Course II

NAGOYA UNIVERSITY

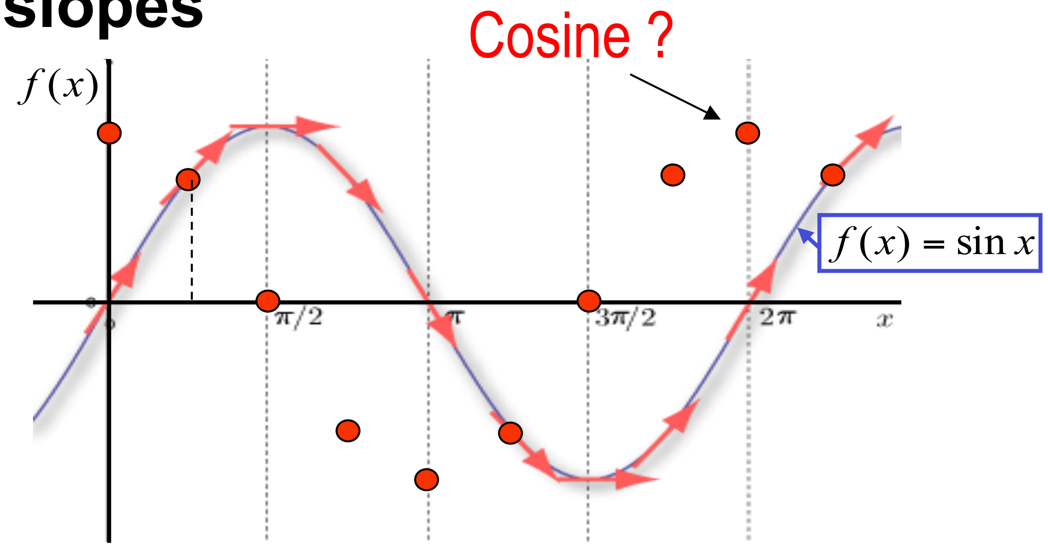
Lesson 4
Derivatives of Trigonometric Functions

4A

- Derivative of Sine Function
- Limit of $\dfrac{\sin x}{x}$
- Derivatives of Basic Trigonometric Function

Derivative of Sine Function

Variation of slopes



Derivative by definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

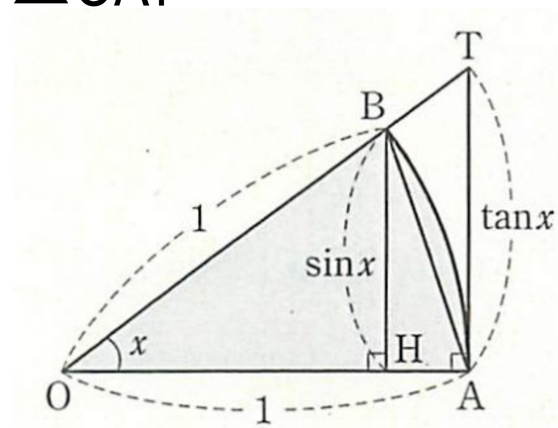
$$= \lim_{h \to 0} \left(\cos x \frac{\sin h}{h} - \sin x \frac{1 - \cos h}{h} \right)$$

Limit of $\sin x / x$

Consider a sector with central angle x

Compare the areas of △OAB, sector OAB, and △OAT

$$\frac{1}{2}\cdot 1\cdot \sin x < (\pi \cdot 1^2)\cdot \frac{x}{2\pi} < \frac{1}{2}\cdot 1\cdot \tan x$$

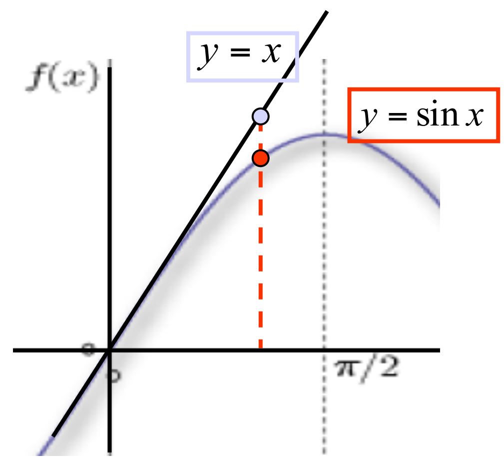


$$\therefore \quad \sin x < x < \tan x$$

Divide by $\sin x (> 0)$

$$\therefore \quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\therefore \quad 1 > \frac{\sin x}{x} > \cos x$$



As $x \to 0$ $\qquad \cos x \to 1$

$$\lim_{h \to 0} \frac{\sin x}{x} = 1$$

Derivative of Sine Function—Cont.

$$f'(x) = \lim_{h\to 0}\left(\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right)$$

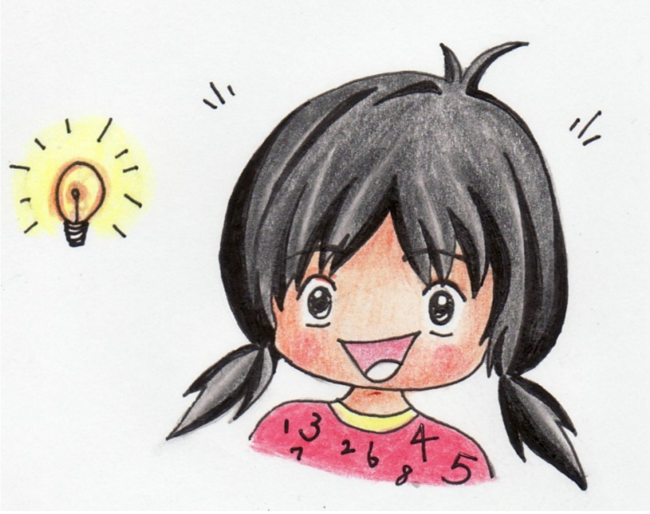
$\frac{\sin h}{h} \to 1$

$$\frac{1-\cos h}{h} = \frac{(1-\cos h)(1+\cos h)}{h(1+\cos h)} = \frac{1-\cos^2 h}{h(1+\cos h)} = \frac{\sinh}{h}\frac{\sinh}{(1+\cos h)}$$

$\frac{\sinh}{h} \to 1$, $\frac{\sinh}{(1+\cos h)} \to 0$

Therefore

$$\boxed{(\sin x)' = \cos x}$$

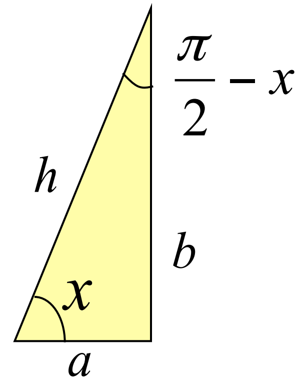


That makes sense!

[Example 4-1] Derive the derivative of $\cos x$ and $\tan x$.

Ans.

(1) From the triangle in the right side



$$\cos x = \sin\left(\frac{\pi}{2}-x\right)$$

Therefore

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)' = \cos\left(\frac{\pi}{2}-x\right)\cdot\left(\frac{\pi}{2}-x\right)'$$

$$= \sin x \times (-1) = -\sin x$$

(2) From the quotient rule

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Summary of Derivatives of Tri. Functions

(1) The basic trigonometric derivatives (Memorize!)

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x$$

(2) Other standard relationships

(Derive from (1) if necessary)

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

[note]

These formula are valid only when the angle x is measured in radians.

[Example 4.2] Find the derivatives of the following functions:

(1) $y = \cos^2 x$ (2) $y = x\sin x + \cos x$

Ans.

(1) Chain rule

$$y' = 2\cos x \cdot (\cos x)' = 2\cos x \cdot (-\sin x)$$
$$= -2\sin x \cos x = -\sin 2x$$

(2) Product rule

$$y' = (x \cdot \sin x)' + (\cos x)'$$
$$= (1 \cdot \sin x + x\cos x) - \sin x = x\cos x$$

Exercise

[Ex.4.1] Find the derivatives of the following functions:

(1) $y = \sin ax^2$ (2) $y = \dfrac{1}{\tan x}$ (3) $y = \cos\left(\dfrac{x}{2} + \dfrac{\pi}{6}\right)$

Pause the video and solve the problem by yourself.

Answer to Exercise

[Ex.4.1] Find the derivatives of the following functions:

(1) $y = \sin ax^2$ (2) $y = \dfrac{1}{\tan x}$ (3) $y = \cos\left(\dfrac{x}{2} + \dfrac{\pi}{6}\right)$

(1) $$\frac{d}{dx}\sin(ax^2) = \frac{d}{du}\sin u \frac{d}{dx}(ax^2) = \cos u \cdot (2ax) = 2ax\cos(ax^2)$$

(2) $$y' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(-\sin x)\sin x - \cos x(\cos x)}{\sin^2 x} = -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

(3) $$y' = \frac{d}{du}\cos u \frac{d}{dx}\left(\frac{x}{2} + \frac{\pi}{6}\right) = -\frac{1}{2}\sin\left(\frac{x}{2} + \frac{\pi}{6}\right)$$

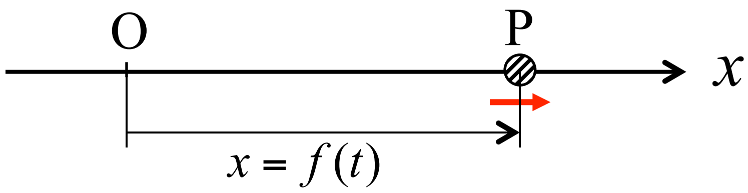
Lesson 4
Derivatives of Trigonometric Functions

4B

- Derivatives and Motions
- Position, Velocity and Acceleration
- Simple Harmonic Motion

Velocity and Acceleration

Point **P** is moving on the straight line :



Its position is given by

$$x = f(t)$$

The average velocity between t_1 and t_2

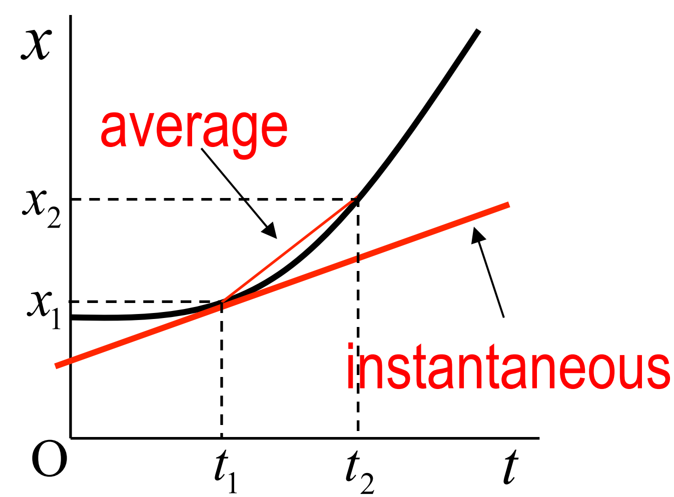
$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

The instantaneous velocity at $t = t_1$

$$v(t_1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad v = \frac{ds}{dt} = f'(t)$$

The instantaneous acceleration at $t = t_1$

$$\alpha = \frac{dv}{dt} = f''(t)$$



[Example 4-3] The position of the mass moving on the x-axis is given by $s(t) = t^3 - 3t^2 - 9t + 10$

.

(1) Find the velocity and the acceleration at $t = 2$.

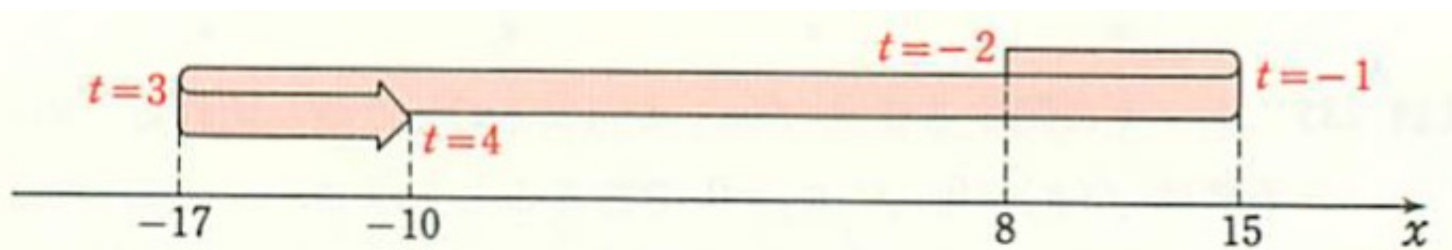
(2) Investigate the motion during $-2 \le t \le 4$

Ans. (1) Velocity : $v = \dfrac{ds}{dt} = 3t^2 - 6t - 9 = 3(t+1)(t-3)$ $\therefore\ v(2) = -9$

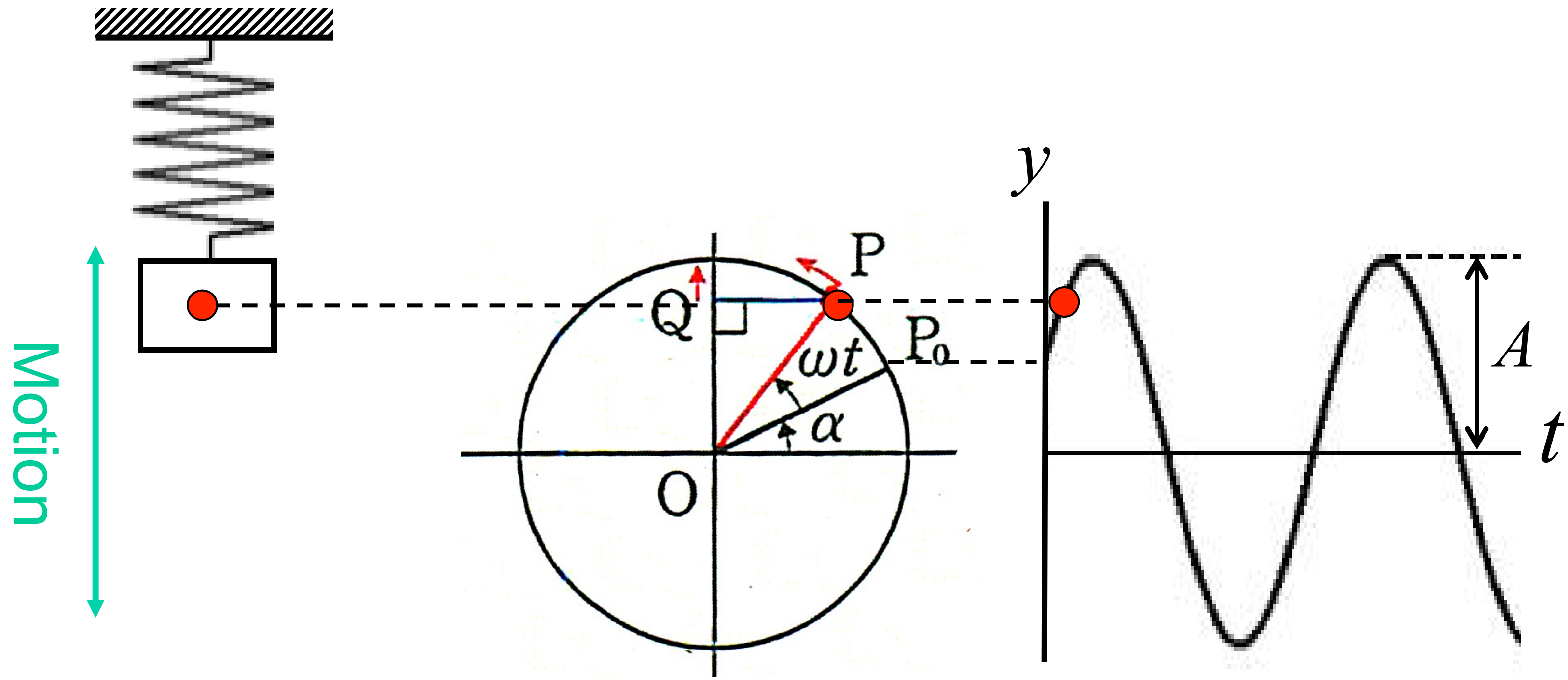
Acceleration : $a = \dfrac{dv}{dt} = 6(t-1)$ $\therefore\ a(2) = 6$

(2)

t	-2	……	-1	……	3	……	4
v		$+$	0	$-$	0	$+$	
s	8	↗	15	↘	-17	↗	-10



Simple Harmonic Motion



Vertical position : $$y = A\sin(\omega t + \alpha)$$

Velocity : $$v = \frac{dy}{dt} = A\omega\cos(\omega t + \alpha)$$

Acceleration : $$a = \frac{dv}{dt} = \frac{d^2 v}{dt^2} = -A\omega^2\sin(\omega t + \alpha)$$

Exercise

[Exercise.4.2] Point P is moving on the x-axis. Its position is given by $x = 2t + \cos t$. Find the time when the point has the maximum velocity and its maximum velocity.

Pause the video and solve the problem by yourself.

[Exercise.4.2] Point P is moving on the x-axis. Its position is given by $x = 2t + \cos t$. Find the time when the point has the maximum velocity and its maximum velocity.

Ans.

Velocity $v = \dfrac{dx}{dt} = 2 - \sin t$

Maximum velocity occurs at

$$\sin t = -1$$

Therefore

$$t = \frac{3}{2}\pi + 2n\pi$$

Maximum velocity is 3.