NAGOYA UNIVERSITY

Lesson 8
Approximation of a Function

8A

- Linear Approximation

Linear Approximation

Definition of the Derivative

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

When $|h|$ is small

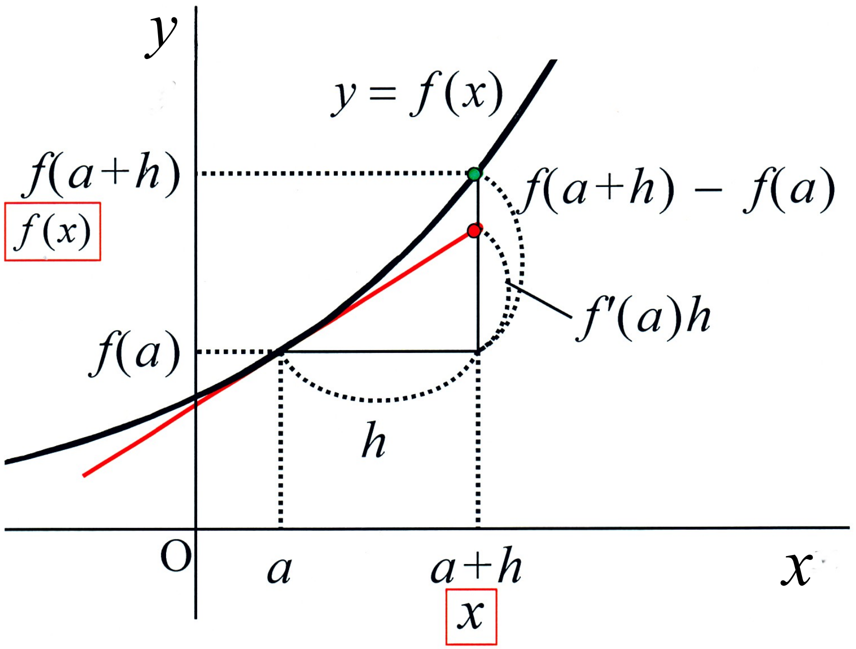
$$\frac{f(a+h)-f(a)}{h}\approx f'(a)$$

Rearrange

$$f(a+h)\approx f(a)+f'(a)h$$

If we put $a+h=x$

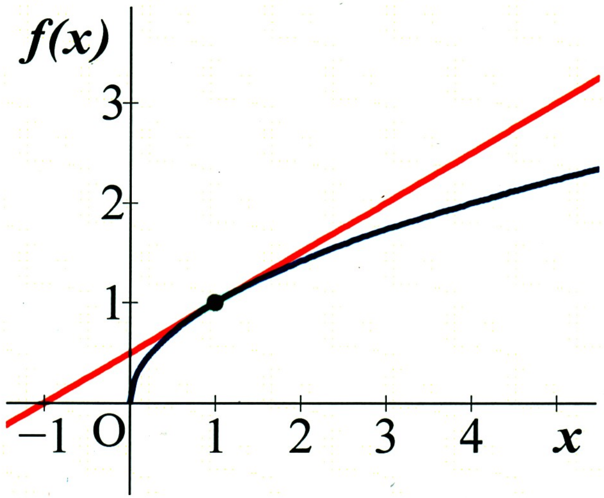
$$\boxed{f(x)\approx f(a)+f'(a)(x-a)}$$



Example

Example 8-1 Derive the approximate expression of $f(x)=\sqrt{x}$ in the neighborhood of $a=1$.

Ans. $f'(x)=\dfrac{1}{2}x^{-\frac{1}{2}}$



At $a=1$

$$f(1)=\sqrt{1}=1 \text{ and } f'(1)=\frac{1}{2}$$

Therefore

$$\sqrt{x}\approx f(1)+f'(1)(x-1)=1+\frac{1}{2}(x-1)=\frac{1}{2}x+\frac{1}{2}$$

For example

$x=1.1$ Calculator $\sqrt{1.1}=1.0488...$

This approximation $\sqrt{1.1}=\dfrac{1}{2}\times 1.1+\dfrac{1}{2}=1.05$

} Error 1.1%

Example

Example 8-2 (1) When h is small, find the linear approximation of $\sin(a+h)$.

(2) Find the approximate value of $\sin 31°$.

Ans. (1) $f(x)=\sin x$ $\quad \therefore \; f'(x)=\cos x$

$$\sin(a+h)=f(a+h)\approx f(a)+f'(a)h=\sin a+h\cos a$$

(2)

$$31°=\frac{\pi}{6}+\frac{\pi}{180}$$ We put $h=\frac{\pi}{180}$

$$\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)\approx\sin\frac{\pi}{6}+\frac{\pi}{180}\cos\frac{\pi}{6}=\frac{1}{2}+\frac{\pi}{180}\cdot\frac{\sqrt{3}}{2}$$

Using $\pi=3.142,\ \sqrt{3}=1.732$

We have $$\sin 31°\approx\frac{1}{2}+\frac{3.142}{180}\cdot\frac{1.732}{2}\approx 0.5151$$

Case that $|x|$ is small

$a + h = x$ $$f(x) \approx f(a) + f'(a)(x-a)$$

When x is small We put $a = 0$.

$$\boxed{f(x) \approx f(0) + f'(0)x}$$

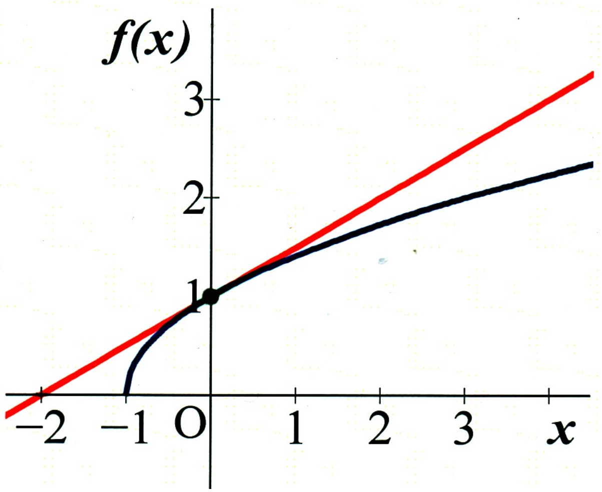
Examples 8-3 When $|x|$ is small enough, derive the approximate expression of $\sqrt{1+x}$

Ans.

$$\left(\sqrt{1+x}\right)' = \frac{1}{2\sqrt{1+x}}$$

Therefore

$$f(x) \approx f(0) + f'(0)x = 1 + \frac{1}{2}x$$



Exercise

Ex.8-1 Find the approximate value of $\cos 29°$ in the following two ways.

(1) Use the formula $f(x) = f(a+h) \approx f(a) + f'(a)h$

(2) Use the formula $f(x) \approx f(0) + f'(0)x$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

Ex.8-1 Find the approximate value of $\cos 29°$ in the following two ways.

(1) Use the formula $f(x) = f(a+h) \approx f(a) + f'(a)h$

(2) Use the formula $f(x) \approx f(0) + f'(0)x$

Ans. (1) We put $f(x) = \cos x$

Then $f'(x) = -\sin x$

$$\cos 29° = \cos(\frac{\pi}{6} - \frac{\pi}{180}) \approx \cos(\frac{\pi}{6}) - \frac{\pi}{180}\sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{180} \cdot \frac{1}{2}$$

$$\approx \frac{1.732}{2} - \frac{3.142}{180} \cdot \frac{1}{2} = 0.875$$

(2) We put $f(x) = \cos\left(\frac{\pi}{6} + x\right)$ Then $f'(x) = -\sin\left(\frac{\pi}{6} + x\right)$

$$\cos 29° = f(-\frac{\pi}{180}) \approx \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{180} \cdot \frac{1}{2} \approx \frac{1.732}{2} + \frac{3.142}{180} \cdot \frac{1}{2} = 0.875$$

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Lesson 8
Application of a Function

8B

- Tayler Expansion

Power Series Expansion

Suppose that $f(x)$ is expanded in power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

By differentiating term by term, we have

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \ldots$$

$$f''(x) = 2a_2 + 2\cdot 3a_3(x-c) + 3\cdot 4a_4(x-c)^2 + \ldots$$

$$\ldots\ldots$$

Setting $x = c$, we have

$$f(c) = a_0,\quad f'(c) = a_1,\quad f''(c) = 2a_2,\ \ldots\ldots\quad f^{(k)}(c) = k!a_k,\ \ldots\ldots$$

Namely, the coefficients are given by

$$a_k = \frac{f^{(k)}(c)}{k!}$$

Taylor series

For a function $f(x)$ that has continuous derivatives in the neighborhood of c, the Taylor series expansion of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

For $c = 0$

Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Example

Example 8-3 Derive the approximate expression for $\sin x$ near $x = 0$ up to the seventh order.

Ans. We put $f(x) = \sin x$

Then $f'(x) = \cos x \quad f''(x) = -\sin x$

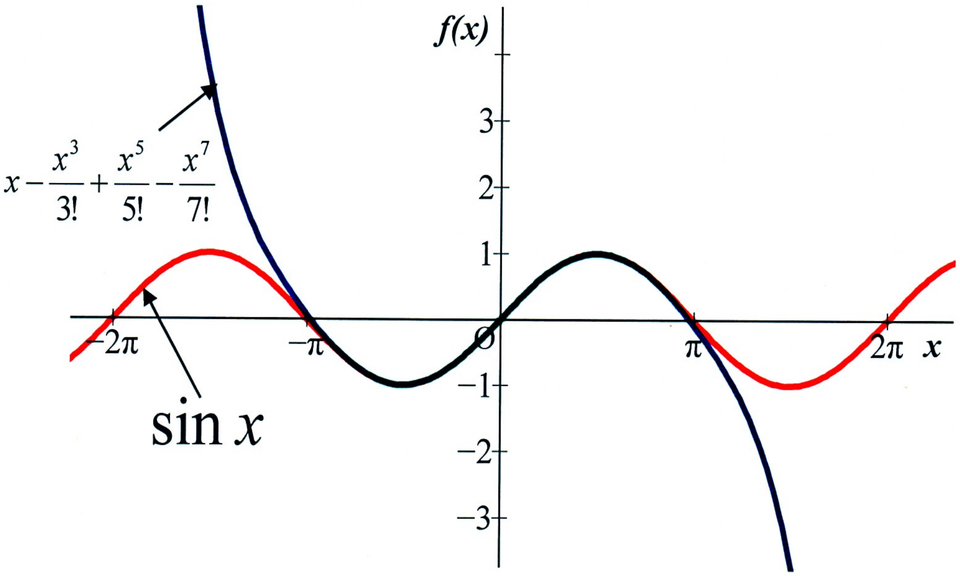
In general $f^{(2n)}(x) = (-1)^n \sin x, \quad f^{(2n+1)}(x) = (-1)^n \cos x$

For $x = 0$: $f^{(2\mathrm{n})}(0) = 0$

$f^{(2\mathrm{n}+1)}(0) = (-1)^n$

Using these values

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$



Exercise

Ex.8-2 Find Maclaurin series for $f(x) = e^x$

Ans.

Pause the video and solve the problem by yourself.

Exercise

Ex.8-2 Find Maclaurin series for $f(x) = e^x$

Ans.

The n -th derivative is $f^{(n)}(x) = e^x$ for all n .

Thus, these values for $x = 0$ are

$$f(0) = f'(0) = f''(0) = \ldots = e^0 = 1$$

The coefficients are

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$$

Therefore

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$