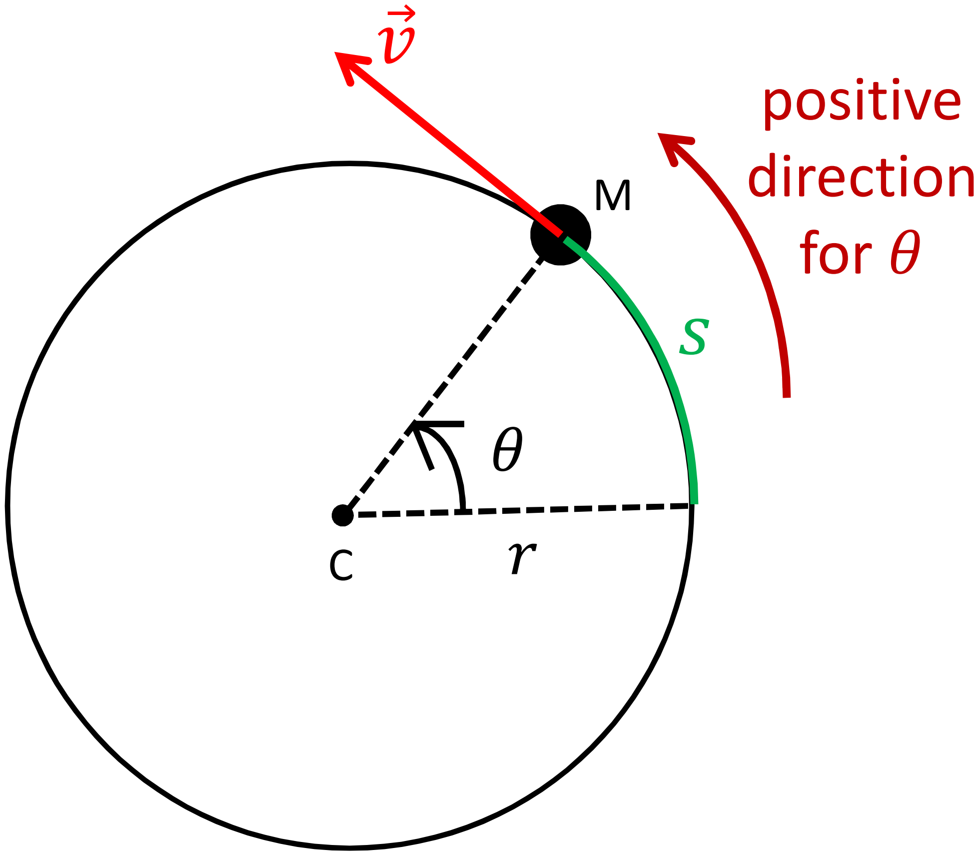
POSITIVE DIRECTION

<u>Convention:</u> An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

Here $\theta > 0$ and $s > 0$

$$\theta = \frac{s}{r} \; [rad]$$

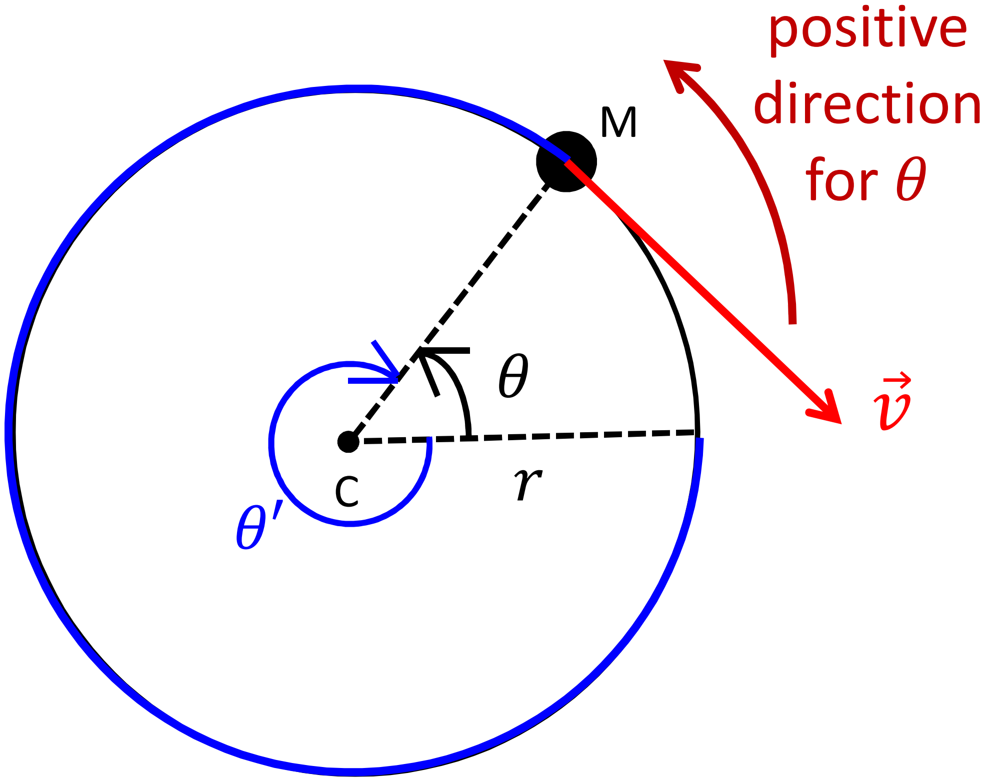
θ: algebraic variable; can be negative!



EXAMPLE OF CLOCKWISE ROTATION

Here $\theta' < 0$ and $s < 0$

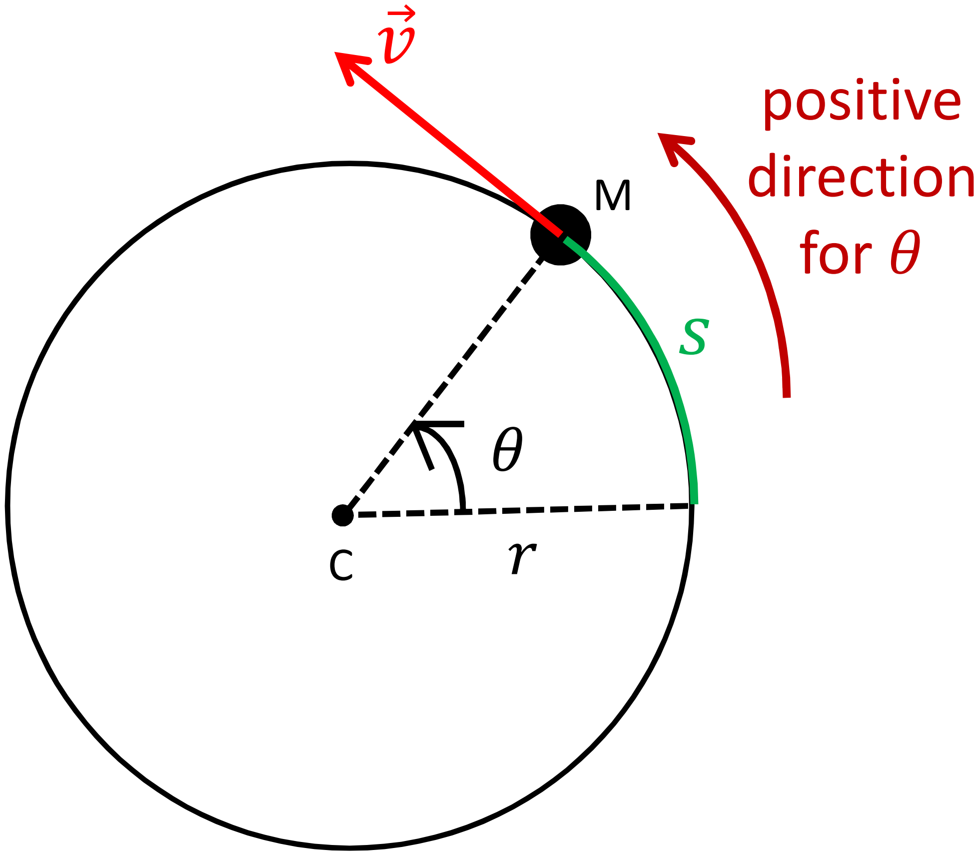
$$\theta' = \frac{s}{r} \ [rad]$$



CHARACTERIZATION OF CIRCULAR MOTION

How does θ changes with time t?
Equation of $\theta(t)$?

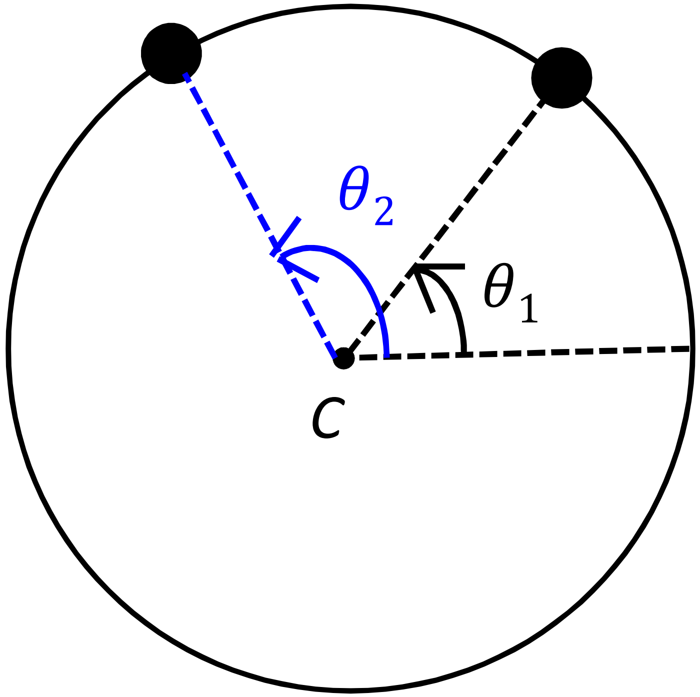
➔ Rate of change of θ?



ANGULAR VELOCITY

Rotating body is

- at angular position θ_1 at time t_1
- at angular position θ_2 at time t_2.



➔ Angular displacement:

$$\Delta\theta = \theta_2 - \theta_1$$

➔ *Average* angular velocity of the body in the time interval Δt from t_1 to t_2 :

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

ω can be negative!
When?

➔ *Instantaneous* angular velocity ω:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

ANGULAR VELOCITY AS A VECTOR $\vec{\omega}$

For a given point in space

- For *linear motion*, velocity $\vec{V}$ gives:
  - The speed
  - The **direction** of motion

- For *rotational motion*, angular velocity $\vec{\omega}$ gives:
  - The angular speed (rate of change of angle)
  - The **rotation direction**

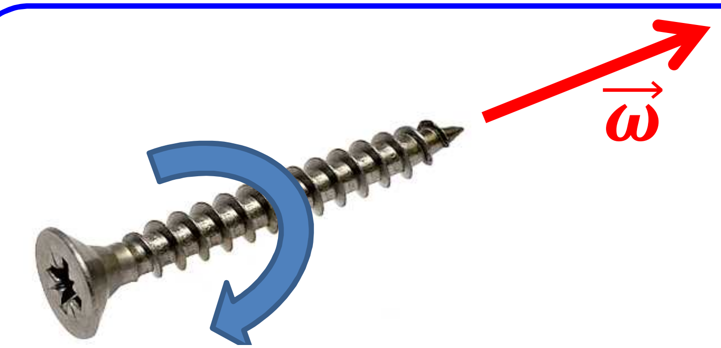
⇩

How do we know if clockwise or counterclockwise?

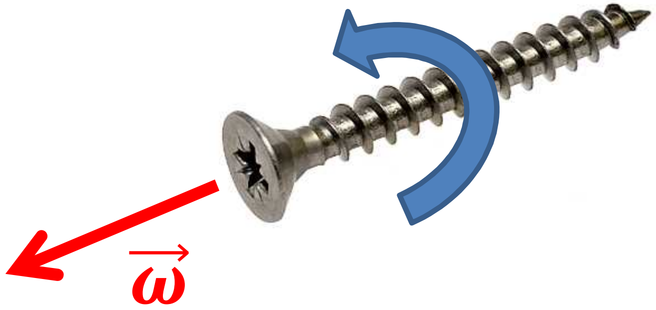
Use the "right-hand rule" or the "screw rule"

ANGULAR VELOCITY AS A VECTOR GIVING THE ROTATION DIRECTION

$\vec{\omega}$ follows the direction of the screw progression



Clockwise rotation:
Screw goes forward



Counterclockwise rotation:
Screw goes backward

Pointing towards us, out of page. $\odot\,\vec{\omega}$

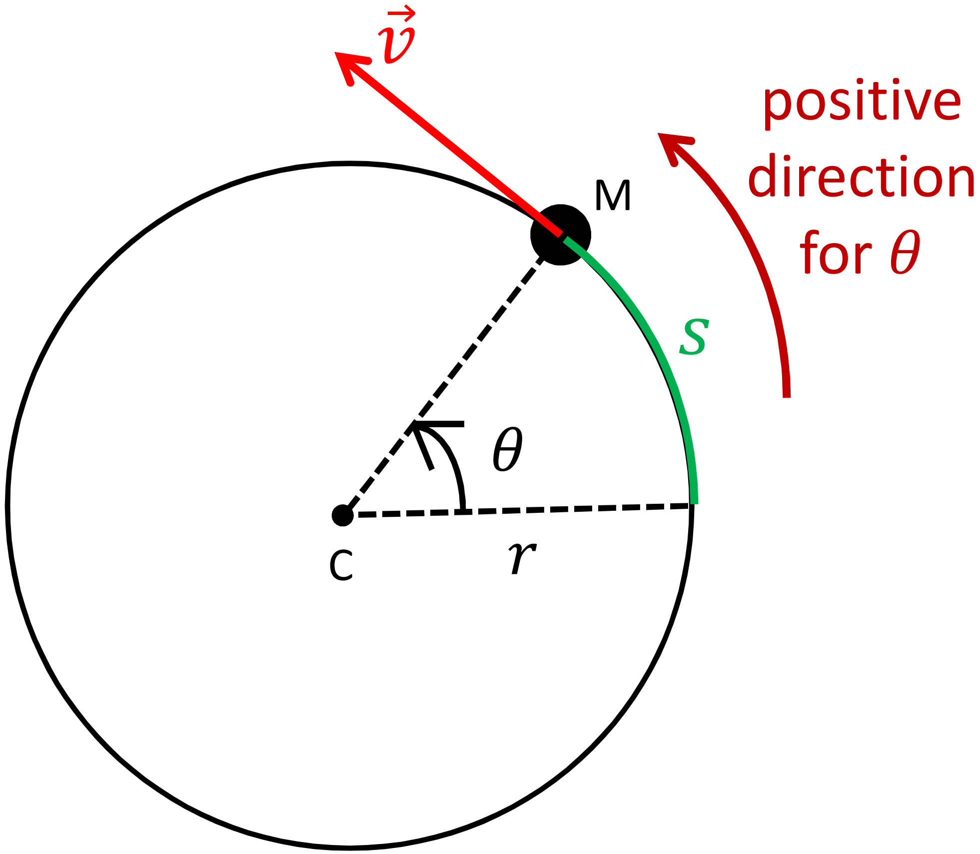
Pointing away from us, in page. $\otimes\,\vec{\omega}$

CHARACTERIZATION OF CIRCULAR MOTION

How does θ changes with time t?
Equation of $\theta(t)$?

➔ Rate of change of θ: ω

➔ Rate of change of ω?
(acceleration of θ)



ANGULAR ACCELERATION

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration.

Angular velocity at times t_1: ω_1
Angular velocity at times t_2: ω_2

Average angular acceleration of the rotating body in the interval from t_1 to t_2 is defined as:

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration α :

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Unit: rad/s^2

α can be negative!

WHAT IS NEXT? RATE OF CHANGE OF α?

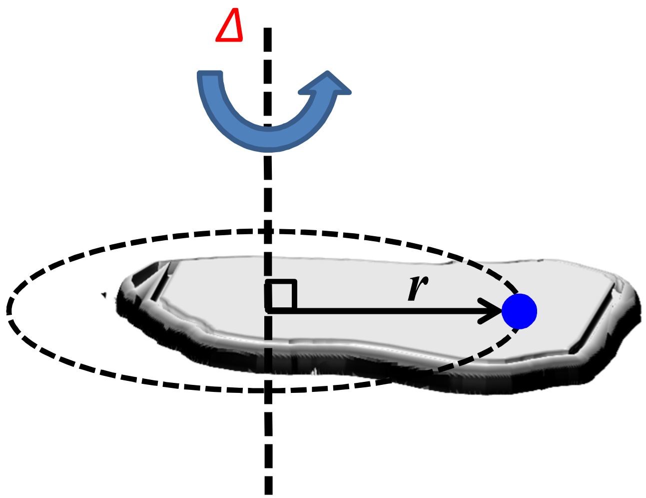
Angular variables: θ, ω, α

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

What about the rate of change of α?

- Not needed. Up to α is enough.
- Furthermore, in this course, we are only interested in **constant** angular acceleration ($\frac{d\alpha}{dt} = 0$).

ANGULAR VARIABLES OF DIFFERENT POINTS IN A RIGID BODY



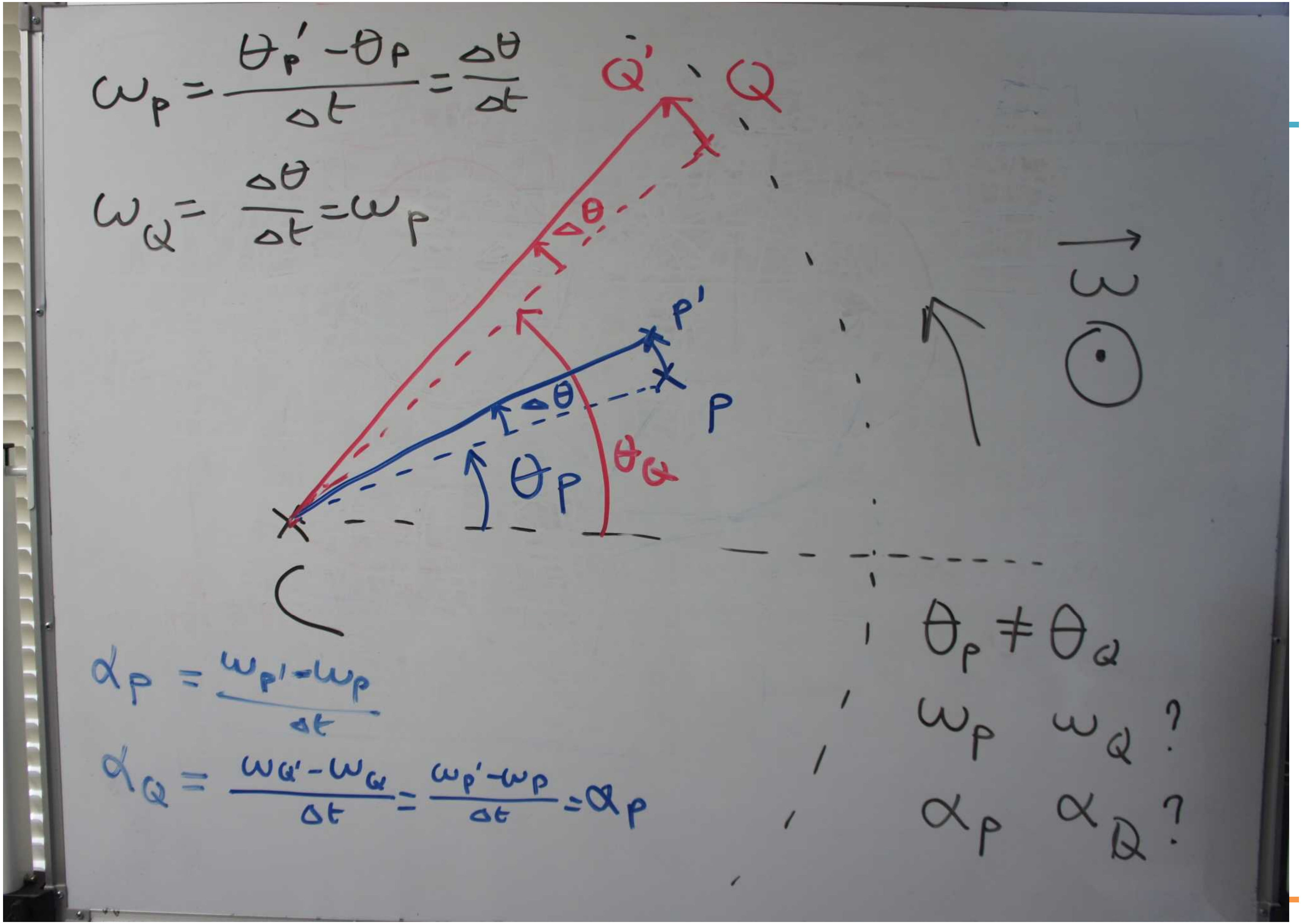
Angular variables: θ, ω, α

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

➔ ω and α do not depend on distance *r* from rotation axis

➔ All particles (points) in a rigid body have the same angular velocity ω and acceleration α.



$\omega_P = \frac{\theta_P' - \theta_P}{\Delta t} = \frac{\Delta\theta}{\Delta t}$
$\omega_Q = \frac{\Delta\theta}{\Delta t} = \omega_P$
Q'
Q
P'
P
$\Delta\theta$
θ_P
θ_Q
$\vec{\omega}$
$\alpha_P = \frac{\omega_{P'} - \omega_P}{\Delta t}$
$\alpha_Q = \frac{\omega_{Q'} - \omega_Q}{\Delta t} = \frac{\omega_{P'} - \omega_P}{\Delta t} = \alpha_P$
$\theta_P \neq \theta_Q$
ω_P ω_Q ?
α_P α_Q ?

ROTATION WITH CONSTANT ANGULAR ACCELERATION

Equation of $\theta(t)$ when α is constant?

Mathematically, the treatment of α is the same as that of the linear acceleration *a*,

… or any constant acceleration, e.g. maybe the population of some bacteria in a culture; number of infected patient during an epidemic; number of fishes in some oceans (would be a negative acceleration!), etc.

➔ the equations of motion for **constant angular acceleration** are the same as those for **constant linear acceleration**.

*The variable is the **angle** instead of being the **distance**.*

$$\alpha = \frac{d\omega}{dt} \quad \rightarrow \quad \omega = \alpha t + \omega_0$$

$$\omega = \frac{d\theta}{dt} \quad \rightarrow \quad \theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$$

ROTATION WITH CONSTANT ANGULAR ACCELERATION

Linear Equation	Missing Variable		Angular Equation
$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

The equations for *constant* ***angular*** *acceleration* are similar to those for *constant* ***linear*** *acceleration*.

RELATING THE LINEAR AND ANGULAR VARIABLES

Pure rotation: r = constant

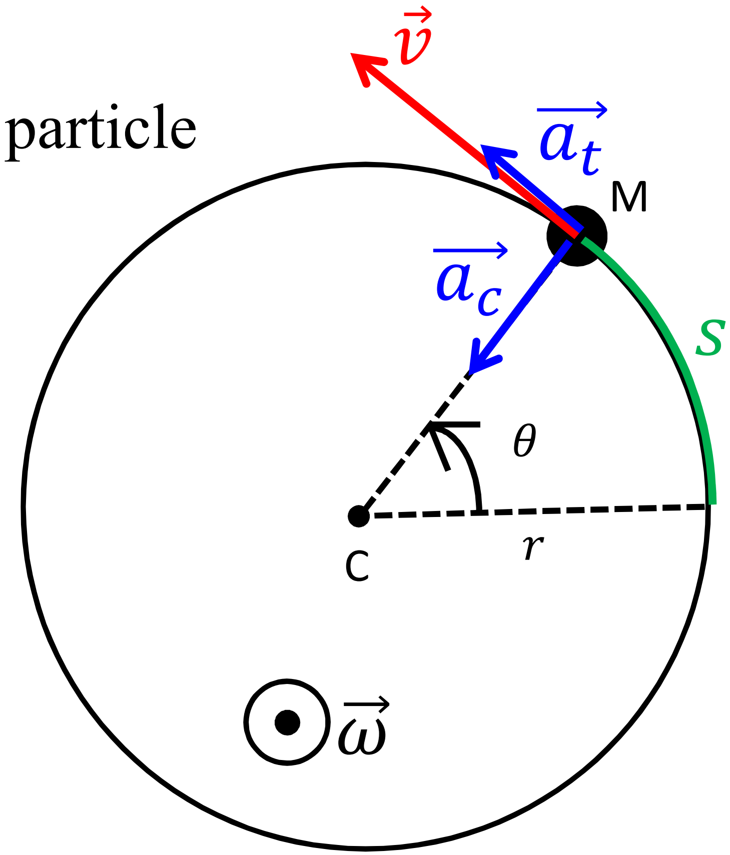
- Angular variables: θ, ω, α
- Linear variables: s, v, a (a_t and a_c) of particle

Pure rotation ➔ v always tangential

(any component of v ⊥ to tangent would lead to a change of r)

$$s = r\theta$$

$$\xrightarrow{Differentiate} \quad v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



Period of revolution T for the motion of each point (no dependence on r) and for the rigid body itself:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

ACCELERATION FOR **UNIFORM** CIRCULAR MOTION

$|\vec{v}|$ = constant $\Rightarrow \dfrac{dv}{dt} = 0$

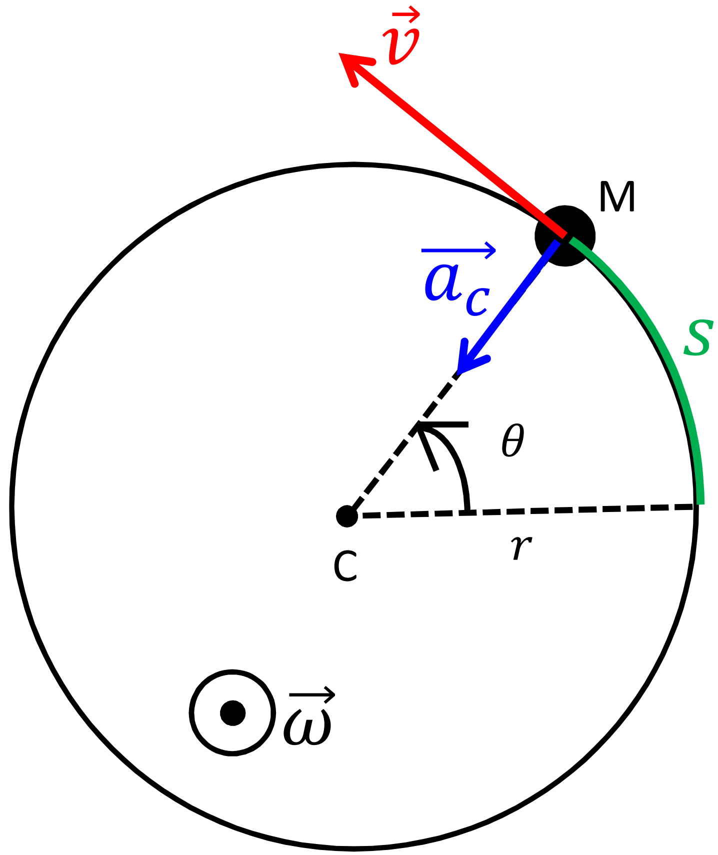
(ω = constant, $\alpha = 0$)

$\vec{v} \neq$ constant $\Rightarrow \dfrac{d\vec{v}}{dt} \neq 0$

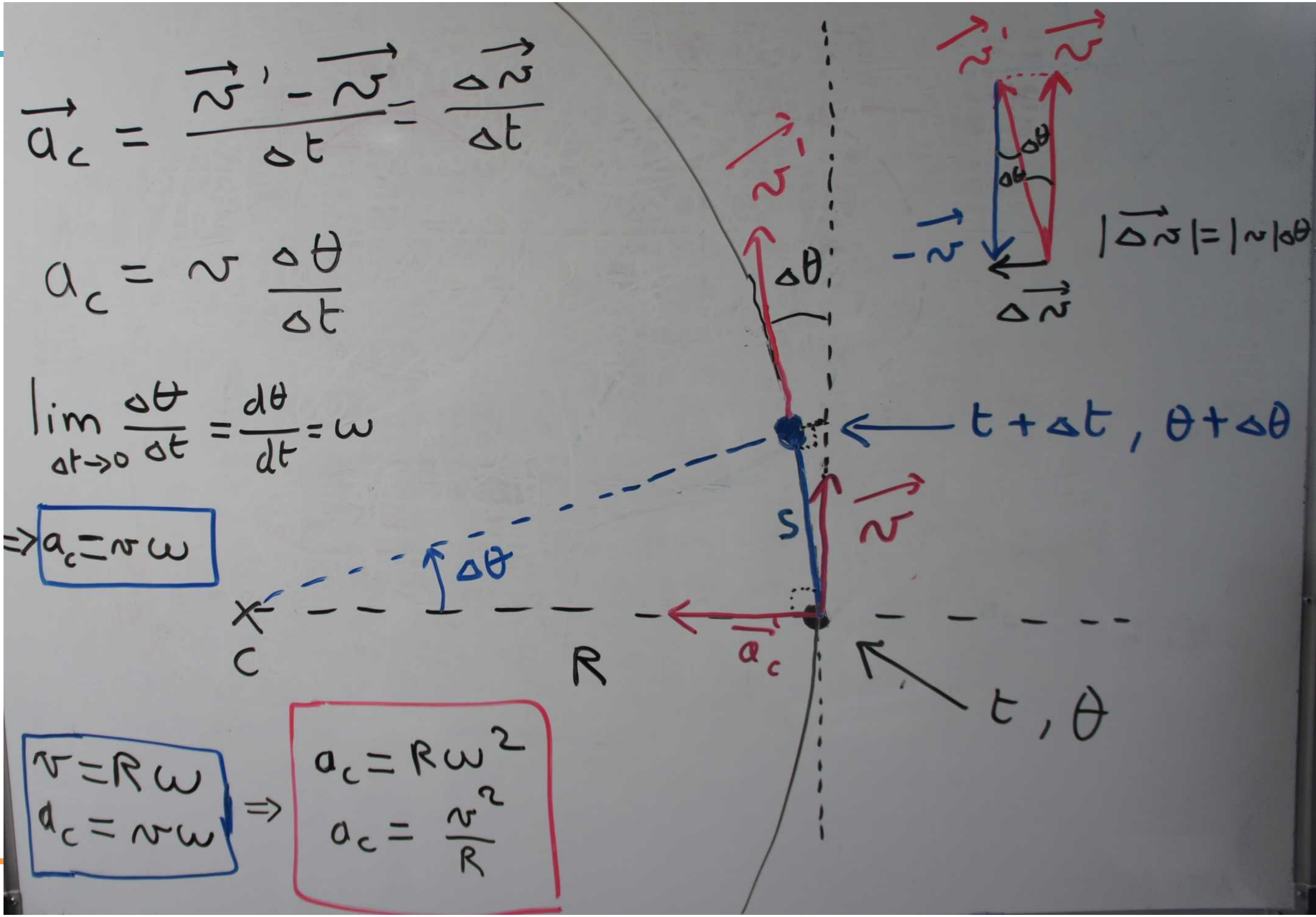
$$\overrightarrow{a_c} = \frac{d\vec{v}}{dt}$$

If there were a tangential component of $\vec{a}$, $|\vec{v}|$ would change!

$\overrightarrow{a_c} \perp \vec{v}$ ➔ $\overrightarrow{a_c}$ only originates from change in $\vec{v}$ direction

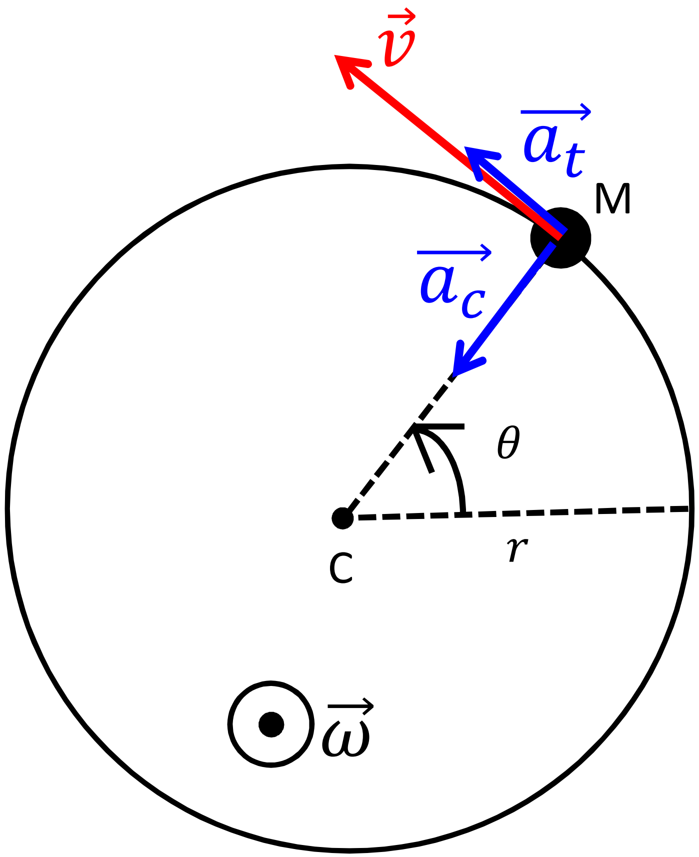


CENTRIPETAL ACCELERATION



$|\vec{v}|$ CHANGES ➔ TANGENTIAL ACCELERATION $\overrightarrow{a_t}$

The velocity vector is always tangent to the circle around the rotation axis.



$$\vec{a} = \frac{d\vec{v}}{dt}$$

➔ There must be a component of $\vec{a}$ along $\vec{v}$ direction!

$\overrightarrow{a_t} // \vec{v}$ ➔ $\overrightarrow{a_t}$ only originates from change in $\vec{v}$ magnitude

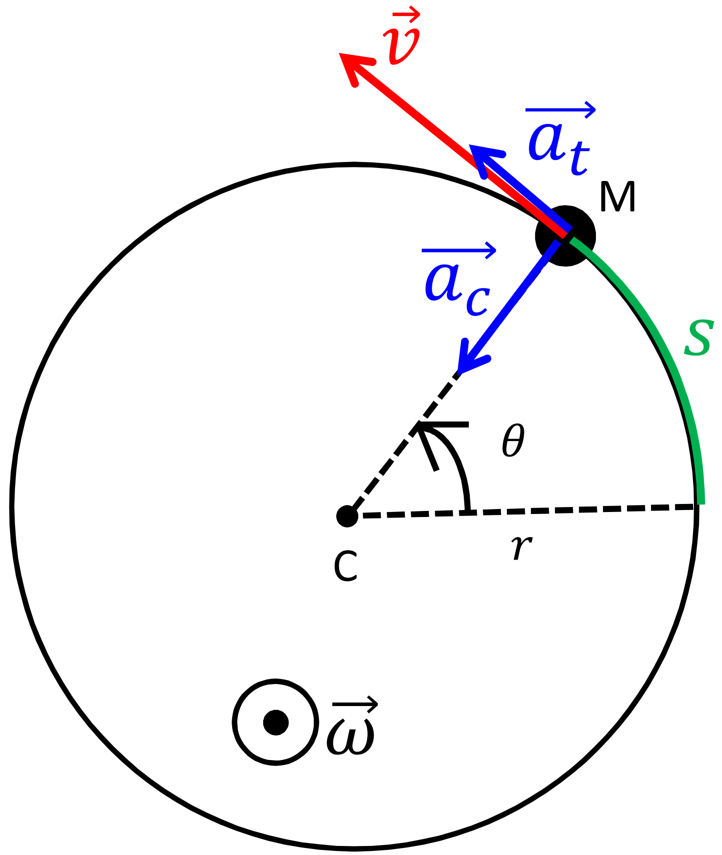
$$\Rightarrow \ a_t = \frac{dv}{dt}$$

RELATING THE LINEAR AND ANGULAR VARIABLES

Pure rotation: r = constant

- Angular variables: θ, ω, α
- Linear variables: s, v, a (a_t and a_c)

$$s = r\theta \qquad v = r\omega$$



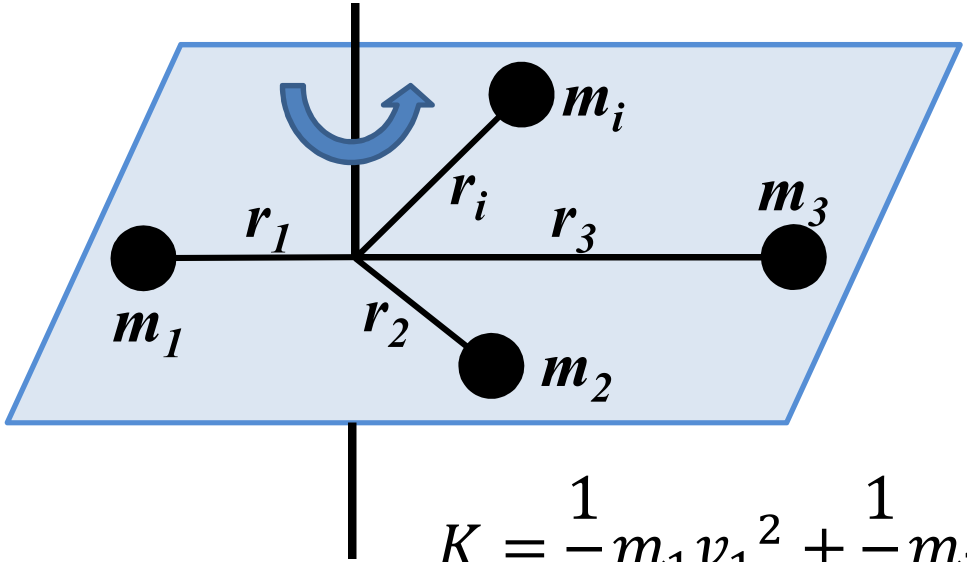
Acceleration:

$$a_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\frac{d\omega}{dt} = r\alpha$$

$$a_c = r\omega^2 = \frac{v^2}{r}$$

$\overrightarrow{a_c} \perp \vec{v}$ ➔ $\overrightarrow{a_c}$ only originates from change in $\vec{v}$ direction

KINETIC ENERGY OF A RIGID SYSTEM OF PARTICLES IN ROTATION



What is K?

$$K = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 + \frac{1}{2}m_3{v_3}^2 + \frac{1}{2}m_i{v_i}^2$$

$$K = \sum_i \frac{1}{2}m_i{v_i}^2$$

Expression as a function of rotational variables?

Remember: θ, ω, and α do not depend on position!

➔ simpler expression of K?

KINETIC ENERGY OF A SYSTEM IN ROTATION

$v = r\omega$ *r*: distance from axis

$$K = \sum_i \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2}\underbrace{\left(\sum_i m_i {r_i}^2\right)}\omega^2$$

Rotational inertia:
a property of the system
for a particular axis:

$$I = \sum_i m_i {r_i}^2$$

$$\Rightarrow \quad \boxed{K = \frac{1}{2} I \omega^2} \qquad \text{(radian measure)}$$

Memory hint: K = ½ Inertia Velocity²

MOMENT OF INERTIA WITH RESPECT TO AXIS Δ

Rotational inertia: $I_\Delta = \sum_i m_i {r_i}^2$

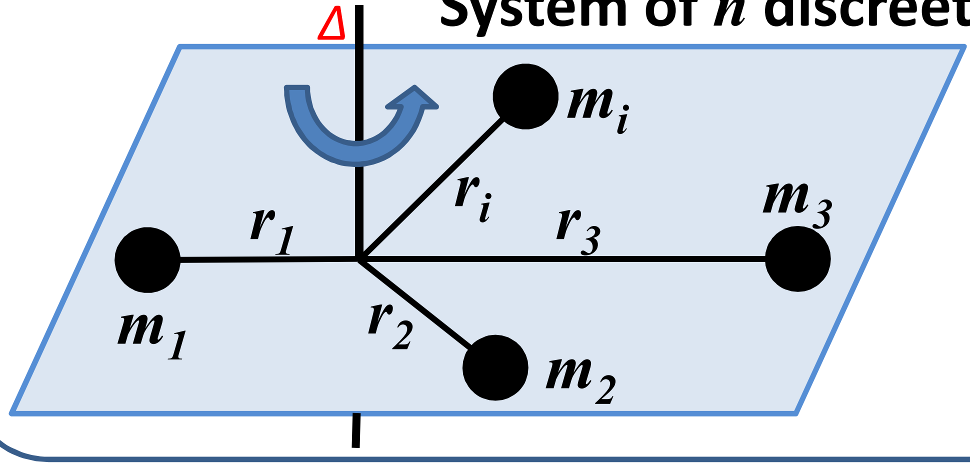
r_i is distance of m_i from axis Δ.

I is called the rotational inertia (or moment of inertia) of the body ***with respect to the axis of rotation*** Δ.

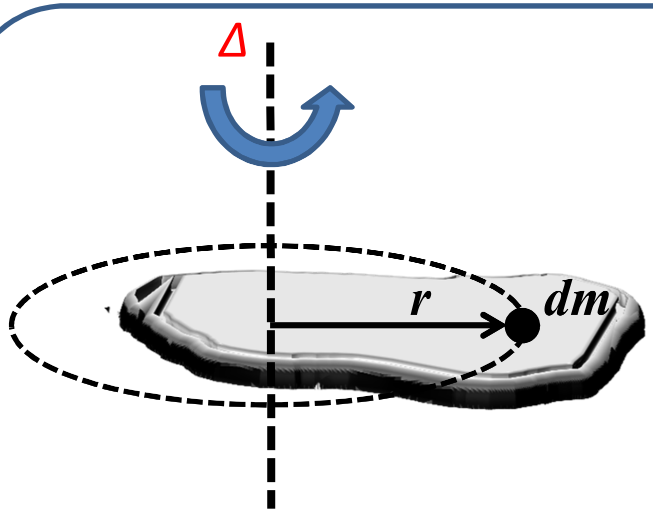
It is a constant for a particular rigid body and a particular rotation axis Δ (That axis must always be specified).

MOMENT OF INERTIA WITH RESPECT TO AXIS Δ

System of *n* discreet point masses



$$I_\Delta = \sum_{i=1}^{n} m_i {r_i}^2$$



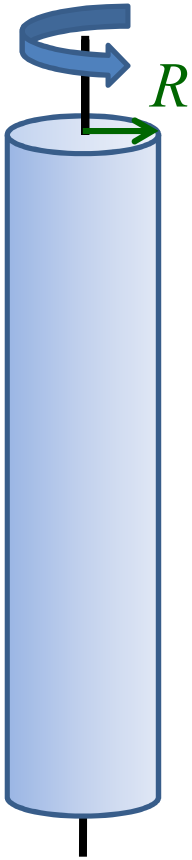
Rigid body

If a rigid body consists of a great many adjacent particles (it is continuous, like a Frisbee), we consider an integral to define the moment of inertia of the body with respect to axis Δ

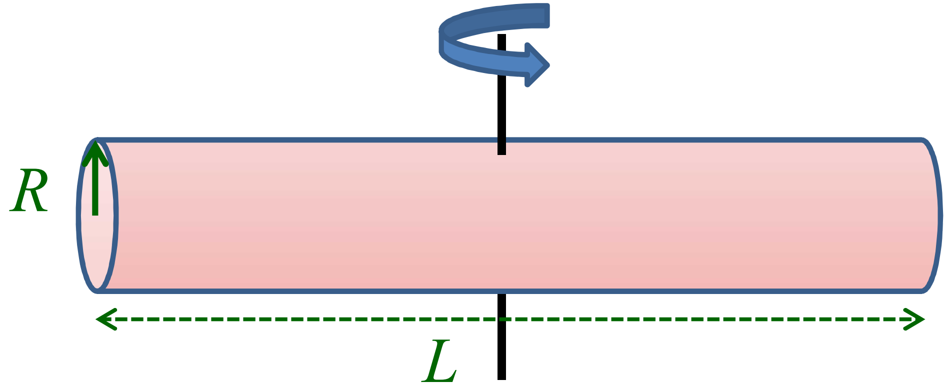
$$I_\Delta = \int_{Body} r^2 dm$$

***r*: distance of *dm* from axis Δ**

ROTATIONAL INERTIA



$$I = \frac{1}{2}MR^2$$



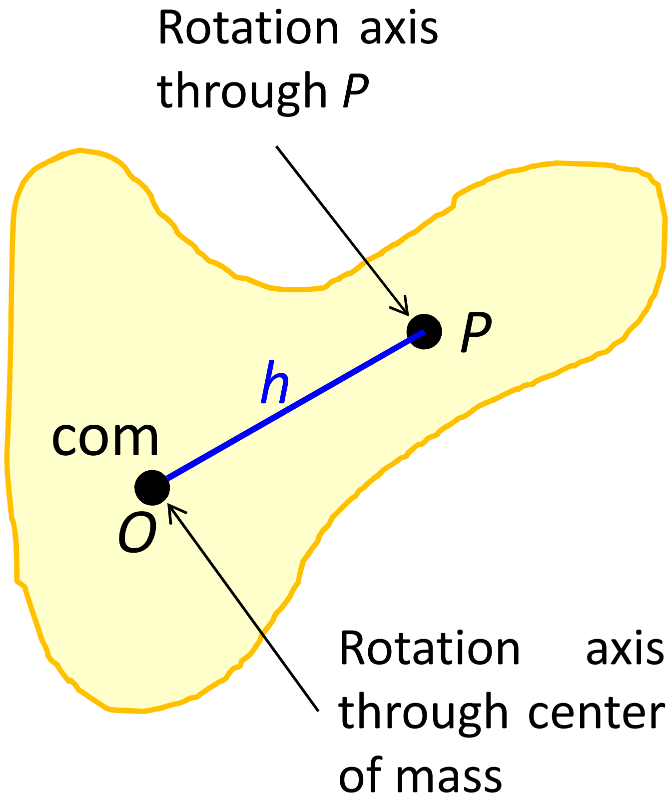
$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

Mass distributed away from the axis
➔higher inertia (when $L > \sqrt{3}R$)
➔Need more force to twirl

Mass concentrated around the axis
➔Low inertia
➔Easy to twirl

PARALLEL AXIS THEOREM

$$I_P = I_{com} + Mh^2$$



$\boldsymbol{I_P}$: rotational inertia about a given axis through *P*.

$\boldsymbol{h}$: perpendicular distance between the given axis and the axis through the center of mass (these two axes being parallel).

$\boldsymbol{M}$: total mass of rigid body.

FORMULA SUMMARY FOR PURE ROTATION

$$\omega = \omega_0 + \alpha t \qquad (10\text{-}12)$$

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad (10\text{-}13)$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad (10\text{-}14)$$

Moment of inertia About axis Δ

$$I_\Delta = \sum_{i=1}^{n} m_i {r_i}^2 \qquad I_\Delta = \int_{Body} r^2 dm$$

Parallel axis theorem

$$I_P = I_{com} + Mh^2$$

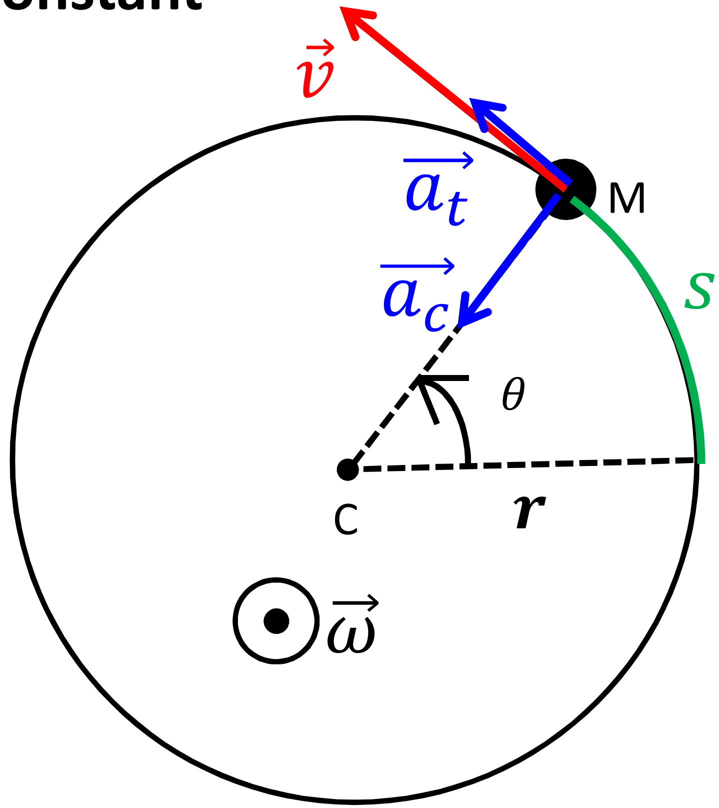
Kinetic energy of pure rotation

$$K = {}^1\!/_2\, I\omega^2$$

FORMULA SUMMARY FOR PURE ROTATION

Pure rotation: r = **constant**

- Angular variables: θ, ω, α
- Linear variables: s, v, a (a_t and a_c)



$$s = r\theta$$

$$v = r\omega$$

$$a_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\frac{d\omega}{dt} = r\alpha$$

$\overrightarrow{a_t} \,//\, \vec{v}$

If $|\overrightarrow{a_t}| \neq 0$ then $|\vec{v}|$ changes

$$a_c = r\omega^2 = \frac{v^2}{r}$$

$\overrightarrow{a_c} \perp \vec{v}$ and $\overrightarrow{a_t}$

$\overrightarrow{a_c}$ is responsible for change in $\vec{v}$ direction

RELATING THE LINEAR AND ANGULAR VARIABLES
VECTOR ANALYSIS (COMMENTS FOR CULTURE ONLY)

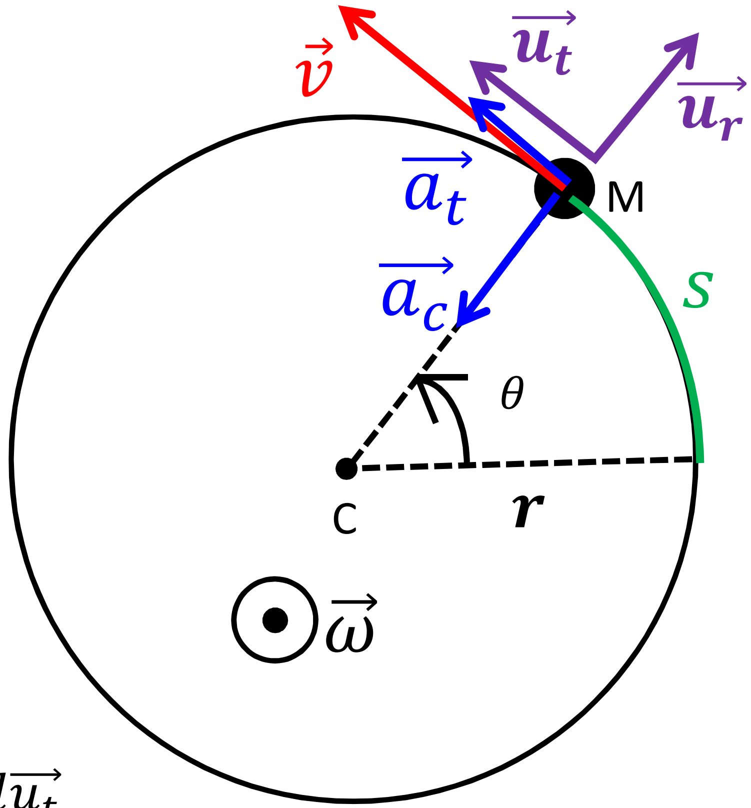
Pure rotation: r = constant $\qquad \overrightarrow{u_r}$ *and* $\overrightarrow{u_t}$: *polar unit vectors*

$$\overrightarrow{CM} = \vec{r} = r\overrightarrow{u_r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt} = r\frac{d\overrightarrow{u_r}}{dt}$$

$$d\overrightarrow{u_r} = d\theta\overrightarrow{u_t} \Rightarrow \frac{d\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\overrightarrow{u_t} = \omega\overrightarrow{u_t}$$

$$\Rightarrow \vec{v} = r\omega\overrightarrow{u_t}$$



$$\vec{a} = \frac{d\vec{v}}{dt} = r\frac{d\omega}{dt}\overrightarrow{u_t} + r\omega\frac{d\overrightarrow{u_t}}{dt}$$

$$= r\alpha\overrightarrow{u_t} + r\omega\frac{d\overrightarrow{u_t}}{dt}$$

$$d\overrightarrow{u_t} = -d\theta\overrightarrow{u_r} \Rightarrow \frac{d\overrightarrow{u_t}}{dt} = -\omega\overrightarrow{u_r}$$

$$\Rightarrow \vec{a} = r\alpha\overrightarrow{u_t} - r\omega^2\overrightarrow{u_r}$$

General formula: $\dfrac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$