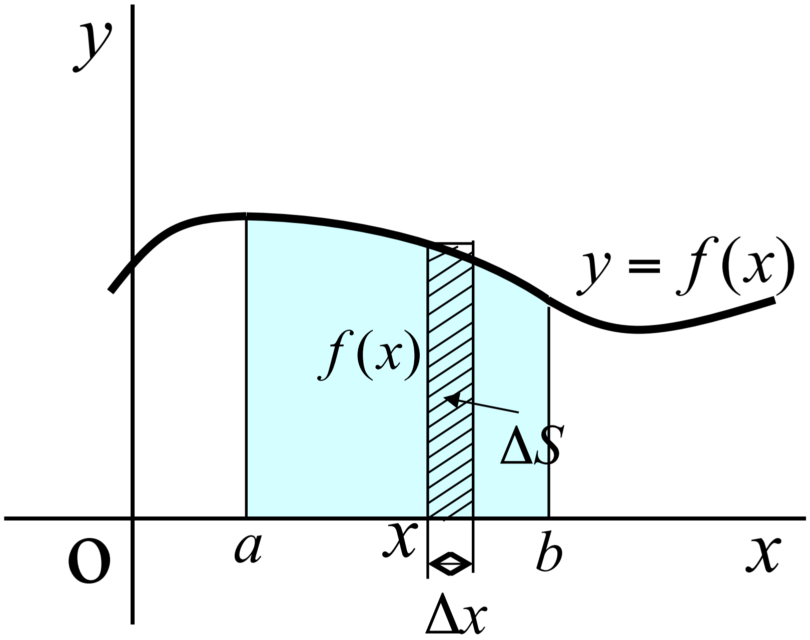
NAGOYA UNIVERSITY

Lesson 12
Application of Integrals （1）

12A

- Areas of plane regions

【Review】 Area of a Plane Region



Area of the strp $\Delta S = f(x)dx$ where $f(x) > 0$

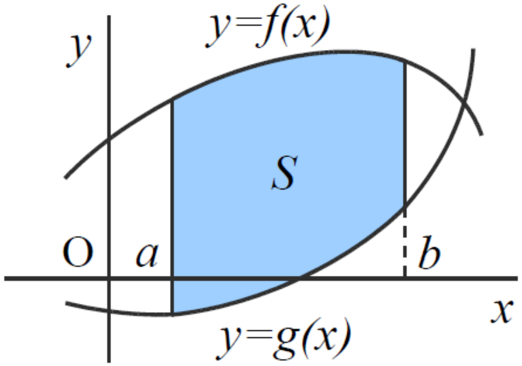
Total area over $[a,b]$

$$S \approx \sum f(x)\Delta x \quad \rightarrow \quad S = \int_a^b f(x)dx$$

Area Between Two Graphs

When $f(x) \geq g(x)$ in [a, b], area S is

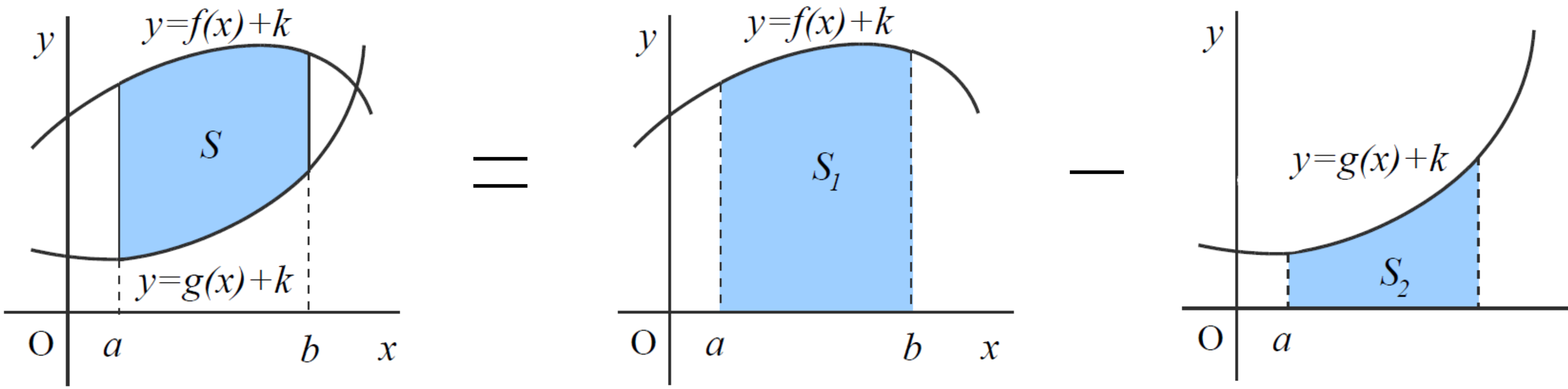
$$S = \int_a^b \{f(x) - g(x)\}dx$$



[Proof]

Consider $y = f(x) + k$ and $y = g(x) + k$

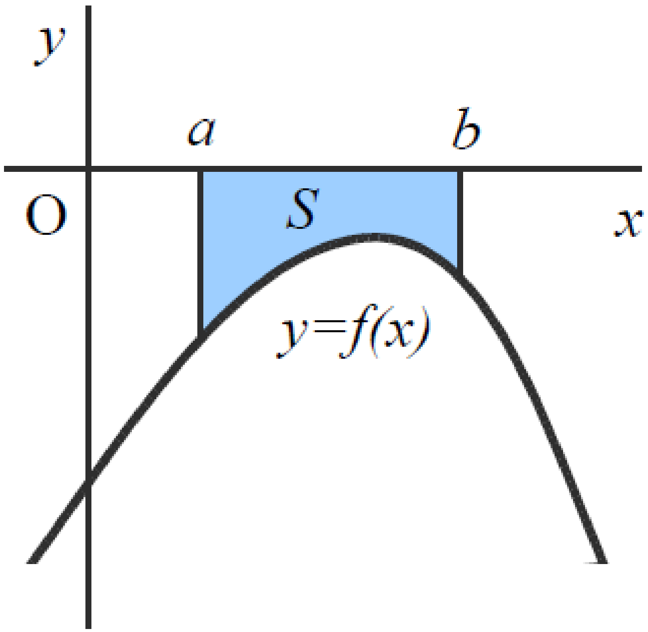
$$S = S_1 - S_2 = \int_a^b \{f(x) + k\}dx - \int_a^b \{g(x) + k\}dx = \int_a^b \{f(x) - g(x)\}dx$$



Case of a Negative Function

When $f(x) < 0$ in the domain $[a, b]$

$$S = \int_a^b \{0 - f(x)\} dx = -\int_a^b f(x) dx$$

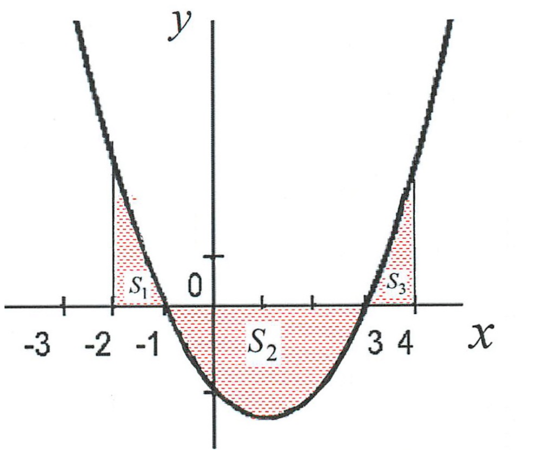


Ahh! That's so easy!

Example

Example 12-1 Find the area between the graph of $y = x^2 - x - 2$ and the x-axis, from $x = -2$ to $x = 3$.

Ans.



$$y = x^2 - x - 2 = (x-2)(x+1) = 0$$

Let the areas be S_1, S_2 and S_3

$$S_1 = \int_{-2}^{-1} (x^2 - x - 2)dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = 1.833$$

$$S_2 = -\int_{-1}^{2} (x^2 - x - 2)dx = 4.5$$

$$S_3 = -\int_{2}^{3} (x^2 - x - 2)dx = 1.833$$

Therefore, $S = S_1 + S_2 + S_3 \approx 1.833 + 4.5 + 1.833 = 8.166$

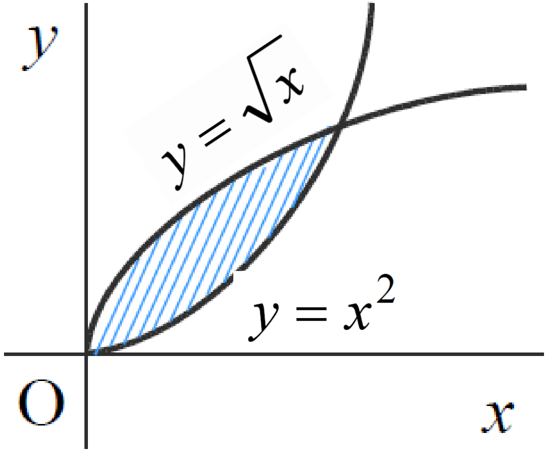
Example

Example 12-2 Determine the area of the region enclosed by the functions

and $y = \sqrt{x}$ $y = x^2$

Ans.

The area is given by the blue region in the figure.
Cross points :



$$\sqrt{x} = x^2$$

$$x = x^4, \quad \therefore \quad x(x-1)(x^2+x+1) = 0$$

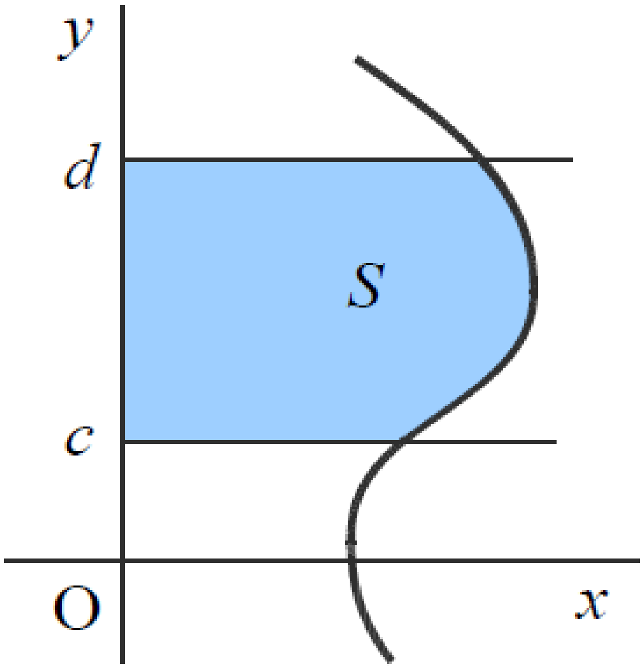
$$(0,0), \quad (1,1)$$

$$S = \int_0^1 (\sqrt{x} - x^2)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{3}$$

Area Between the graph and the y-Axis

The area between the graph $x = g(y)$ and the y-axis

$$S = \int_c^d g(y)dy$$



Exercise

Ex.12-1 Find the area of the region enclosed by $y = \sin 2x$ and $y = \cos x$ in the domain $0 \le x \le \dfrac{\pi}{2}$.

Ans.

Pause the video and solve by yourself.

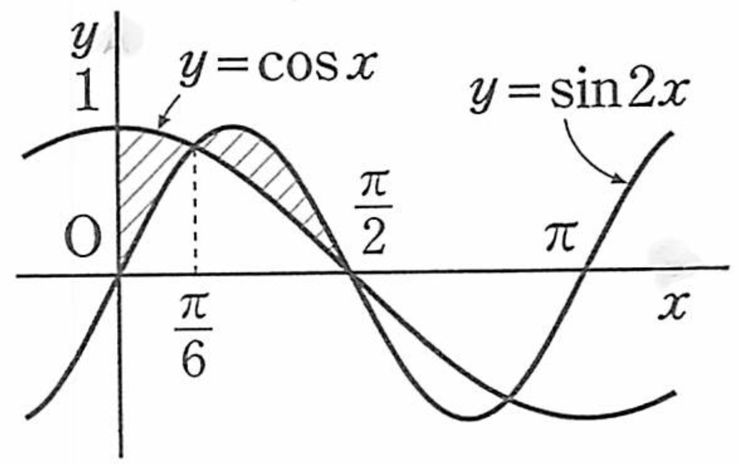
Ex.12-1 Find the area of the region enclosed by $y = \sin 2x$ and $y = \cos x$ in the domain $0 \le x \le \frac{\pi}{2}$.

Ans. We put $\sin 2x = \cos x$

$\therefore\ \cos x(2\sin x - 1) = 0,$

$\therefore\ \cos x = 0$ or $\sin x = \frac{1}{2}$

$\therefore\ x = \frac{\pi}{2}, \frac{\pi}{6}$



$$S = \int_0^{\frac{\pi}{6}} (\cos x - \sin 2x)dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)dx$$

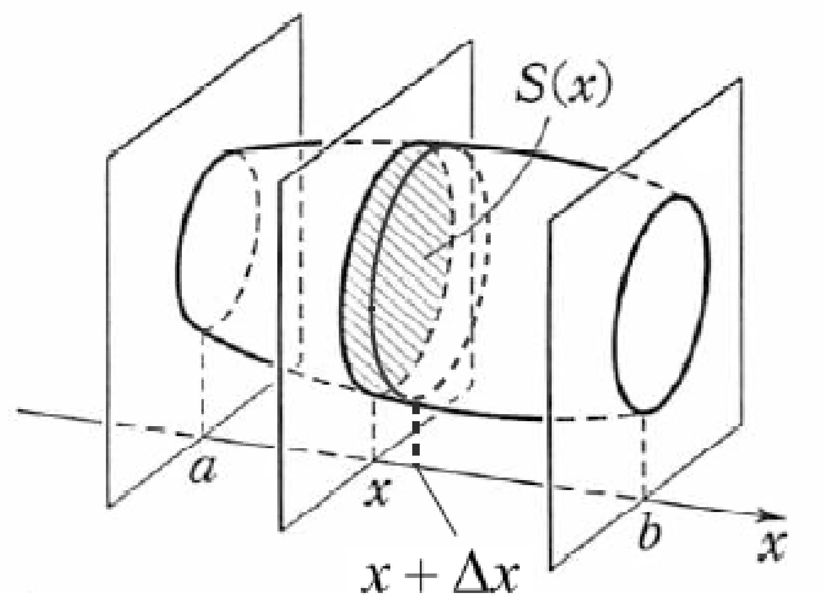
$$= \left[\sin x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}$$

Lesson 12
Application of Integrals (1)

12B

- Volumes of Solids

Volumes of Solids



Volume of the slab $\Delta V \approx S(x)\Delta x$

Total volume of the solid between x=a and x=b

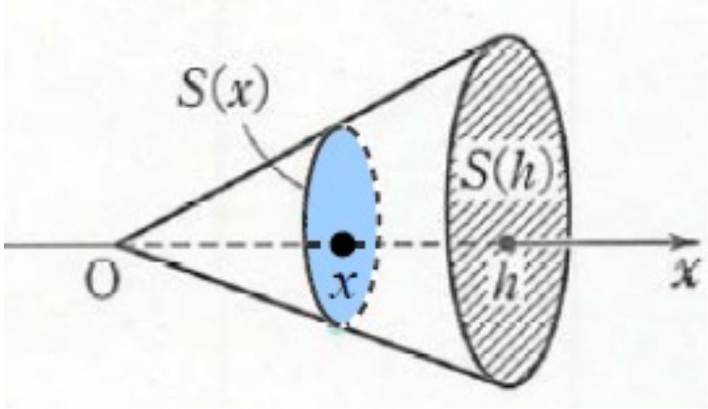
$$V \approx \sum S(x)\Delta x \;\rightarrow\; \boxed{V = \int_a^b S(x)\,dx}$$

Example

[Example 12-2]

Find the volume of a cone with bottom radius r and height h.

Ans.



Set the x-axis as shown in the figure.

Area of the bottom $S(h) = \pi r^2$

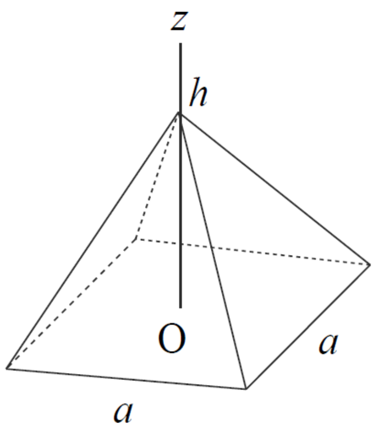
From $S(x) : S(h) = x^2 : h^2$, we have $S(x) = \dfrac{\pi r^2}{h^2} x^2$

Therefore

$$V = \int_0^h \left(\frac{\pi r^2}{h^2} x^2 \right) dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \frac{\pi}{3} r^2 h$$

Exercise

Ex.12-2 Find the volume of the pyramid having a horizontal square cross section. The bottom side length is a and the height is h.



Ans.

Pause the video and solve by yourself.

Exercise

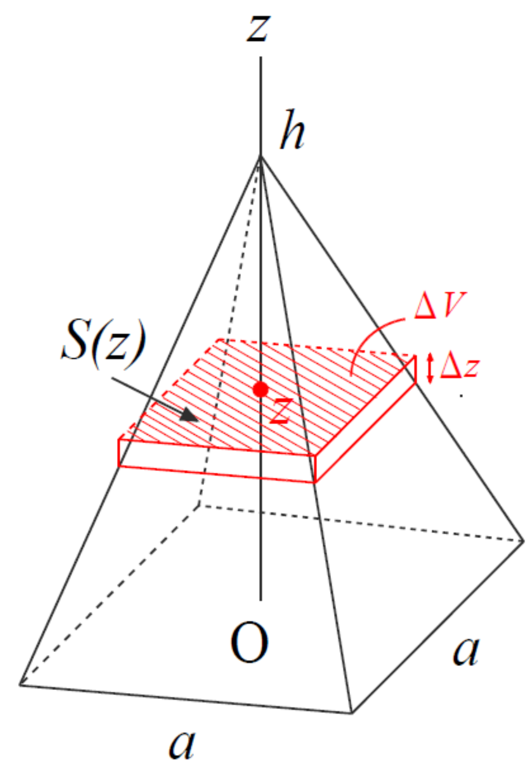
Ex.12-2 Find the volume of the pyramid having a horizontal square cross section. The bottom side length is a and the height is .

Ans. We consider the z-axis vertically.

The horizontal cross section at z :

$$S(z) : a^2 = (h-z)^2 : h^2$$

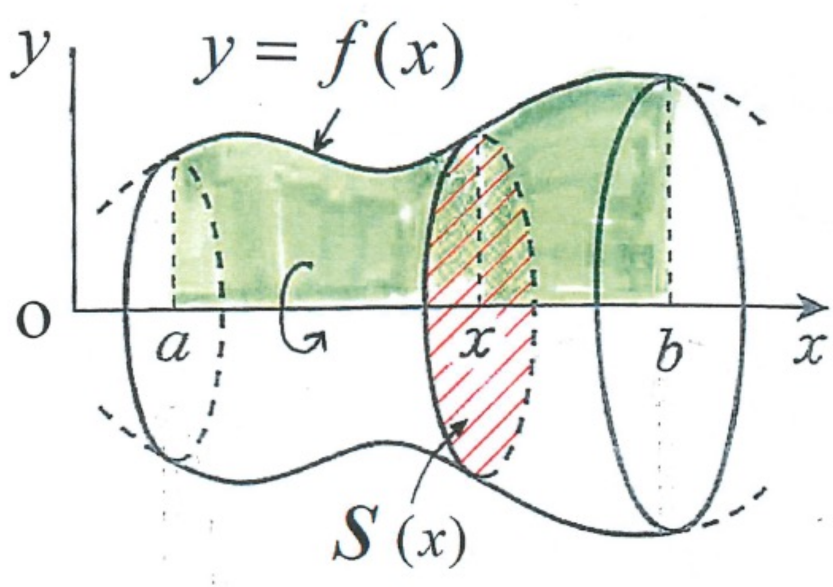
$$\therefore \quad S(z) = \frac{a^2(h-z)^2}{h^2}$$



$$V = \int_0^h \frac{a^2(h-z)^2}{h^2} dz = \frac{a^2}{h^2}\int_0^h (h^2 - 2hz + z^2$$

$$= \frac{a^2}{h^2}\left[h^2 z - hz^2 + \frac{1}{3}z^3\right]_0^h = \frac{1}{3}a^2 h$$

Volumes of Solids of Revolution



When the solid is generated by revolving a region about the x-axis

$$V = \int_a^b S(x)dx = \int_a^b \pi r^2 dx = \int_a^b \pi f(x)^2 dx$$

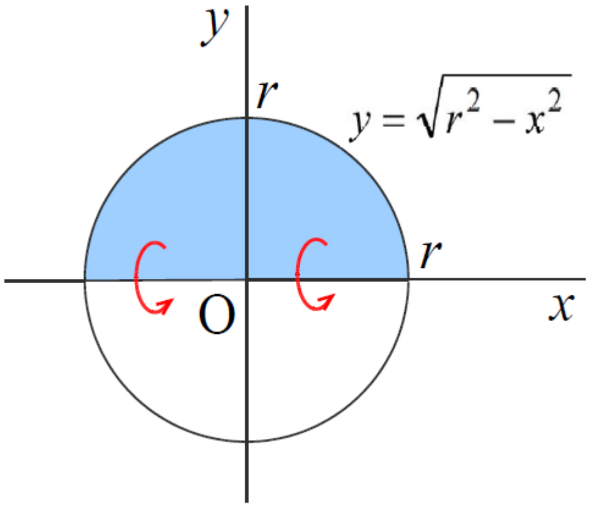
Example

[Examples 12-3]
Find the volume of a sphere with radius r.

Ans.

The upper half of the circle .

$$y = \sqrt{r^2 - x^2}$$



By rotating this blue area, we have a circle.

Volume of a circle

$$V = \int_{-r}^{r} \pi y^2 dx = \int_{-r}^{r} \pi (r^2 - x^2) dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4}{3}\pi r^3$$

That makes sense!

Exercise

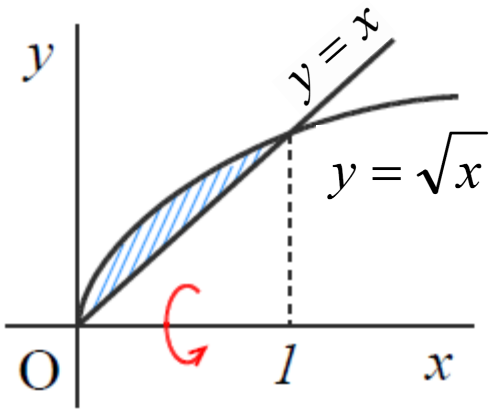
Ex.12-3 Find the volume of the solid made by rotating the region surrounded by $f(x) = \sqrt{x}$ and y=x.

Ans.

Pause the video and solve by yourself.

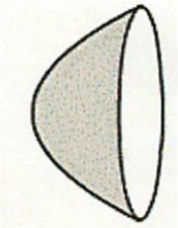
Exercise

Ex.12-3 Find the volume of the solid made by rotating the region surrounded by $y = \sqrt{x}$ and y=x.

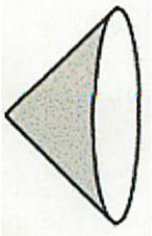


Ans.

This volume can be obtained by subtracting B from A, where .

A 

$$V_A = \pi \int_0^1 \left(\sqrt{x}\right)^2 dx = \pi \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}\pi$$

B 

$$V_B = \pi \int_0^1 (x)^2 dx = \pi \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\pi$$

$$V = V_B - V_A = \frac{1}{3}\pi - \frac{1}{2}\pi = \frac{1}{6}\pi$$