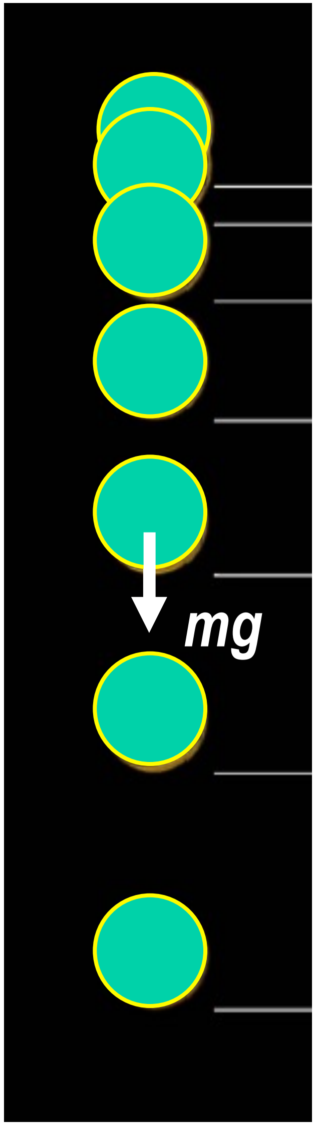
Course II

NAGOYA UNIVERSITY

Lesson 14
Differential Equations (1)

14A
- General introduction

Example : Free Falling Body

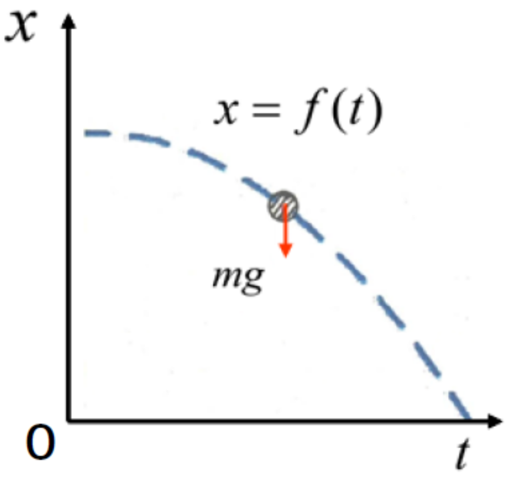


Example 14-1 Motion of a falling ball

Newton's second law

$$m\frac{d^2x}{dt^2} = -mg$$

→ Solve



Differential equation relates the values of the function itself and its derivatives of various orders.

Meaning of Differential Equation

[Examples 14-2] Investigate the D.E. $y' = -\frac{x}{y}$

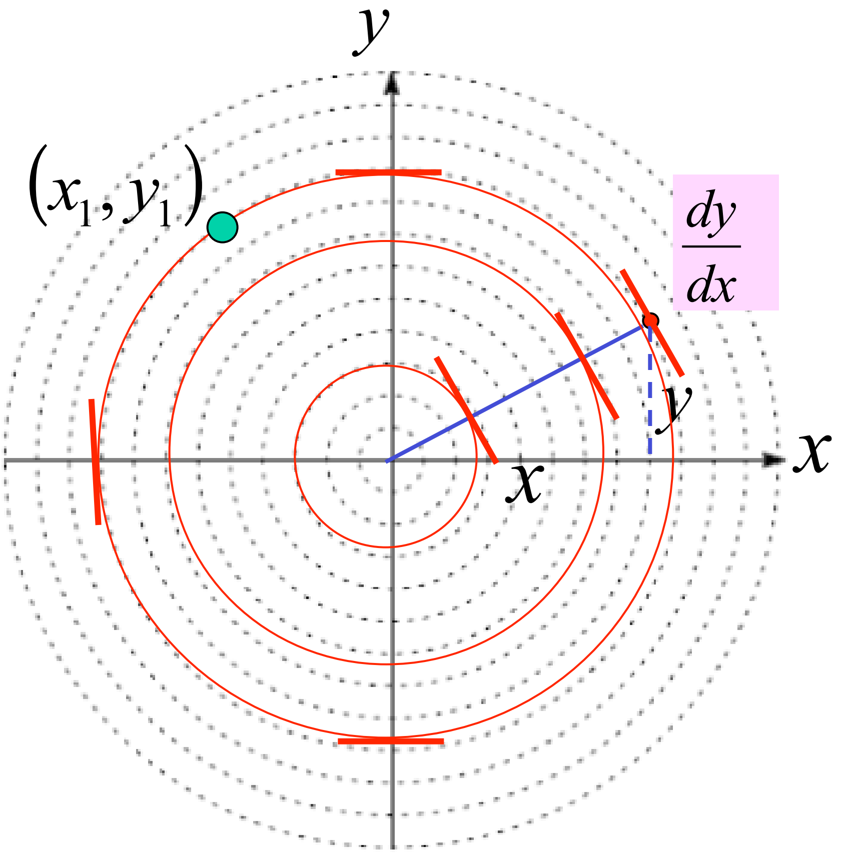
Ans.

$y' = -\frac{x}{y}$ gives a slope $\frac{dy}{dx}$ at point (x, y)

If the slopes are connected smoothly, the solution $x^2 + y^2 = C^2$ is obtained.

(C : Arbitrary constant)

General solution



Boundary condition $x = x_1,\ y = y_1$

determines one solution.

(Special solution)

Some Terminologies on Differential Equations

First order D.E.

D.E. which contains only first derivatives. (Ex.) $y' = y + x$

Second order D.E.

D.E. which contains second derivatives (and possibly first derivatives also.) (Ex.) $y'' + y = 1 + y^2 + x$

Linear D.E.

The general n-th order linear D.E. of the form

$$y^{(n)} + P_n(x)y^{(n-1)} + \cdots + P_2(x)y' + P_1(x)y = Q(x)$$

Nonlinear D.E.

Differential equations which are not linear are called nonlinear D.E.

(Ex.) $yy' + 5x = 0$ $\theta'' + 5\sin\theta = 0$

How to Solve D.E. : Simplest Case

Simplest differential equations $y' = f(x)$

➡ The solution is an antiderivative of $f(x)$,

$$y = \int f(x)\,dx$$: General solution

[Examples 14-3] Answer the following questions

(1) Find the general solution of $y' = -7x$.

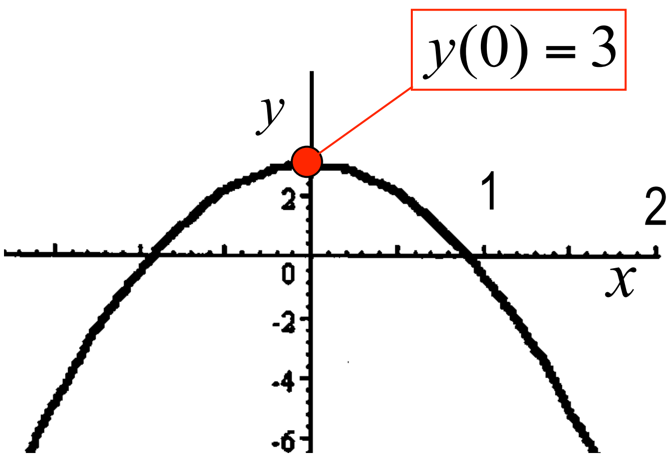
(2) Find the particular solution satisfying the boundary condition $y(0) = 3$.

Ans.

(a) $$y = \int(-7x)\,dx = -\frac{7}{2}x^2 + C$$

(b) The boundary condition is $x = 0, y = 3$.

$\therefore\ 3 = 0 + C \qquad \therefore\ y = -\frac{7}{2}x^2 + 3$



Exercise

[Ex.14-1] Find the particular solution of the following D.E.

(1) $y' = 5$, Boundary condition $x = 0, y = 2$

(2) $y' = \cos 3x$, Boundary condition $x = 0,\ y = 0$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.14-1] Find the particular solution of the following D.E.

(1) $y' = 5$, Boundary condition $x = 0, y = 2$

(2) $y' = \cos 3x$, Boundary condition $x = 0,\ y = 0$

Ans.

(1) Integrating both sides by x, we have $y = \int 5dx \quad \therefore \quad y = 5x + C$

Applying the boundary condition $2 = 5 \times 0 + C \quad \therefore \quad y = 5x + 2$

(2) $y' = \cos 3x \quad \therefore \quad y = \int \cos 3x dx + C \quad \therefore \quad y = \frac{1}{3}\sin 3x + C$

Applying the boundary condition $x = 0,\ y = 0$

$$\therefore \quad y = \frac{1}{3}\sin 3x$$

Lesson 14
Differential Equations

14B

- Some Types of the First Order D.E.

Separable Differential Equations

Separable D.E.

$$\frac{dy}{dx} = f(x)g(y)$$

Rewriting this, we have

$$\frac{1}{g(y)}dy = f(x)dx$$

Integrating both sides, we have

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

After integration, we have the solution (in the implicit expression)

Example

[Examples 14-4] Answer the following questions concerning $\frac{dy}{dx} = -\frac{x}{y}$

(1) Find the general solution.

(2) Find the particular solution which passes $x = 0,\ y = 1$.

Ans. (1) Rewriting the given equation, we have $ydy = -xdx$

$$\therefore \quad \int ydy = -\int xdx \qquad \therefore \quad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1$$

If we put $C = 2C_1$, we have the general solution $x^2 + y^2 = C$

(2) Substituting $x = 0,\ y = 1$, we have $C = 2$.

Therefore, $x^2 + y^2 = 1$.

Exercise

[Ex14-2] Solve the following differential equation.

$$\frac{dy}{dx} = 6y^2x \qquad \text{Boundary condition} \qquad x = 1,\ y = \frac{1}{25}$$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex14-2] Solve the following differential equation.

$$\frac{dy}{dx} = 6y^2x \qquad \text{Boundary condition} \qquad x = 1,\ y = \frac{1}{25}$$

Ans.

Rewriting the given equation, we have $\frac{1}{y^2}dy = 6xdx$

By integrating this, we have $\int \frac{1}{y^2}dy = \int 6xdx$

$$\therefore \quad -\frac{1}{y} = 3x^2 + C$$

Substituting the boundary condition, we have $C = -28$

The particular solution is

$$-\frac{1}{y} = 3x^2 - 28 \qquad \therefore \quad y = \frac{1}{28 - 3x^2}$$

Homogeneous Differential Equations

Homogeneous D.E.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

If we put $\frac{y}{x} = u$

then $y = ux$ $\therefore$ $\frac{dy}{dx} = u + x\frac{du}{dx}$

I am getting confused.

By substitution, we have $u + x\frac{du}{dx} = f(u)$

Therefore, $\frac{du}{f(u) - u} = \frac{dx}{x}$ ⟵ A separable form

[Note] The following ordinary D.E. which has no term containing x alone is also called a homogeneous equation. Their meanings are entirely different.

$$y^{(n)} + P_n(x)y^{(n-1)} + \cdots + P_2(x)y' + P_1(x)y = 0$$

$\therefore$

Example

[Examples 14-5] Find the general solution of $y' = \dfrac{x^2 + y^2}{2xy}$

Ans. We can rewrite the equation to $y' = \dfrac{1 + (y/x)^2}{2(y/x)}$

Put $\dfrac{y}{x} = u$ then $y = xu$

After substitution, we have

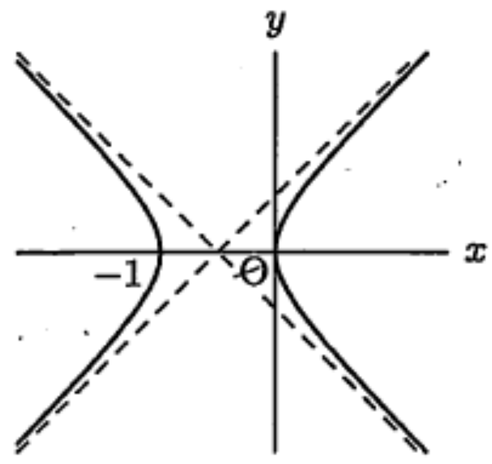
$$u + xu' = \frac{1+u^2}{2u} \quad \therefore \int \frac{2u}{u^2-1}du = -\int \frac{1}{x}dx$$

$$\therefore \quad \ln\left|u^2 - 1\right| = -\ln|x| + C_1$$

Put $\pm e^{C_1} = C$, we have $x(u^2-1) = C$

$$\therefore \quad x\left\{\left(\frac{y}{x}\right)^2 - 1\right\} = C \qquad \therefore \quad y^2 - x^2 = Cx$$

When $C = 1$



$y^2 - x^2 = x$

Exercise

[Ex.14-3] Find the general solution of $y' = \dfrac{x+y}{x}$

Ans.

Pause the video and solve the problem by yourself.

Exercise

[Ex.14-3] Find the general solution of $y' = \dfrac{x+y}{x}$

Ans.

Put $\dfrac{y}{x} = u$ then $y = xu$

After substitution, we have

$$u + xu' = 1 + u \quad \therefore \quad x\frac{du}{dx} = 1 \quad \therefore \quad \int du = \int \frac{1}{x} dx$$

By integrating this, we have

$$u = \ln|x| + C \quad \therefore \quad \frac{y}{x} = \ln|x| + C$$