NAGOYA UNIVERSITY

Lesson 13
Application of Trigonometric Functions

13A

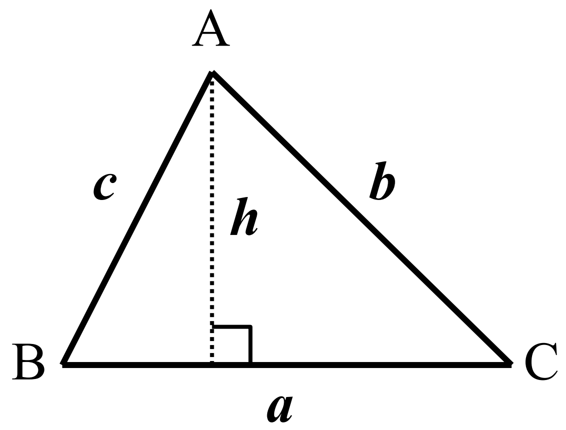
- Area of Triangles
- Heron's Formula
- Circumscribed Circle
- Inscribed Circle

Area Of A Triangle

Area of a Triangle

From two sides and the included angle

$$S = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$



[Proof]

Using the label in the figure on the right,

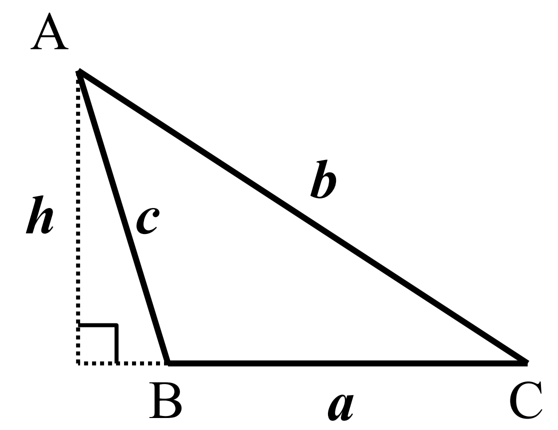
the height is

$$h = c\sin B \qquad (0^\circ \le B \le 90^\circ)$$

or

$$h = c\sin(180^\circ - B)$$

$$= c\sin B \qquad (90^\circ \le B \le 180^\circ)$$



Therefore

$$S = \frac{1}{2}ah = \frac{1}{2}ac\sin B$$

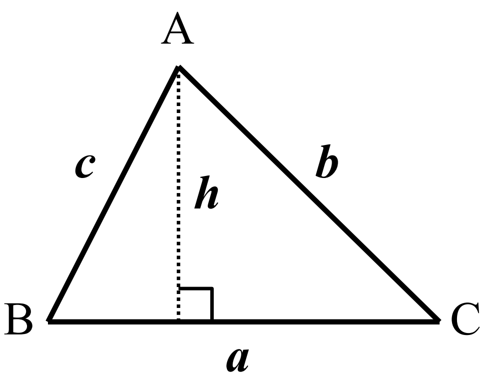
Heron's Formula

[Example 13.1] Prove that the area of a triangle is given by the following formula.

$$S = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{1}{2}(a+b+c)$$

Ans. From the Law of Cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



Therefore

$$\sin A = \sqrt{1 - \cos^2 A} = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2bc}$$

The area is

$$S = \frac{1}{4} bc \sin A = \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}$$

$$= \frac{1}{4}\sqrt{\{2bc - (b^2 + c^2 - a^2)\}\{2bc + (b^2 + c^2 - a^2)\}}$$

$$= \cdots = \sqrt{s(s-a)(s-b)(s-c)}$$

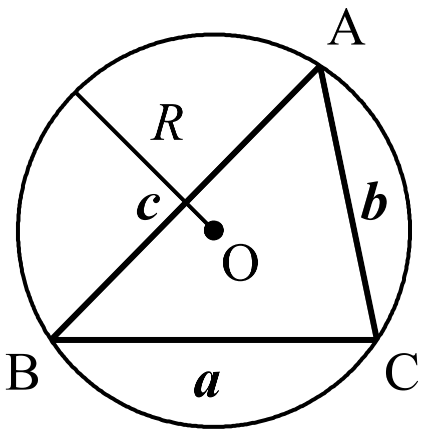
This formula is called "Heron's Formula"

Circumscribed Triangle

[Example 13.2] Prove the following Formula.

$$S = \frac{abc}{4R}$$

where a, b, c are lengths of sides and R is the radius of the circumscribed circle.



Ans. From the Law of Sines

$$\frac{a}{\sin A} = 2R$$

Therefore

$$S = \frac{1}{2}bc\sin A = \frac{1}{2}bc \cdot \frac{a}{2R} = \frac{abc}{4R}$$

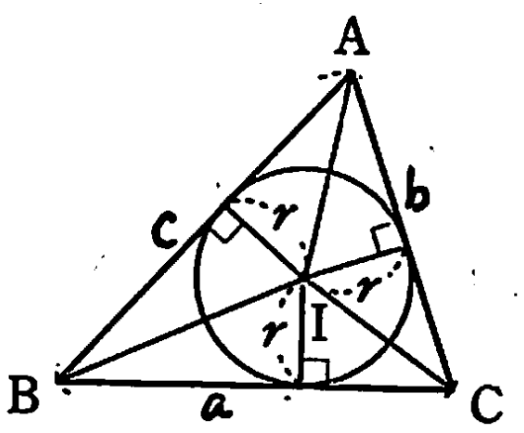
Inscribed Triangle

[Excercise 13.1] Prove the following Formula.

$$S = rs$$

where r is the radius of the inscribed circle

and $s = \dfrac{a+b+c}{2}$



Ans.

Pause the video and solve the problem by yourself.

Inscribed Triangle

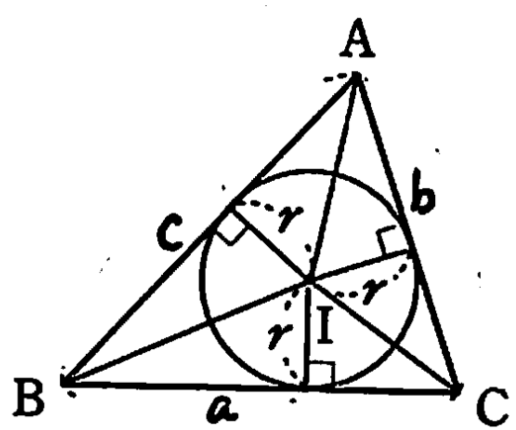
[Excercise 13.1] Prove the following Formula.

$$S = rs$$

where r is the radius of the inscribed circle and $s = \frac{a+b+c}{2}$, a, b, c are the lengths of three sides.

Ans.

Let the center of the inscribed circle be I.



The area is

$$\Delta\mathrm{ABC} = \Delta\mathrm{IBC} + \Delta\mathrm{ICA} + \Delta\mathrm{IAB}$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = rs$$

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Lesson 13
Application of Trigonometric Functions

13B

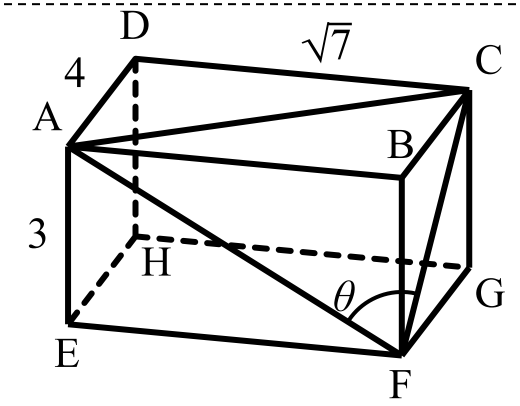
- 3 dimensional figures

Rectangular Parallelpiped

[Example 12.4] In the rectangular parallelepiped ABCD-EFGH, find the area of triangle ACF.

(1) Use the law of cosines.

(2) Use the Heron's formula.



Ans. (1) From Pythagorean Theorem CF=5, AF=4, AC= $\sqrt{23}$

From the law of cosine $23 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos\theta \quad \therefore \cos\theta = \dfrac{9}{20}$

Therefore

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{319}}{20}$$

Area $S = \dfrac{1}{2} 5 \cdot 4 \sin\theta = \dfrac{\sqrt{319}}{2}$

(2) $s = \dfrac{1}{2}(5+4+\sqrt{23}) = \dfrac{1}{2}(9+\sqrt{23}) \quad \therefore \; s - \mathrm{CF} = \dfrac{1}{2}(\sqrt{23}-1)$

$s - \mathrm{AF} = \dfrac{1}{2}(1+\sqrt{23}) \qquad s - \mathrm{AC} = \dfrac{1}{2}(9-\sqrt{23})$

Area $$S = \sqrt{\frac{1}{2}(9+\sqrt{23})\frac{1}{2}(\sqrt{23}-1)\frac{1}{2}(1+\sqrt{23})\frac{1}{2}(9-\sqrt{23})} = \frac{\sqrt{319}}{2}$$

Minimum Length of a Thread on a Cone

[Example 12.5] The dimensions of the cone is :

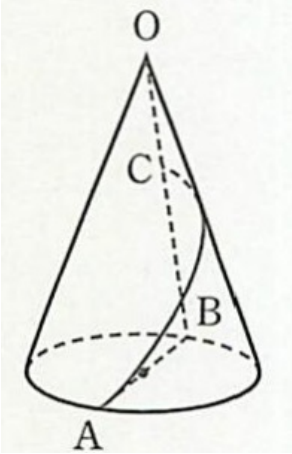
the diameter of the bottom circle AB=20 cm,

the length of OB=30 cm

the distance OC=10 cm (C lies on OB)

A thread is stretched from point A to point C.

What is the minimum length of the thread ?



Ans. The angle of the sector

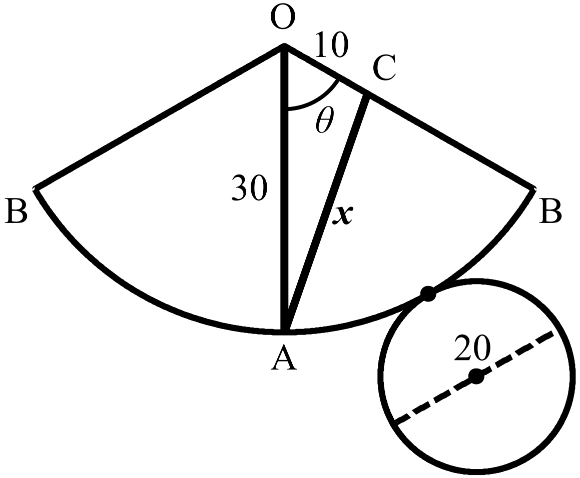
$$\frac{\theta}{360^\circ} = \frac{20\pi/2}{2\pi\cdot 30}, \quad \therefore\ \theta = 60^\circ$$

From the law of cosine

$$x^2 = 10^2 + 30^2 - 2\cdot 10\cdot 30\cos 60^\circ = 700$$

Therefore

$$x = 10\sqrt{7} \approx 26.5 \quad \text{cm}$$



Sphere and Quadrangular Pyramid

[Example 13.6] A sphere in a quadrangular pyramid has contacts with all planes. The area of the bottom is 4 m^2 and that of side plane is 5 m^2. What is the radius of this sphere ?

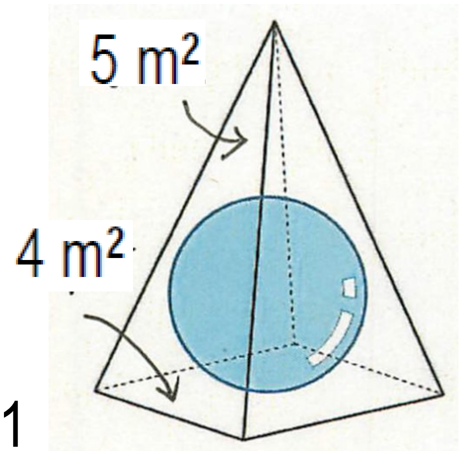


Fig.1

Ans.

The length of the side of the bottom square is 2 m

The height of the side triangle is 5 m.

The cross section along the read line.

Its height is $h = \sqrt{5^2 - 1^2} = 2\sqrt{6}$

From Ex.13.1, the area of a triangle is

Fig.2

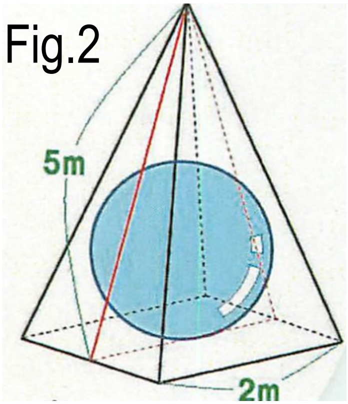
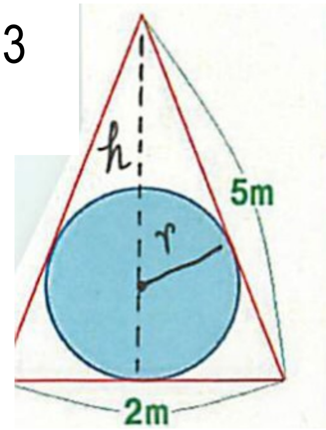


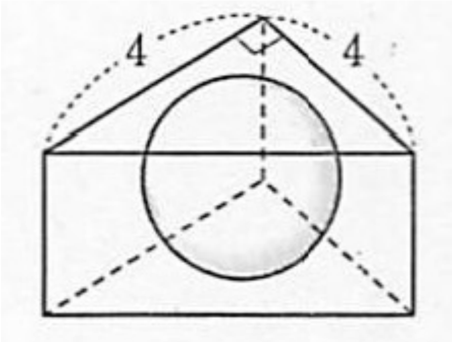
Fig.3



$$S = rs = r\left(\frac{a+b+c}{2}\right) \qquad \frac{1}{2}\cdot 2\cdot 2\sqrt{6} = r\left(\frac{2+5+5}{2}\right), \quad \therefore r = \frac{\sqrt{6}}{3}$$

Sphere and Quadrangular Pyramid

[Exercise 13.2] A sphere in a triangular prism has contact with all surface planes of the prism.
The bottom of the prism is right-angled isosceles triangle.



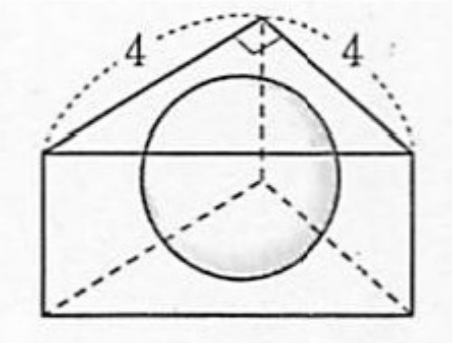
(1) What is the radius of the sphere ?
(2) What is the area of the surface of the sphere ?
(3) What is the volume of the sphere ?

Ans.

Pause the video and solve the problem by yourself.

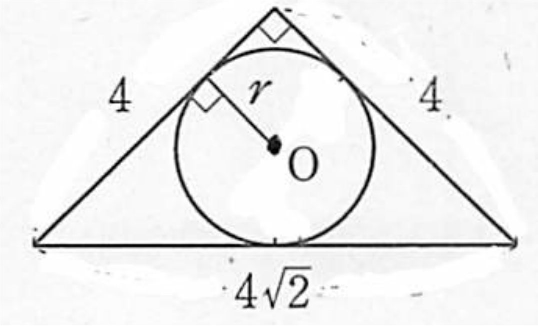
Answer to the Exercise

[Exercise 13.2] A sphere in a triangular prism has contact with all surface planes of the prizm.
The bottom of the prism is right-angled isosceales triangle.
(1) What is the radius of the sphere ?
(2) What is the area of the surface of the sphere ?
(3) What is the volume of the sphere ?



Ans. (1) From Ex.13.1

$$\frac{1}{2}\cdot 4\cdot 4 = r\left(\frac{4+4+4\sqrt{2}}{2}\right), \quad \therefore r = \frac{4}{2+\sqrt{2}} = 4-2\sqrt{2}$$



(2) $S = 4\pi r^2 = 4\pi(4-2\sqrt{2})^2 = 32\pi(3-2\sqrt{2})$

(3) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4-2\sqrt{2})^3 = \frac{64}{3}\pi(10-7\sqrt{2})$