NAGOYA UNIVERSITY

Lesson 5
Derivatives of Logarithmic Functions and Exponential Functions

5A

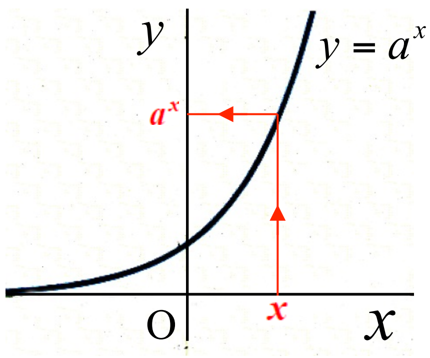
- **Derivative of logarithmic functions**

Review of the Logarithmic Function

Exponential function

$$y = a^x \quad (a > 0,\ \ a \neq 1)$$

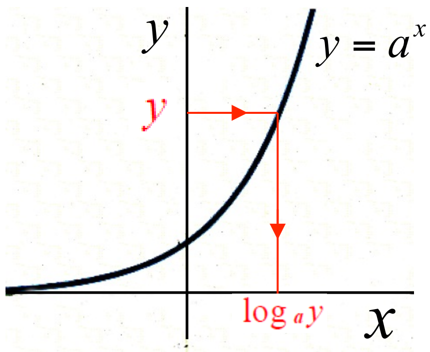
Fig.1



Logarithmic function

$$y = a^x$$

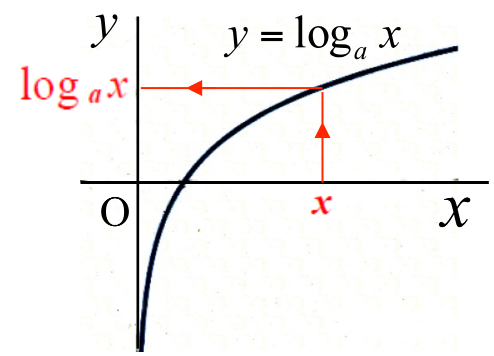
Fig.2



We replace the notation

$$x = a^y \Longleftrightarrow y = \log_a x$$

Fig.3



Derivative of the Logarithmic Function

From the definition

$$(\log_a x)' = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\left\{\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right\}$$

We put $\dfrac{h}{x} = k$. When $h \to 0,\ k \to 0.$

$$(\log_a x)' = \lim_{k\to 0}\left\{\frac{1}{x}\cdot\frac{1}{k}\log_a(1+k)\right\}$$

?

$$= \frac{1}{x}\lim_{k\to 0}\log_a(1+k)^{\frac{1}{k}} = \frac{1}{x}\log_a\left[\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right]$$

Napier's Constant

Trial

k	$(1+k)^{\frac{1}{k}}$	k	$(1+k)^{\frac{1}{k}}$
0.1	2.59374······	−0.1	2.86797······
0.01	2.70481······	−0.01	2.73199······
0.001	2.71692······	−0.001	2.71964······
0.0001	2.71814······	−0.0001	2.71841······
0.00001	2.71826······	−0.00001	2.71829······

We expect that $(1+k)^{\frac{1}{k}}$ approaches one value as $k \longrightarrow 0$

Napier's Constant

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = 2.718281828459045\cdots\cdots$$

Important Mathematical Constants

"π=3.1415…" was known 4000 years ago

"e=2.7182…" was found in 17th century

Natural Logarithm

Then

$$(\log_a x)' = \frac{1}{x}\log_a e = \frac{1}{x\log_e a}$$

If the base is e, we have

$$(\log_e x)' = \frac{1}{x}\log_e e = \frac{1}{x}$$

Natural logarithm is the logarithm to the base e.

Notation: $\log_e x \to \ln x$

Summary

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

[Example 5.1] Find the derivative of the following functions.

$$(1)\quad y=\ln 2x,\qquad (2)\quad y=\log_2(3x+2),\qquad (3)\quad y=x\ln 3x$$

Ans.

$$(1)\quad y'=\frac{1}{2x}\cdot(2x)'=\frac{2}{2x}=\frac{1}{x}$$

$$(2)\quad y'=\frac{1}{(3x+2)\ln 2}\cdot(3x+2)'=\frac{3}{(3x+2)\ln 2}$$

$$(3)\quad y'=x'\ln 3x+x(\ln 3x)'=\ln 3x+x\cdot\frac{3}{3x}=\ln 3x+1$$

Examples

[Example 5.2] Calculate the money which you can receive one year later using various compound systems. The principal is 10000 yen. (1) Annual interest is 100%. (2) Half a year interest is 50%, (3) Monthly interest is 100/12%, (4) Daily interest is 100/360%.

Ans.

(1) $10000 \times (1+1)^1 = 20{,}000$ yen

(2) $10000 \times (1+1/2)^2 = 22{,}500$ yen

(3) $10000 \times (1+1/12)^{12} = 26{,}130$ yen

(4) $10000 \times (1+1/365)^{365} = 27{,}148$ yen

[Note] Napier's Constant

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7182$$

[Ex.5.1] Find the derivatives of the following functions.

(1) $y = (\ln x)^2$, (2) $y = x \ln x$, (3) $y = \log_{10} x$

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.5.1] Find the derivatives of (1) $y = (\ln x)^2$ and (2) $y = \ln(x^3 + 1)$

Ans.

(1) $$\frac{d}{dx}(\ln x)^2 = 2\ln x \cdot \frac{d}{dx}\ln x = \frac{2\ln x}{x}$$

(2) $$\frac{d}{dx}(x\ln x) = (x)' \cdot \ln x + x \cdot (\ln x)' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

(3) $$\log_{10} x = \frac{\ln x}{\ln 10}$$

$$\frac{d}{dx}\log_{10} x = \frac{d}{dx}\frac{\ln x}{\ln 10} = \frac{1}{\ln 10}\frac{d}{dx}\ln x = \frac{1}{(\ln 10)x}$$

Lesson 5
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5B

- **Derivative of exponential functions**

Derivative of Inverse Functions

Let f and g be inverse functions. Then

$$y = f(x) \longleftrightarrow x = g(y)$$

Differentiate both sides of (1) by y and from the chain rule, we have

$$1 = \frac{df(x)}{dy} = \frac{df(x)}{dx}\frac{dx}{dy} = \frac{df(x)}{dx}\frac{dg(y)}{dy}$$

$$1 = \frac{df(x)}{dx}\frac{dg(y)}{dy}$$

Therefore

$$\frac{df(x)}{dx} = \frac{1}{\left(\dfrac{dg(y)}{dy}\right)}$$

or

$$\boxed{\frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}}$$

Derivative of the Exponential Function

Exponential function of base e

$$y = f(x) = e^x \longleftrightarrow x = g(y) = \ln y$$

Therefore, from the previous slide we have

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{1}{\left(\frac{dg(y)}{dy}\right)} = \frac{1}{\left(\frac{1}{y}\right)} = y$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Case $y = a^x$

If $a = e^x$, then $x = \ln a$. Therefore

$$y = a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

From the Chain Rule

$$\frac{d}{dt}\left(a^x\right) = \frac{d}{dt}\left(e^{x \ln a}\right) = e^{x \ln a}\frac{d}{dt}(x \ln a) = a^x \ln a$$

$$\boxed{\frac{d}{dt}\left(a^x\right) = a^x \ln a}$$

[Example 5.3] Find the derivative of the following functions.

$$(1)\ y = e^{2x}, \qquad (2)\quad y = a^{-2x}$$

Ans.

(1) Chain rule

$$y' = e^{2x} \cdot (2x)' = 2e^{2x}$$

(2) Chain rule

$$y' = \left(a^{-2x} \log a\right) \cdot (-2x)' = -2a^{-2x} \log a$$

Exercise

[Ex.5.2] Find the derivatives of the following functions.

(1) $y = x a^x$ (2) $y = 2^{\ln x}$ (3) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.5.2] Find the derivatives of the following functions.

(1) $y = x a^x$ (2) $y = 2^{\ln x}$ (3) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(1) Product rule

$$y' = (x)' a^x + x(a^x)' = a^x + x \cdot a^x \log a = a^x (1 + x \log a)$$

(2) Chain rule

$$y' = 2^{\ln x} \ln 2 \cdot (\frac{1}{x}) = \frac{2^{\ln x} \ln 2}{x}$$

(3) Quotient rule

$$y' = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$