Lesson 13
Application of Integrals (2)

13A

- Arc length of a curve

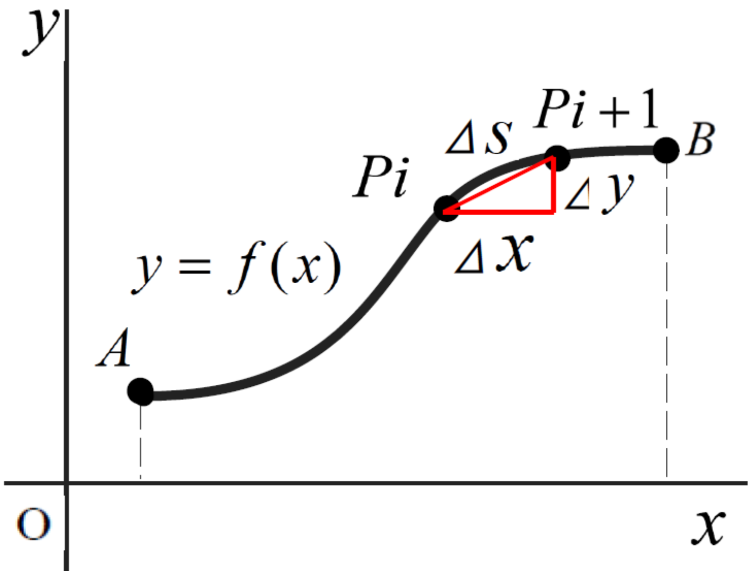
Finding Arc Length by Integration (1)

Curve in a plane

$$y = f(x)$$

Small segment

$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2} \approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$



Total length

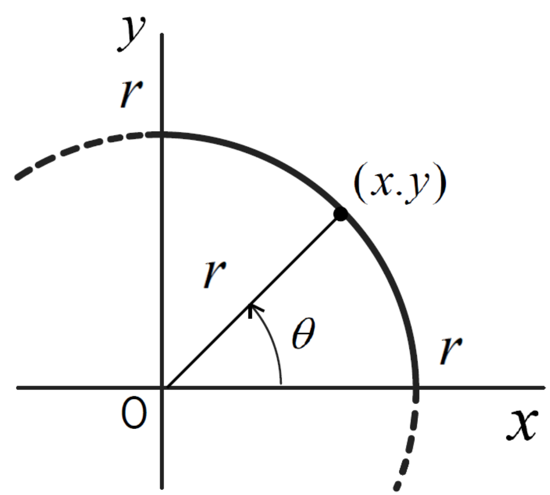
$$s \approx \sum \Delta s \approx \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

$$\rightarrow \quad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + \{f'(x)\}^2} dx$$

Example

Example 13-1 Find the length of circumference of a circle $x^2 + y^2 = r^2$

Ans. Consider the length in quadrant I.



Graph $y = \sqrt{r^2 - x^2}$

Length $s_1 = \int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^r \sqrt{1 + \left(-\frac{x}{y}\right)^2}\,dx$

$$2x + 2y\frac{dy}{dx} = 0$$

Polar coordinates $x = r\cos\theta,\ y = r\sin\theta$

$$s_1 = \int_0^r \sqrt{1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2}(-r\sin\theta)d\theta = -r\int_{\frac{\pi}{2}}^{0} d\theta = -r[\theta]_{\frac{\pi}{2}}^{0} = \frac{\pi}{2}r$$

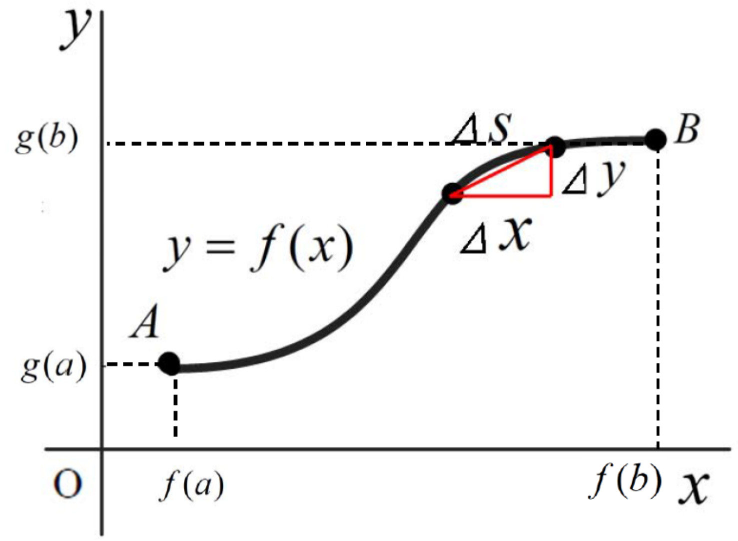
Length of circumference $s = 4s_1 = 2\pi r$

Finding Arc Length by Integration (2)

Curve in a plane

Expression using a parameter

$$x = f(t),\ y = g(t)$$



Small segment approximation

$$s \approx \sum \Delta s = \sum \sqrt{\Delta x^2 + \Delta y^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t$$

Total arc length

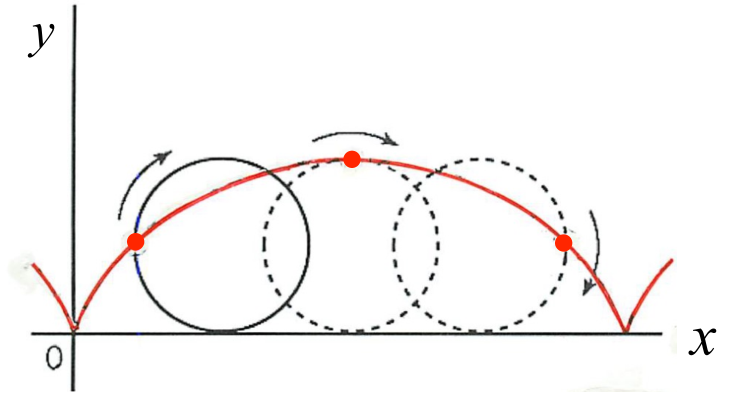
$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_a^b \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2}\ dt$$

Example

Example 13-2 Find the length of the following cycloid line.

$$x = 2(t - \sin t), \quad y = 2(1 - \cos t) \quad (0 \le t \le 2\pi)$$

[Note] A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line.



Example 13-2 Find the length of the following cycloid line.

$$x = 2(t - \sin t), \quad y = 2(1 - \cos t) \quad (0 \le t \le 2\pi)$$

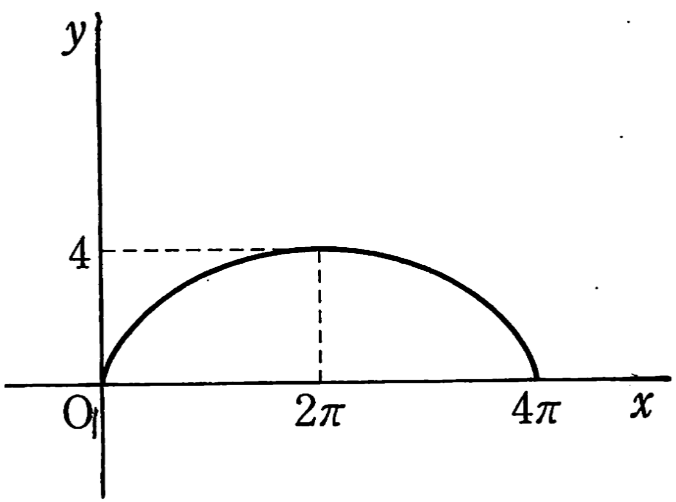
Ans.

$$\frac{dx}{dt} = 2(1 - \cos t), \qquad \frac{dy}{dt} = 2\sin t$$

Total arc length

$$s = \int_0^{2\pi} \sqrt{4(1-\cos t)^2 + 4\sin^2 t}\, dt = 2\int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt$$

$$= 4\int_0^{2\pi} \sqrt{\sin^2 \frac{t}{2}}\, dt$$



Since $\sin \frac{t}{2} \ge 0$ in $0 \le t \le 2\pi$

we have

$$s = 4\int_0^{2\pi} \sin \frac{t}{2}\, dt = 4\left[-2\cos\frac{t}{2}\right]_0^{2\pi} = 16$$

Exercise

Ex.13-1 Answer the questions about the curve $r = 2\sin\theta \ \left(0 \le \theta \le \pi\right)$ expressed by the polar coordinates.

(1) Represent this curve by the rectangular coordinate (x, y).

(2) Find the length of the curve.

Ans.

Pause the video and solve by yourself.

Answer to the Exercise

Ex.13-1 Answer the questions about the curve $r = 2\sin\theta \ \left(0 \le \theta \le \pi\right)$ expressed by the polar coordinates.

(1) Represent this curve by the rectangular coordinate (x, y) .

(2) Find the length of the curve.

Ans. (1) $x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$

$$y = r\sin\theta = 2\sin^2\theta = 1 - \cos 2\theta$$

This curve is a circle because

$$x^2 + (y-1)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$$

(2)

$$s = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, dt = \int_0^{\pi} \sqrt{\{2\cos 2\theta\}^2 + \{2\sin 2\theta\}^2}\, dt = 2\int_0^{\pi} d\theta = 2\pi$$

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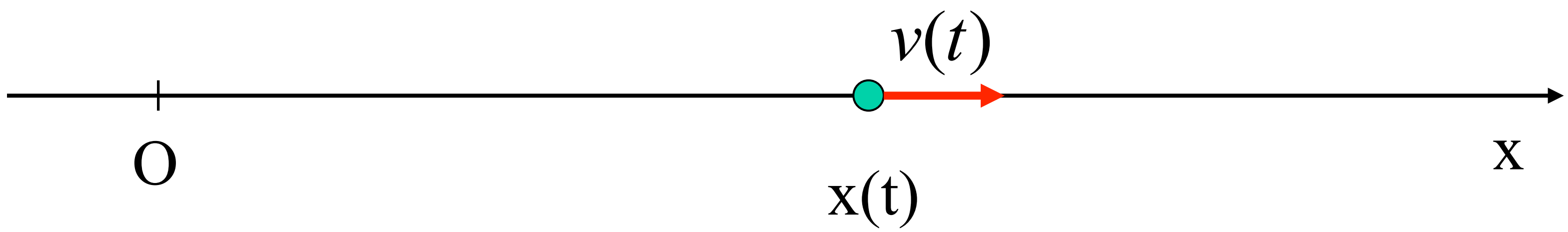
Lesson 13
Application of Integrals

13B

- Application to Physics

Position, Velocity and Acceleration

Point P moving on the x-axis



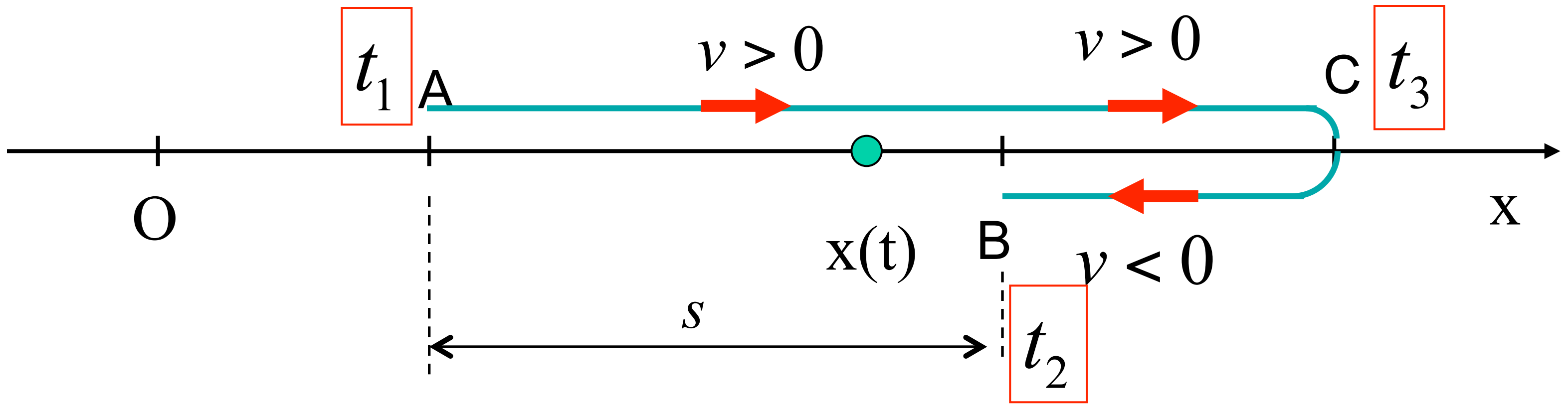
$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

Position	Velocity	Acceleration
$x = x(t)$	$v = v(t)$	$a = a(t)$

$$x(t) = \int_{t_1}^{t} v\,dt + x_1 \qquad v(t) = \int_{t_1}^{t} a\,dt + v_1$$

Distance Traveled in a Straight Line

Movement of an object P



Displacement from t=t_1 to t=t_2

$$s = \int_{t_1}^{t_2} v(t)\,dt = x(t_2) - x(t_1)$$

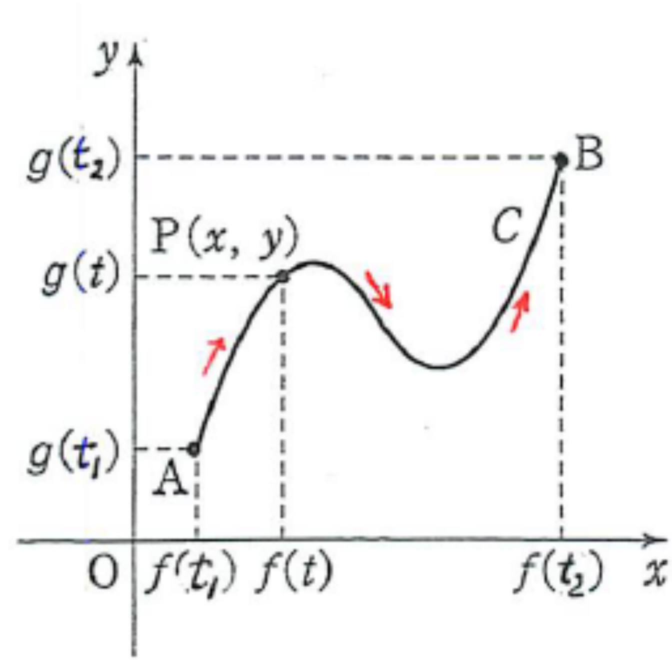
Distance traveled in a straight line t=a to t=b

$$l = \int_{t_1}^{t_3} v(t)\,dt + \int_{t_3}^{t_2} \{-v(t)\}dt = \int_{t_1}^{t_2} |v(t)|\,dt$$

Distance Traveled in a Plane

Point P moves in curve C

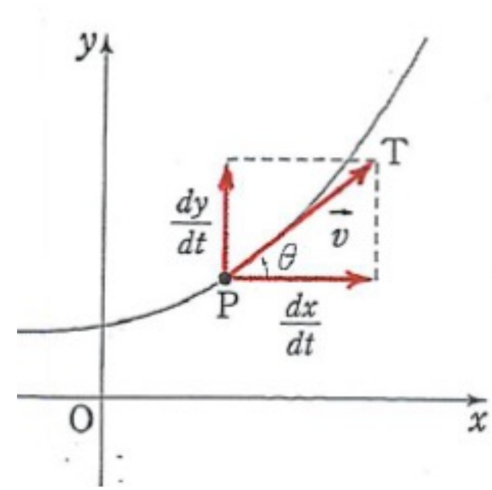
$$x = f(t),\ y = g(t)$$



Total arc length

$$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Velocity $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$



Distance traveled

$$l = \int_{t_1}^{t_2} |\vec{v}|\, dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example

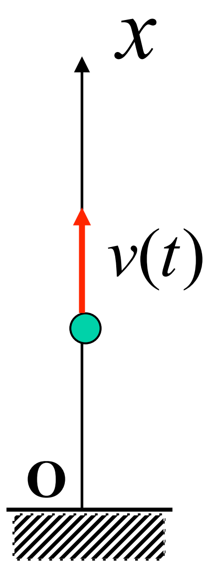
[Examples 13-3] Consider a ball being tossed upward from the ground with an initial velocity of 29.4m/s. The x-axis is taken vertically and its origin is located at the ground. The acceleration is $-9.8 m/s^2$. Answer the following questions.

(1) Express the velocity as a function of time.

(2) When the ball reached the highest point ?

(3) When did the ball fall on the ground ?

(4) Find the total distance traveled.



Ans. (1) Velocity $v(t) = 29.4 - 9.8t$

(2) From $v = 29.4 - 9.8t = 0 \quad \therefore \quad t = 3$ sec

(3) The position is given by $x = 29.4t - 4.9t^2$. Therefore, from

$$x = -4.9t(t-6) = 0 \quad \therefore \quad t = 6 \text{ sec}$$

(4) The total distance traveled

$$l = \int_{t_1}^{t_2} |\vec{v}|\, dt = \int_0^3 (29.4 - 9.8t)\, dt + \int_3^6 (-29.4 + 9.8t)\, dt = 88.2 \text{ m}$$

Exercise

Ex.12-2 The velocity of a point P moving on the x-axis is given by $v(t) = t^2 - 4t + 3$. Find the distance traveled between t=0 and t=6.

Ans.

Pause the video and solve by yourself.

Exercise

Ex.12-2 The velocity of a point P moving on the x-axis is given by $v(t) = t^2 - 4t + 3$. Find the distance traveled between t=0 and t=6.

Ans. From $v(t) = (t-1)(t-3) = 0$

We find $v(t) \geq 0$ when $t \leq 1$ and $t \geq 3$

$v(t) \leq 0$ when $1 \leq t \leq 3$

Therefore, $|v(t)| = t^2 - 4t + 3$ when $t \leq 1$ and $t \geq 3$

when $1 \leq t \leq 3$

The distance traveled

$$l = \int_0^6 |\vec{v}|\, dt = \int_0^1 (t^2 - 4t + 3)\, dt - \int_1^3 (t^2 - 4t + 3)\, dt + \int_3^6 (t^2 - 4t + 3)\, dt$$

$$\therefore \quad l = \frac{62}{3}$$