Course I

Nagoya University

Lesson 15
Graphs and Equations (II)

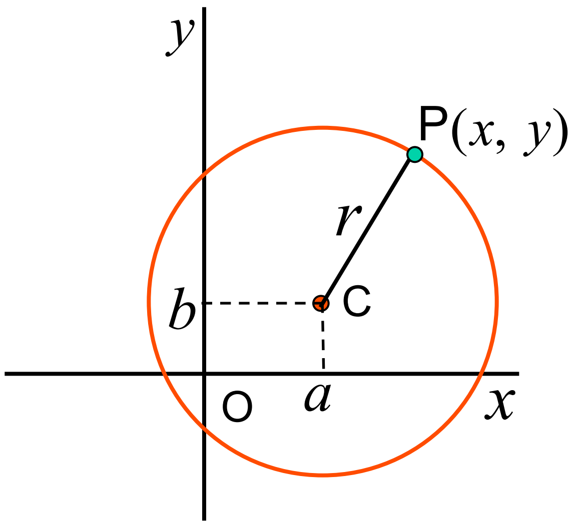
15A

- Circle
- Circle and a Straight Line

Circle

Condition

$$\sqrt{(x-a)^2+(y-b)^2}=r \quad (\text{where } r>0)$$



Standard expression

$$(x-a)^2+(y-b)^2=r^2$$

By expanding this, we obtain

$$x^2-2ax+y^2-2by+a^2+b^2=r^2$$

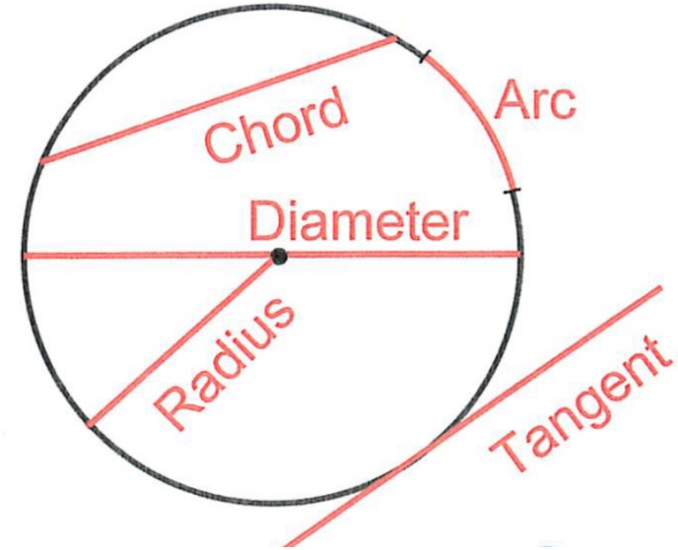
General expression

$$x^2+y^2+lx+my+n=0 \quad (\text{where } l^2+m^2>4n)$$

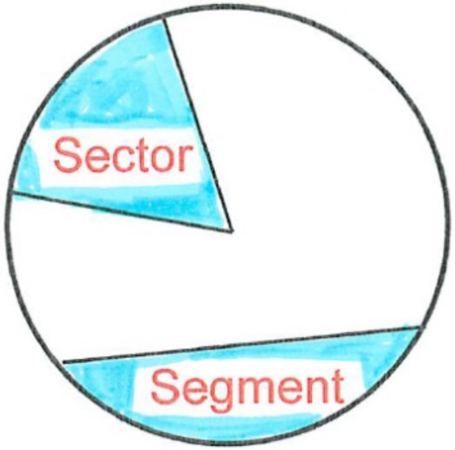
$$\left(x+\frac{l}{2}\right)^2+\left(y+\frac{m}{2}\right)^2=\frac{l^2+m^2-4n}{2}$$

Terminologies

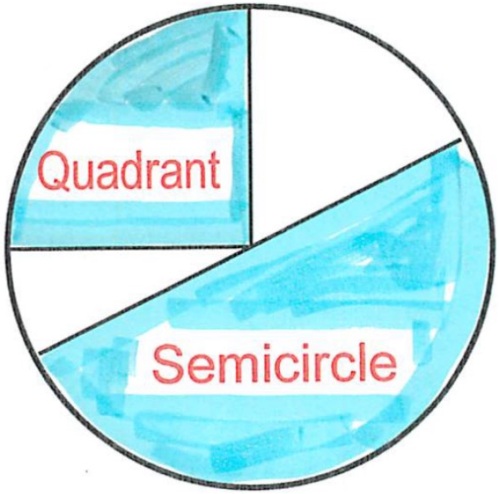
Lines



Slices



Special types of sector

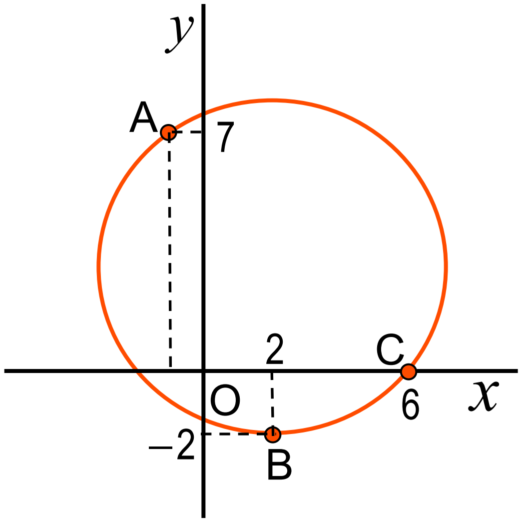


Example

[Example 15.1] Answer the following questions.
(1) Find the circle which passes points A(-1, 7), B(2, -2) and C(6, 0).
(2) Find the center of this circle.

Ans.

$$x^2 + y^2 + lx + my + n = 0$$



(1) Because the circle passes points A, B, C

$$-l + 7m + n = -50$$
$$2l - 2m + n = -8$$
$$6l + n = -36$$

Therefore

$$l = -4,\ m = -6,\ n = -12$$ and $$x^2 + y^2 - 4x - 6y - 12 = 0$$

(2) This equation is rearranges to the standard form as follows.

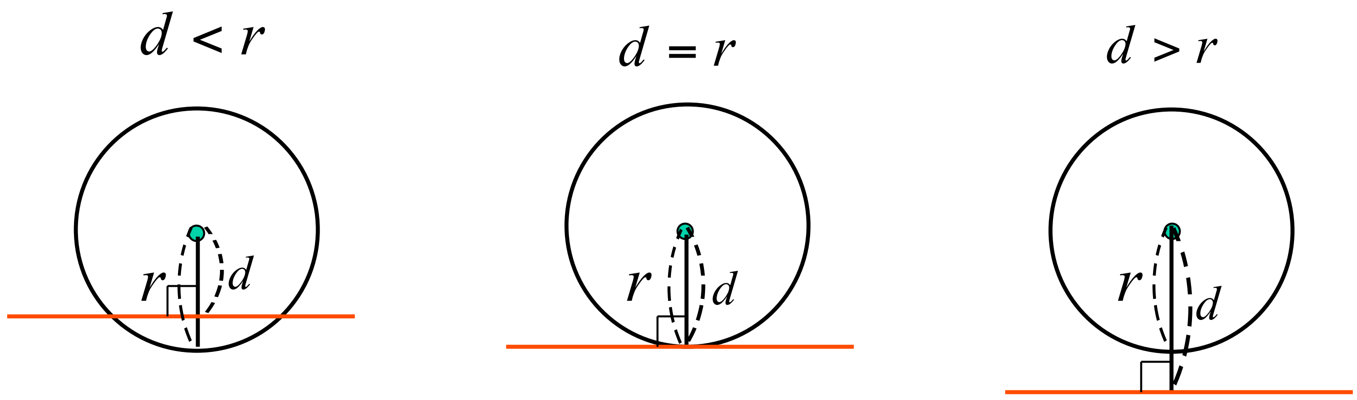
$$(x^2 - 4x + 4) + (y^2 - 6y + 9) - 4 - 9 - 12 = 0$$

$$\therefore \quad (x-2)^2 + (x-3)^2 = 5^2$$

Therefore, the center is (2, 3)

Relationship between a Circle and a Straight Line

Three cases



(a) Cross (b) Contact (c) No contact

Simultaneous Equation Problem

Circle $x^2 + y^2 + lx + my + n = 0$

Line $y = px + q$

$$\Rightarrow ax^2 + bx + c = 0$$

Discriminant $D = b^2 - 4ac$

(a) Cross : $D > 0$ (b) Contact : $D = 0$ (c) No contact : $D < 0$

[Example 15.2] Find the coordinates of the common points of a circle and a straight line in the following.

(1) Circle $x^2 + y^2 = 5$ (i) and line $y = x - 1$ (ii) .

(2) Circle $x^2 + y^2 = 5$ (iii) and line $2x - y + 5 = 0$ (iv) .

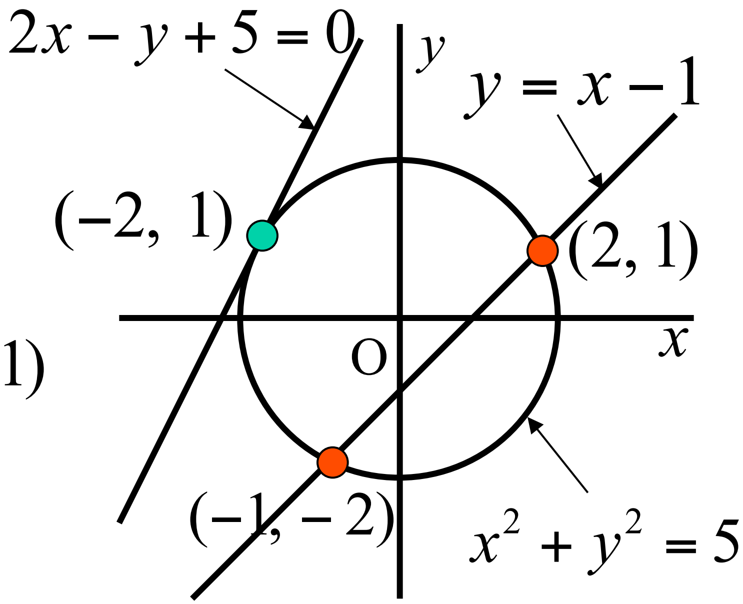
Ans.

(1) Substituting Eq.(ii) into Eq.(i), we have

$$x^2 - x - 2 = (x+1)(x-2) = 0$$

$$\therefore x = -1,\ 2 \qquad \therefore y = -2,\ 1$$

The common points are $(-1, -2)$ and $(2, 1)$



(2) Substituting Eq.(iv) into Eq.(iii), we have

$$x^2 + 4x + 4 = (x+2)^2 = 0$$

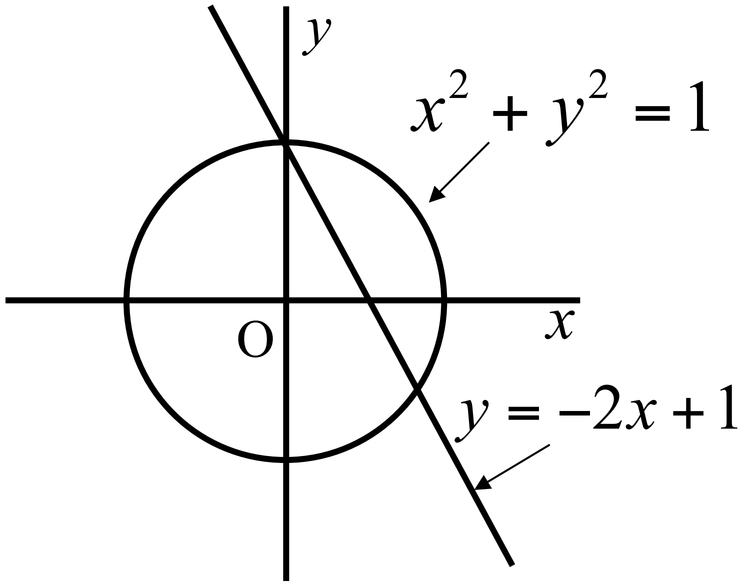
$$\therefore \quad x = -2, \quad \therefore \quad y = 1$$

Therefore, the contact point is $(-2,\ 1)$

[Exercise 15.1] The straight line $y = -2x + 1$ crosses with the circle $x^2 + y^2 = 1$. Find the chord length.

Ans.

Pause the video and solve the problem by yourself.



Answer to the Exercise

[Exercise 15.1] The straight line $y = -2x + 1$ crosses with the circle $x^2 + y^2 = 1$. Find the chord length.

Ans.

By eliminating y, we have

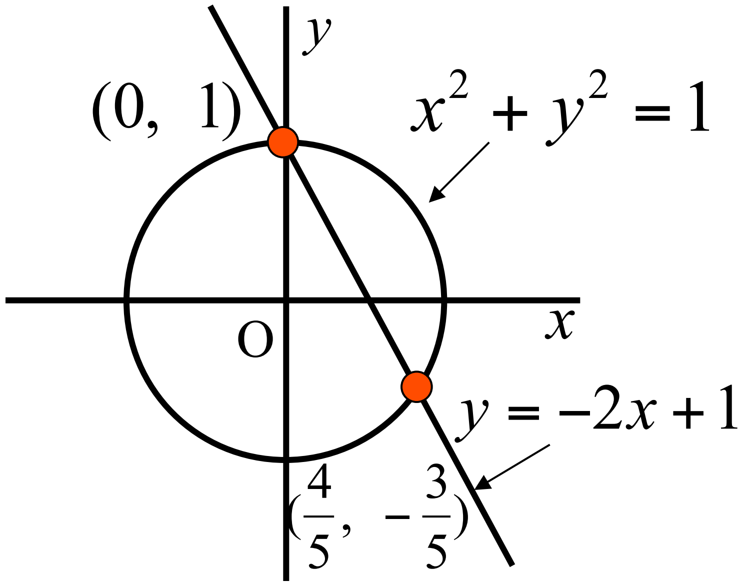
$$x^2 + (-2x + 1)^2 = 1$$

$$\therefore\ 5x^2 - 4x = 5x(x - \frac{4}{5}) = 0$$

$$\therefore\ x = 0,\ \frac{4}{5}$$

Then, the two cross points are

$$(0,\ 1) \quad (\frac{4}{5},\ -\frac{3}{5})$$



The chord length is $\sqrt{\left(0 - \frac{4}{5}\right)^2 + \left(1 + \frac{3}{5}\right)^2} = \frac{4\sqrt{5}}{5}$

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Lesson 15
Graphs and Equations (II)

15B
- Tangent line to a Circle
- Line through a cross point
- Circle through cross points
- Region defined by inequalities

Tangent Line to a Circle

Tangent Line

- A tangent line to a circle intersects at a single point T.
- The radius of a circle is perpendicular to the tangent line.

- The equation of the tangent line is given by

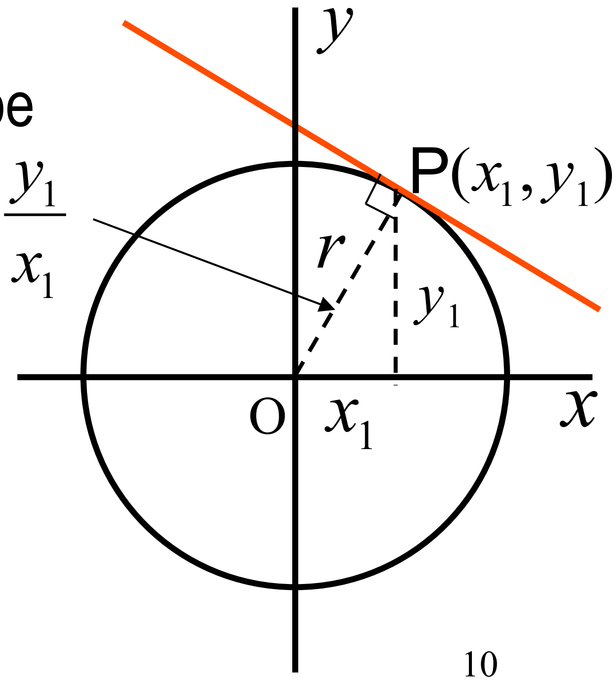
$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

That is,

$$\boxed{x_1 x + y_1 y = r^2}$$

Slope

$$m = \frac{y_1}{x_1}$$



Uhh…. What?

[Example 15.3] Find the tangent lines to circle $x^2 + y^2 = 2$, which passes point (3, 1).

Ans. Put tangent line $x_1 x + y_1 y = 2$

Since his line passes (3, 1)

$$3x_1 + y_1 = 2 \qquad \text{(i)}$$

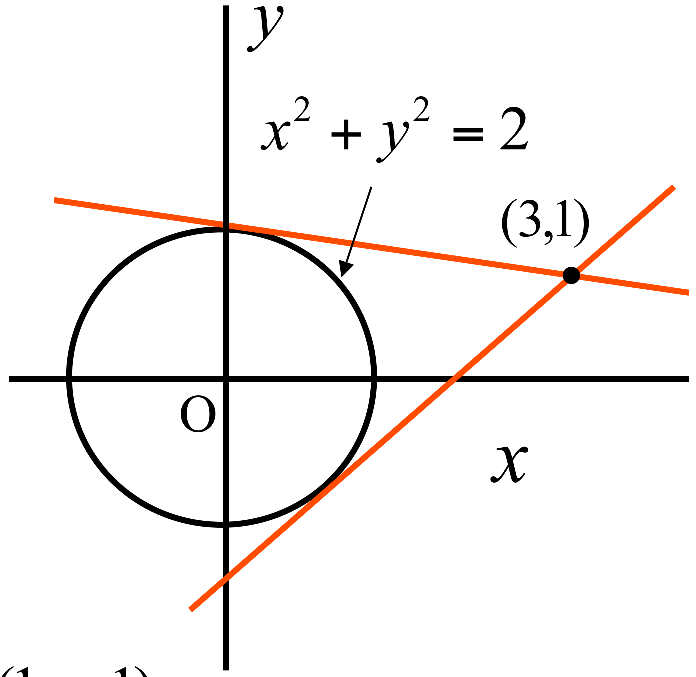
Since (x_1, y_1) is on the given circle

$$x_1^2 + y_1^2 = 2 \qquad \text{(ii)}$$

From (i) and (ii), we have

$$5x_1^2 - 6x_1 + 1 = (5x_1 - 1)(x_1 - 1) = 0$$

$x_1 = \frac{1}{5},\ 1$ Therefore $(x_1, y_1) = (\frac{1}{5}, \frac{7}{5}),\ (1, -1)$



The tangent lines are

$$\frac{1}{5}x + \frac{7}{5}y = 2 \qquad x - y = 2$$

Line and Circle Passing Through Cross Points

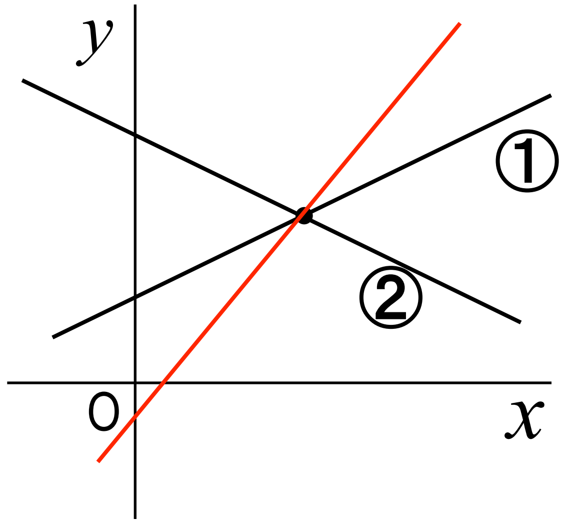
Line

Two lines : $a_1x + b_1y + c_1 = 0$ ①

and $a_2x + b_2y + c_2 = 0$ ②

The line which passes the cross point is

$$k \cdot (a_1x + b_1y + c_1) + a_2x + b_2y + c_2 = 0$$



Circle

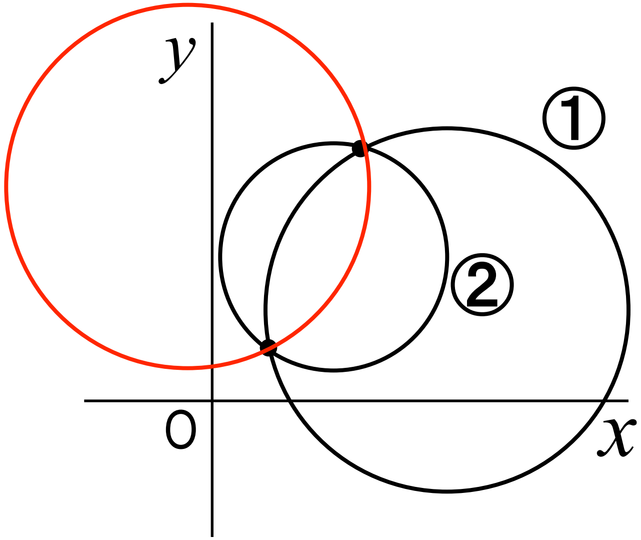
Two circles :

$$f(x, y) = x^2 + y^2 + l_1x + m_1y + n_1 = 0 \quad ①$$

$$g(x, y) = x^2 + y^2 + l_2x + m_2y + n_2 = 0$$

② The circle which passes the cross points

$$k \cdot f(x, y) + g(x, y) = 0$$



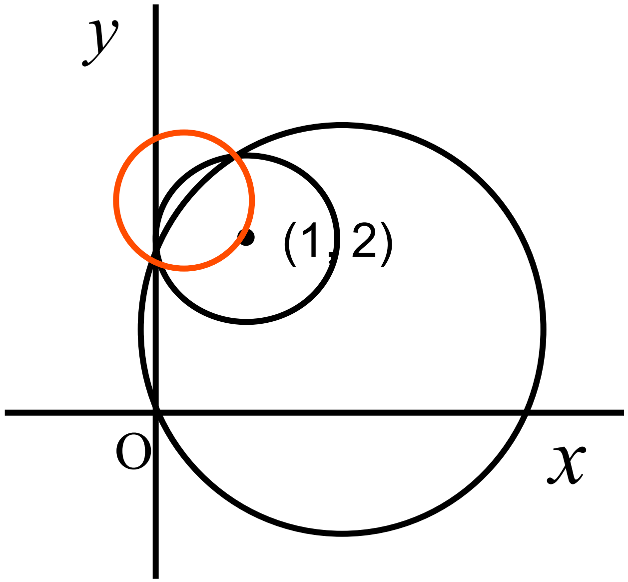
Cross Points of Two Circles

[Example 15.4] Find the expression of a circle which passes the cross points of two circles $x^2 + y^2 - 2x - 4y + 4 = 0$ and $x^2 + y^2 - 4x - 2y = 0$, and also passes point (1, 2).

Ans.

The line which passes the cross points .

$$k(x^2 + y^2 - 2x - 4y + 4) + x^2 + y^2 - 4x - 2y = 0$$



Substituting the coordinate of the point (1, 2), we have

$$-k - 3 = 0 \qquad \therefore k = -3$$

Then

$$-3(x^2 + y^2 - 2x - 4y + 4)$$

$$+ x^2 + y^2 - 4x - 2y = 0$$

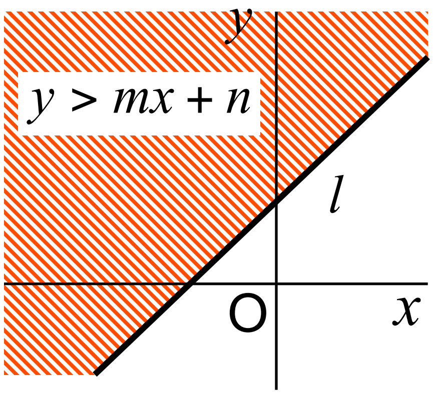
The answer is

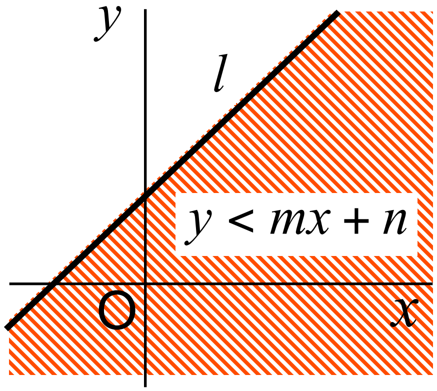
$$(x^2 + y^2) - x - 5y + 6 = 0$$

Region Defined by Inequalities

Line

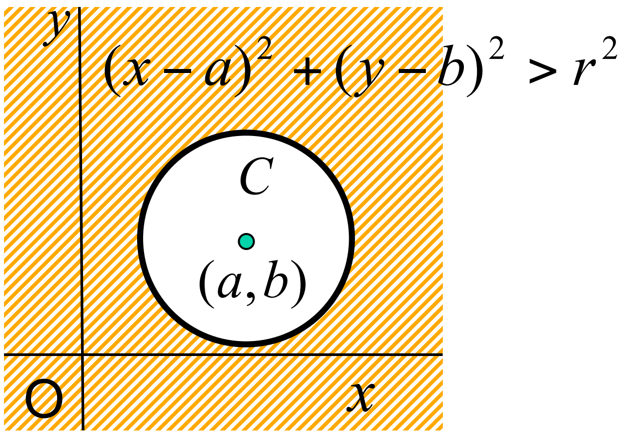
Straight line l : $y = mx + n$

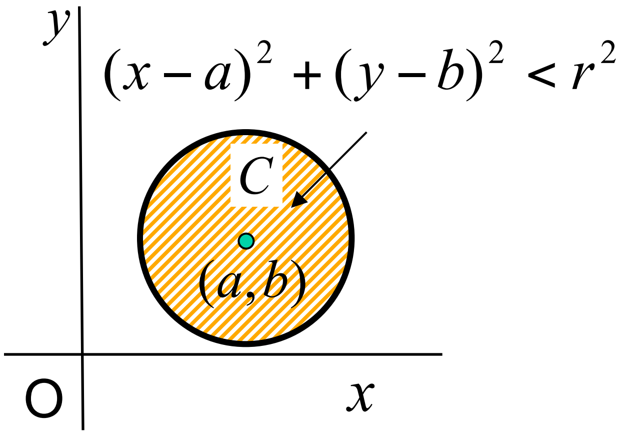




Circle

Circle : $(x - a)^2 + (y - b)^2 = r^2$





Example

[Example 15.4] Illustrate the region given by the following expression.

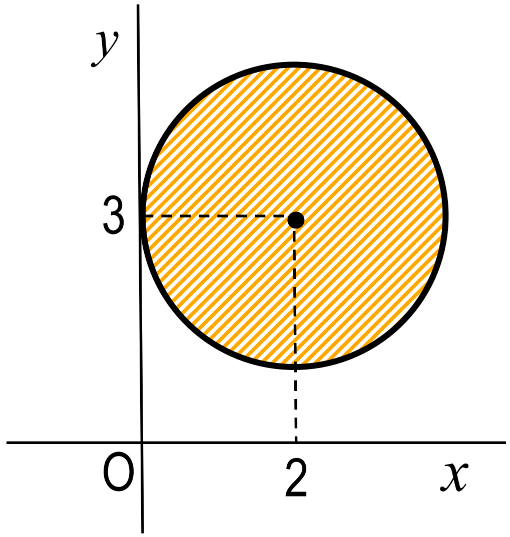
$$x^2 - 4x + y^2 - 6y + 9 < 0$$

Ans. First, we rearrange the expression to the following standard form.

$$(x-2)^2 + (y-3)^2 < 2^2$$

Therefore, the boundary is given by a circle with radius 2 and with center $(2,\ 3)$.

The answer is given by the shaded region.



Exercise

[Exercise 15.2] Illustrate the region which satisfies the following inequality.

$$(y-x+3)(x^2+y^2-12x-6y+20)<0$$

Ans.

Pause the video and solve the problem by yourself.

[Exercise 15.2] Illustrate the region which satisfies the following inequality.

$$(y - x + 3)(x^2 + y^2 - 12x - 6y + 20) < 0$$

Ans. $AB < 0 \iff A > 0,\ B < 0$ or $A < 0,\ B > 0$

We add the following two regions. After rearrangement, we have

(1) $y > x - 3$ and $(x-6)^2 + (y-3)^2 < 5^2$

(2) $y < x - 3$ and $(x-6)^2 + (y-3)^2 > 5^2$

Let the boundary $y = x - 3$ be l

and the boundary

$$(x-6)^2 + (y-3)^2 = 5^2 \text{ be } C.$$

Then,

the range (1) is above l and inside of C.

the range (2) is below l and outside of C.