

Lesson 07

Linear Transformation

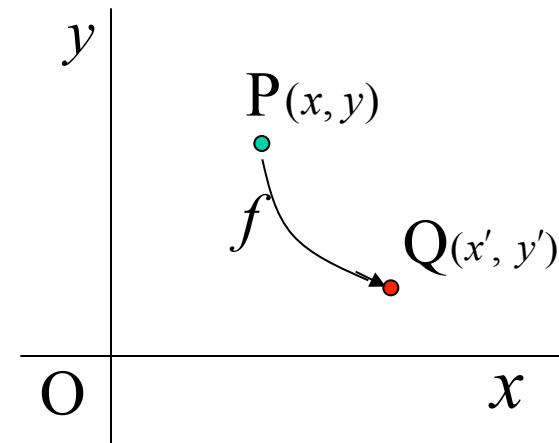
7A

- Linear transformation
- Reflections

Linear Transformation

Transformation

$$f: (x, y) \rightarrow (x', y')$$



Linear transformation

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \rightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Return to the old coordinate

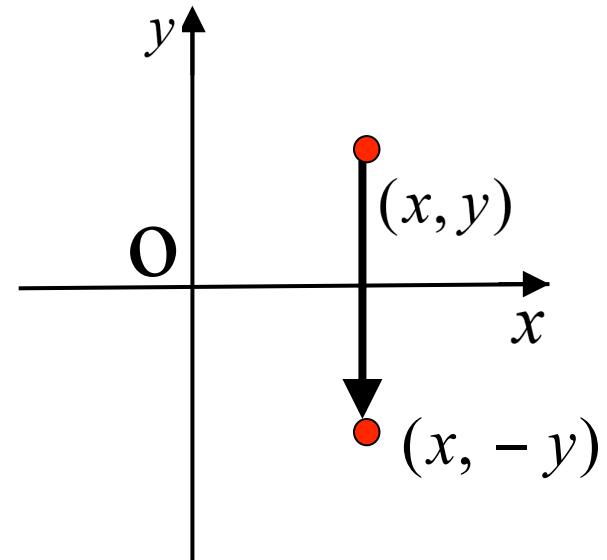
$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Representative Reflections

1. Reflection about the x-axis

$$f : (x, y) \rightarrow (x, -y)$$

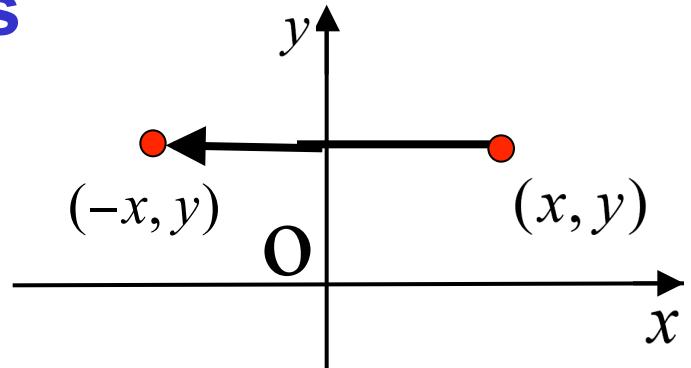
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



2. Reflection about the y-axis

$$f : (x, y) \rightarrow (-x, y)$$

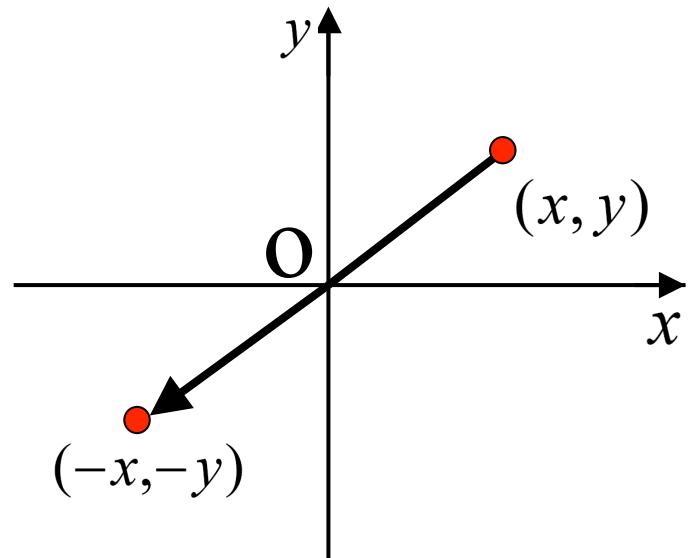
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



3. Reflection about the origin

$$f : (x, y) \rightarrow (-x, -y)$$

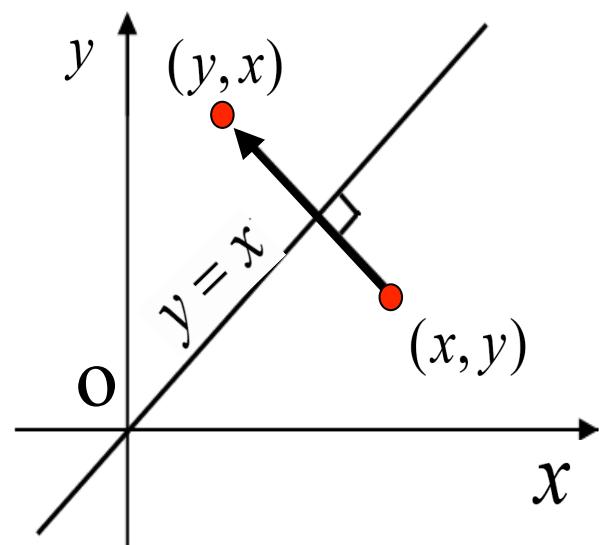
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



4. Reflection about the line y=x

$$f : (x, y) \rightarrow (y, x)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



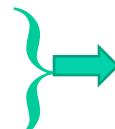
Example

[Example 7-1] By the linear transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, points $(1, 1)$ and $(2, 1)$ are transformed to the points $(3, 2)$ and $(7, 0)$, respectively. Find the transformation matrix A.

Ans.

Transformations : $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$


$$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Since

$$\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \times 1 - 2 \times 1 = -1 \neq 0$$

Inverse matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{(-1)} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

Postmultiplying

$$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -2 & 4 \end{pmatrix}$$

Exercise

[Ex.7-1] Three points $P(-2, -2)$, $Q(0, 2)$ and $R(2, -2)$ are the vertices of a triangle. Consider the following linear transformation. $x'=x+y$, $y'=x-y$
Illustrate the original triangle and the transformed triangle $P'Q'R'$.

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.7-1] Three points $P(-2, -2)$, $Q(0, 2)$ and $R(2, -2)$ are the vertices of a triangle. Consider the following linear transformation. $x'=x+y$, $y'=x-y$
Illustrate the original triangle and the transformed triangle $P'Q'R'$.

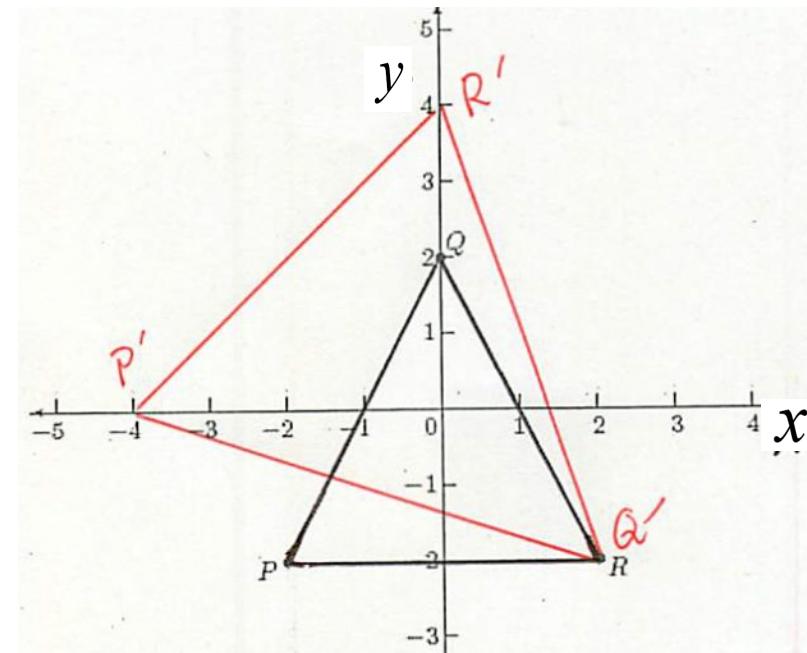
Ans.

The transformed vertices are

$$P', \quad \begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$Q', \quad \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$R', \quad \begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$



Lesson 07

Application: Linear Transformation

7B

- Rotation
-

Rotation

Rotation around the origin by θ

Original point P $x = r \cos \alpha$ (1)
 $y = r \sin \alpha$

Transferred point Q

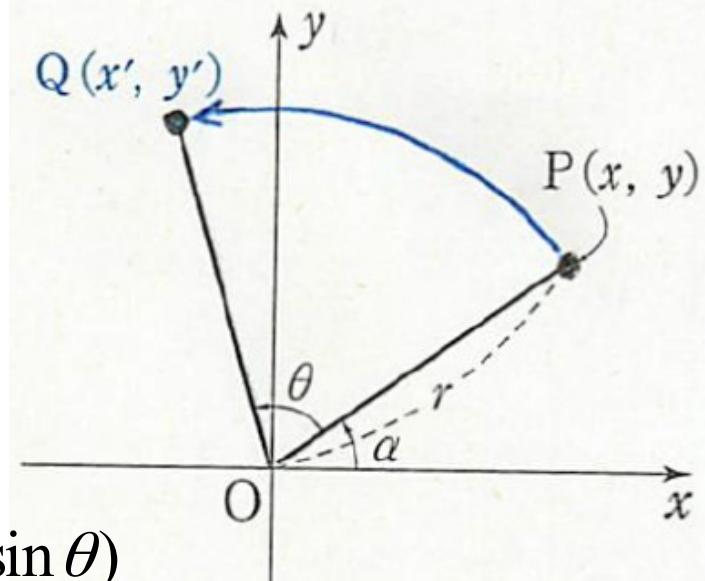
$$\begin{aligned}x' &= r \cos(\alpha + \theta) = r(\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\y' &= r \sin(\alpha + \theta) = r(\sin \alpha \cos \theta + \cos \alpha \sin \theta)\end{aligned}$$

Substituting Eq.(1), we have

$$\left. \begin{aligned}x' &= x \cos \theta - y \sin \theta \\y' &= x \sin \theta + y \cos \theta\end{aligned} \right\}$$

Using matrix, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Example

[Examples 7-2]

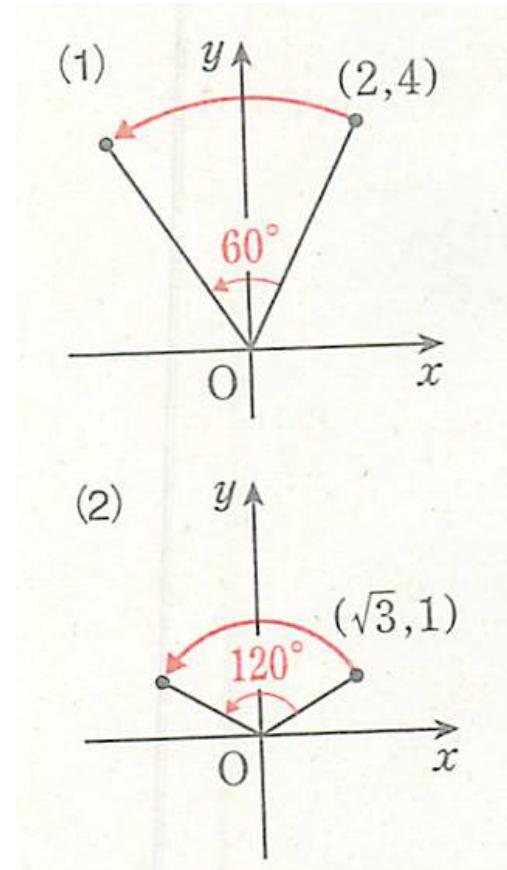
Find the new positions of each point after given rotation counterclockwise.

- Position $(2, 4)$, 60°
- Position $(\sqrt{3}, 1)$, 120°

Ans. From the formula of rotation, we have

$$(1) \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 - 2\sqrt{3} \\ \sqrt{3} + 2 \end{pmatrix}$$

$$(2) \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$



End



Yeah ! I did it !

We appreciate your patience and

Authors
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