Course II

NAGOYA UNIVERSITY

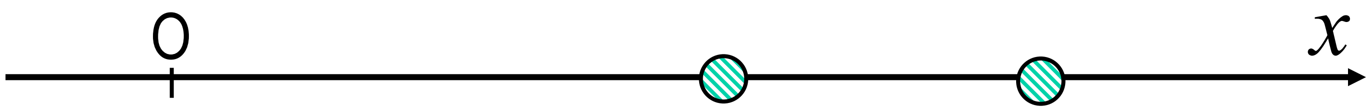
Lesson 7
Applications to Physics

7A

- Velocity and Acceleration of a Particle

Motion in a Straight Line : Velocity

Motion in the x-axis



$$x = 0 \qquad x = x_1 \qquad x = x_2 = x_1 + \Delta x$$

$$t = 0 \qquad t = t_1 \qquad t = t_2 = t_1 + \Delta t$$

Average velocity

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \qquad [\text{ m/s }]$$

Instantaneous velocity

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad [\text{ m/s }]$$

Average acceleration

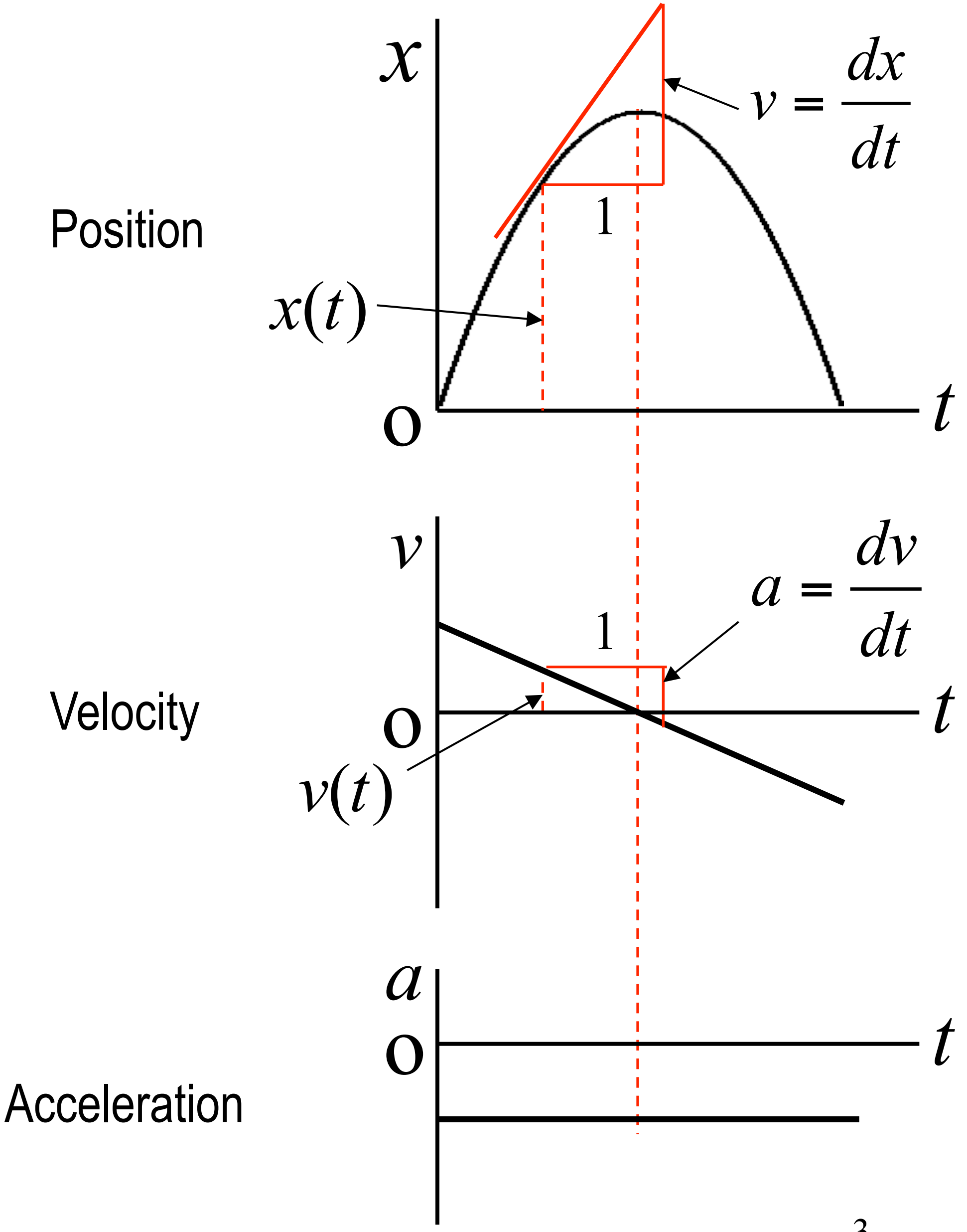
$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

[m/s^2]

Instantaneous velocity

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

[m/s^2]

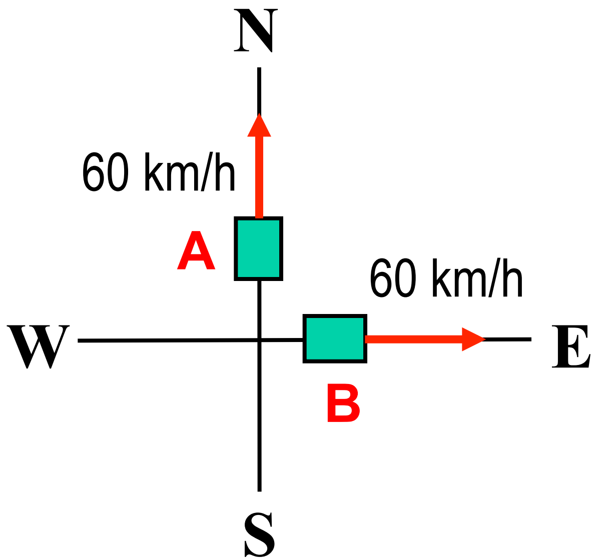


Vector and Scalar

Vector --- A quantity that is described by both a magnitude and a direction.

Scalar --- A quantity that is described by a magnitude alone.

Velocity and Speed



Velocity --- A vector quantity that refer to how fast and in what direction an object is moving.

Speed --- A scalar quantity that refers to how fast an object is moving

Car A and car B have same speed but different velocity

"Velocity is speed with a direction."

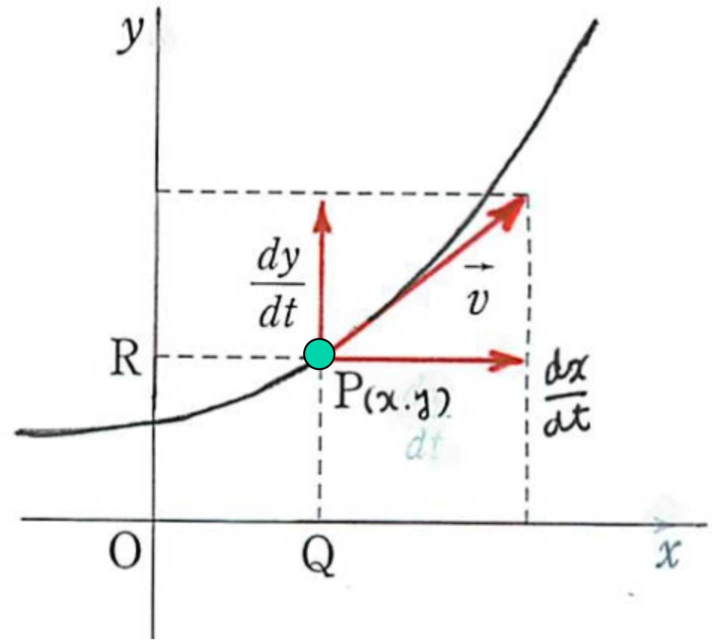
Velocity and Speed

Velocity of point Q : $\dfrac{dx}{dt}$,

Velocity of point R : $\dfrac{dy}{dt}$

Velocity of point P: $\vec{v} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$

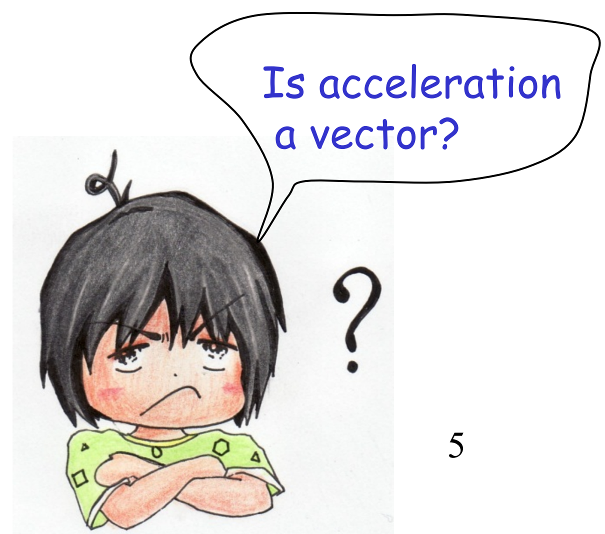
Speed of point P: $v = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$



Acceleration

Acceleration of point P: $\vec{a} = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}\right)$

Magnitude of acceleration $a = \sqrt{\left(\dfrac{d^2x}{dt^2}\right)^2 + \left(\dfrac{d^2y}{dt^2}\right)^2}$



Example : Case of Constant Acceleration

[Examples 7-1] A ball is tossed straight up with an initial speed of 25m/s. Take the y-axis upward with the origin at the ball's release point. If the time t is counted from the instant when the ball is released, the position of the ball is given by

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2$$

where $y_0 = 0$ m, $v_0 = 25$ m/s and $g = 9.8$ m/s². Determine the time when the ball reaches its highest position and what is its height ?

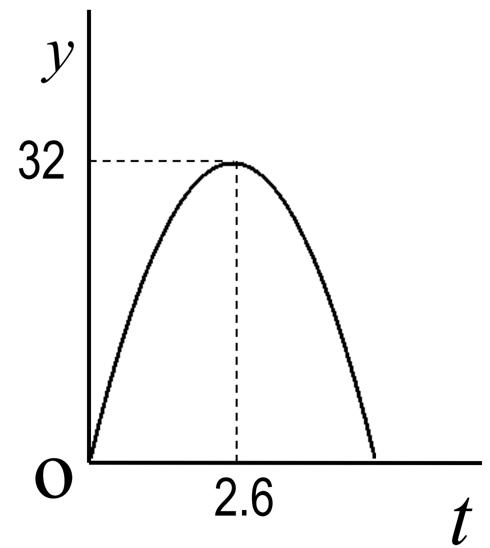
Ans.

The velocity $v = \frac{dy}{dt} = v_0 - gt$

At the top, we have $v_0 - g t_1 = 0$

$$\therefore \; t_1 = \frac{v_0}{g} = \frac{25\text{m/s}}{9.8\text{m/s}^2} = 2.6\text{s}$$

$$y(t_1) = 0 + 25 \times 2.6 - \frac{1}{2} \times 9.8 \times 2.6^2 = 32 \text{ m}$$



Exercises

[Ex.7-1] The position of a particle moving in the x-axis is given by

$$x(t) = t^3 - 6t^2 + 9t + 2$$

(1) Find the velocity and the acceleration at $t = 2$.

(2) Investigate the motion during $0 \le t \le 4$.

Ans.

Pause the video and solve the problem by yourself.

[Ex.7-1] The position of a particle moving in the x-axis is given by

$$x(t) = t^3 - 6t^2 + 9t + 2$$

(1) Find the velocity and the acceleration at $t = 2$.

(2) Investigate the motion during $0 \le t \le 4$.

Ans.

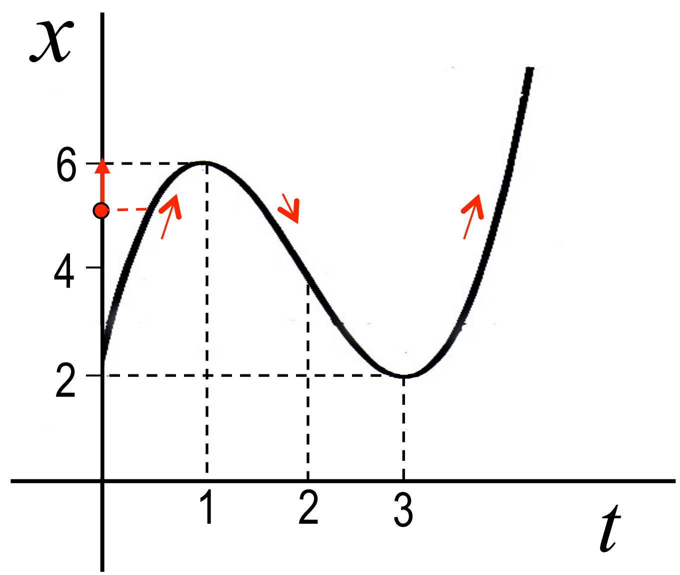
(1) $$v(t) = \frac{dx}{dt} = 3t^2 - 12t + 9 = 3(x-1)(x-3)$$

$$a(t) = \frac{dv}{dt} = 6t - 12$$

Therefore, $v(2) = -3$, $a(2) = 0$

(2)

t	0	…	1	…	3	…	4	4
v	+	+	0	−	0		+	+
x	2	↗	6	↘	2	↗	6	6



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Lesson 7
Applications to Physics

7B

- Related Rates

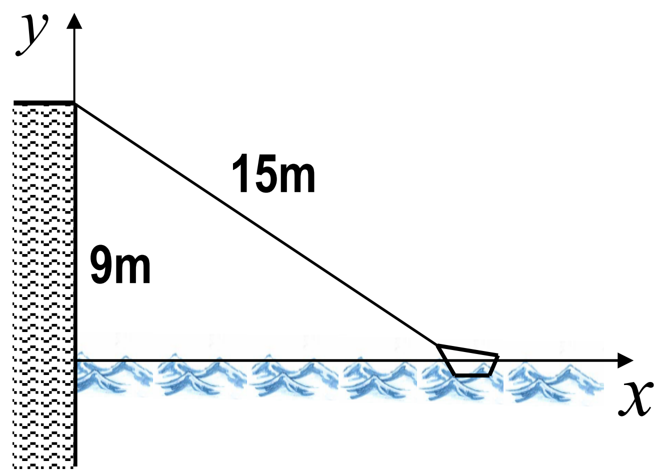
Example

[Example 7-2] A man is pulling a boat from the top of quay using a rope. The height of a man is 9 m from the water surface and the rate of change of the rope length is 2 m/s. What is the speed of the boat when the rope length is 15m.

Ans.

Rope length l [m]

Distance between the quay and the boat u [m]



$$u^2 + 9^2 = l^2$$

Differentiate by t

$$2u\frac{du}{dt} = 2l\frac{dl}{dt}$$

Conditions

$$\frac{dl}{dt} = -2 \text{ m/s}, \quad l = 15 \text{ m}, \quad u = \sqrt{15^2 - 9^2} = 12$$

Then

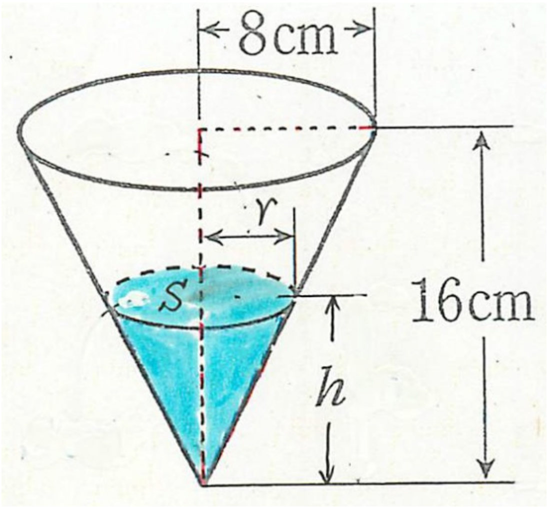
$$12 \times \frac{du}{dt} = 15 \times (-2) \qquad \therefore \frac{du}{dt} = -2.5 \text{ m/s}$$

Example

[Example 7-3] Water is poured into an inverted circular cone of base radius 8 cm and height 16 cm at the rate 3 cm³. What is the rate of increase of the height of the water level when the depth is 6cm ?

Ans.

Volume of the water $V = \frac{1}{3}\pi r^2 h$



Ratio $8 : r = 16 : h$, Therefore $r = \frac{1}{2}h$

$$V = \frac{1}{12}\pi h^3$$

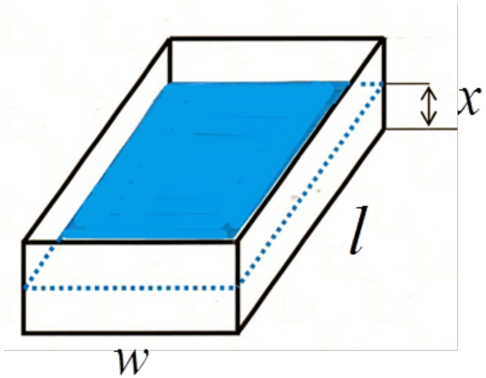
Differentiation by t : $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

Condition : $\frac{dV}{dt} = 3$ cm³/s , $h = 6$ cm

Substitution

$$3 = \frac{1}{4}\pi \times 6^2 \times \frac{dh}{dt}, \quad \therefore \frac{dh}{dt} = \frac{1}{3\pi} \text{ cm/s}$$

[Ex.7-2] A rectangular water tank is being filled at the constant rate of 20 liters/s. The dimensions of the tank are $w = 1$ m and $l = 2$ m. What is the rate of change of the height x of water in the tank ?

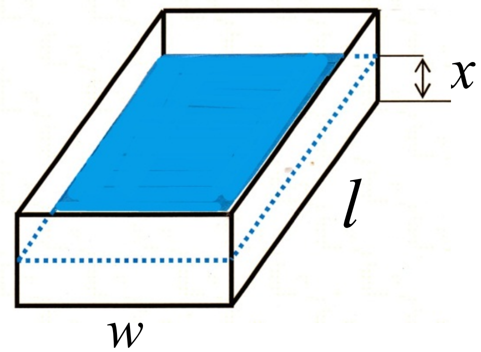


Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercises

[Ex.7-2] A rectsngulsr water tank is being filled at the constant rate of 20 liters/s. The dimensions of the tank are $w = 1$ m and and $l = 2$ m. What is the rate of change of the height x of water in the tank ?



Ans.

$$V(t) = wlx(t)$$

$$\frac{dV}{dt} = wl\frac{dx}{dt} = \frac{1}{wl}\left(\frac{dV}{dt}\right) = \frac{20000}{100\cdot 200} = 1$$

Differentiation by t :

$$\frac{dV}{dt} = wl\frac{dx}{dt}$$

Condition : $\frac{dV}{dt} = 20\text{liter/s} = 20000$ cm³/s

By substitution, we have

$$\frac{dx}{dt} = \frac{1}{wl}\left(\frac{dV}{dt}\right) = \frac{20000}{100\cdot 200} = 1 \quad \text{cm/s}$$