NAGOYA UNIVERSITY

Lesson 12
Law of Sines and Law of Cosines

12A

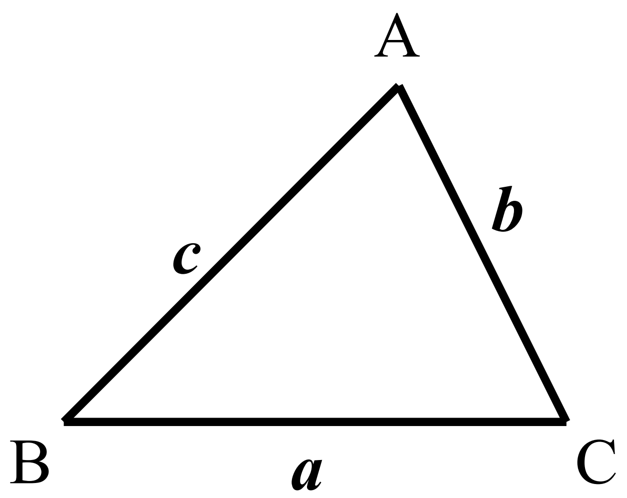
- Fundamental Properties of Triangles
- The Law of Sines

Fundamental Properties of Triangles

Fundamental Properties

(1) $A + B + C = 180^\circ$

(2) $a < b + c,\ b < c + a,\ \ c < a + b$



Determination of a triangle

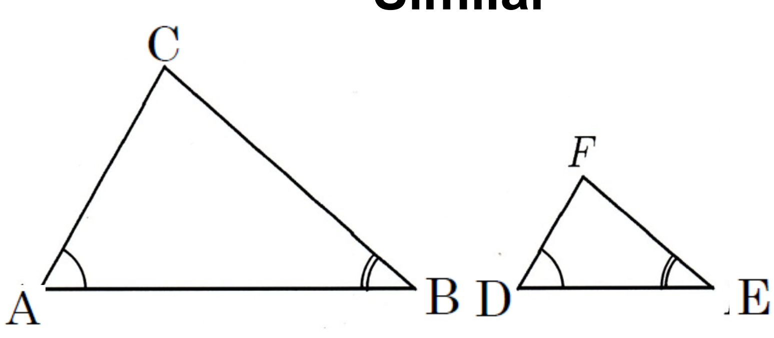
(1) Three sides

(2) Two sides and the included angle.

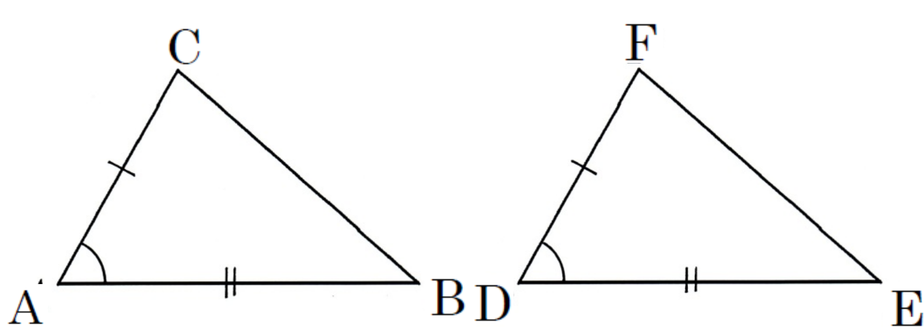
(3) Two internal angles and the included side

Similar and Congruent Triangles

Similar

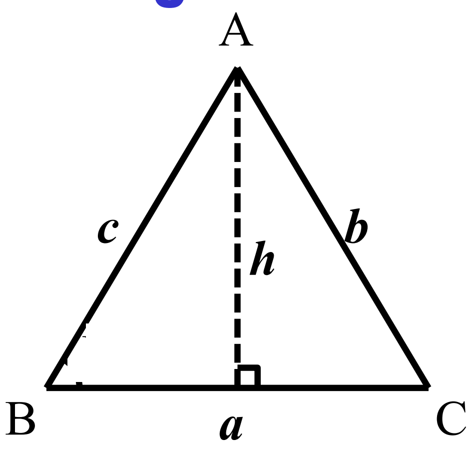


Congruent

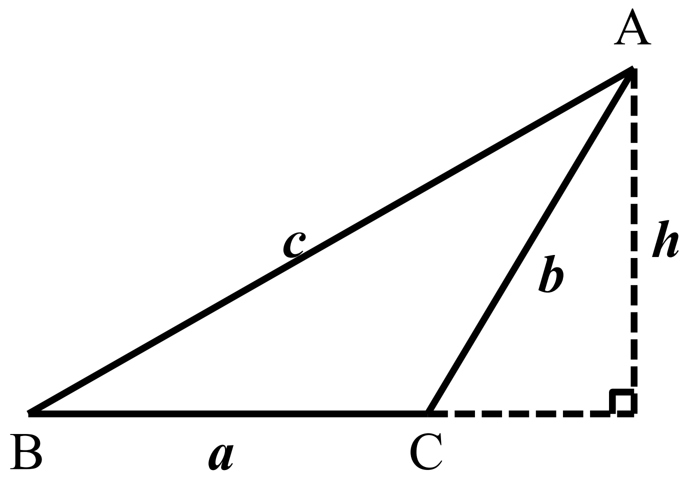


The Law of Sines

Oblique triangles



C is acute



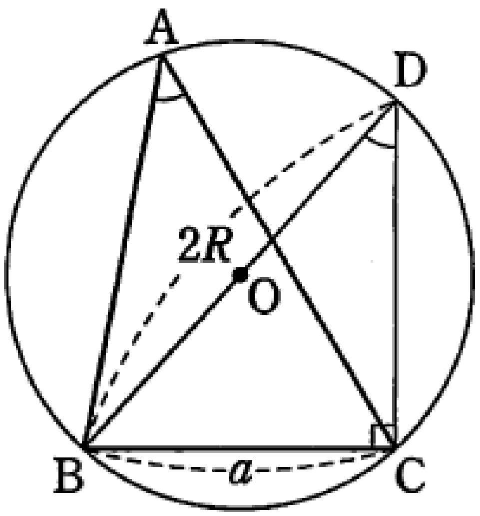
C is obtuse

The Law of Sines

The ratio of the length of a side of a triangle to the sine of the angle opposite that side is the same for all sides and angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof of the Law of Sines [Case 1]



[Case that all angles are acute]

The circle passes three vertices.

Let the diameter of this circle be BD.

From the inscribed angle theorem , we have

$$D = A \quad , \qquad \angle \mathrm{BCD} = 90^\circ$$

Therefore,

$$a = 2R\sin D = 2R\sin A.$$

Similarly

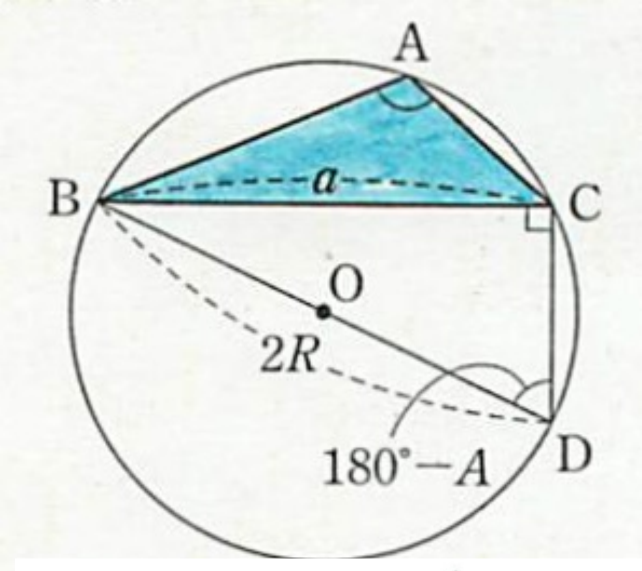
$$b = 2R\sin B, \quad c = 2R\sin C.$$

Therefore, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Proof of the Law of Sines [Case 2]

[Example 12.1] Prove $a = 2R\sin A$. for the case $A > 90°$.



Ans. Draw the line BD which passes the center.

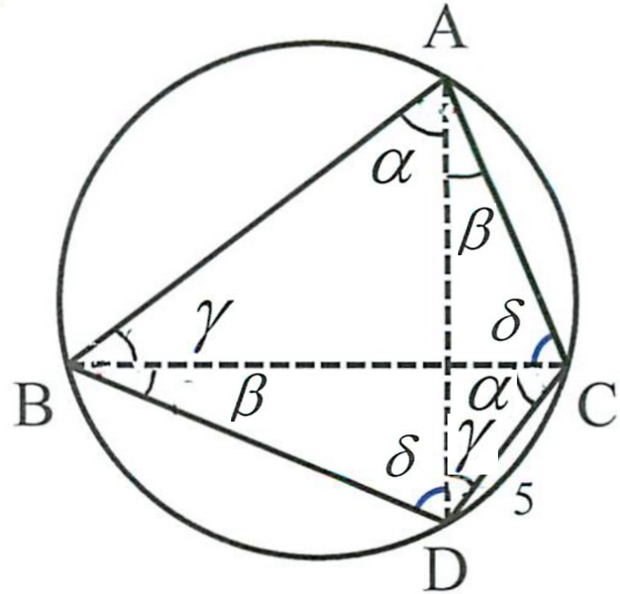
$$\angle BCD = 90°$$

Since quadrangle has contact with the circle, the following relationship holds. [**Note**]

$$A + D = 180°$$

Therefore

$$a = 2R\sin D = 2R\sin(180° - A) = 2R\sin A$$



[Note]

From the inscribed angle theorem, the angles with the same symbol has the same magnitude.
Therefore

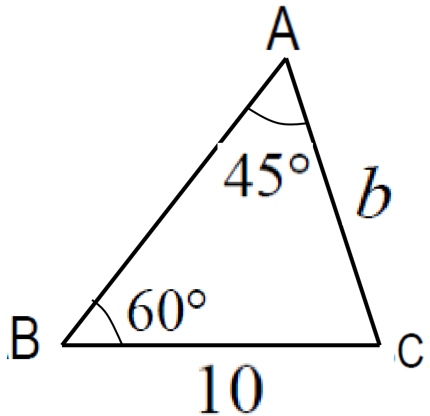
$$A + D = \alpha + \beta + \gamma + \delta = 180°$$

How Can We Use Sine Law ?

When two angles and one side of an acute triangle is given, we can know the other sides.

[Example 12.2] In the triangle ABC, $A = 45°, \quad B = 60°, \quad a = 10$ are given. Find the lengths b and c.

Ans. $C = 180° - 45° - 60° = 75°$



From the Law of Sines

$$\frac{10}{\sin 45°} = \frac{b}{\sin 60°} = \frac{c}{\sin 75°}$$

Value of each sine is

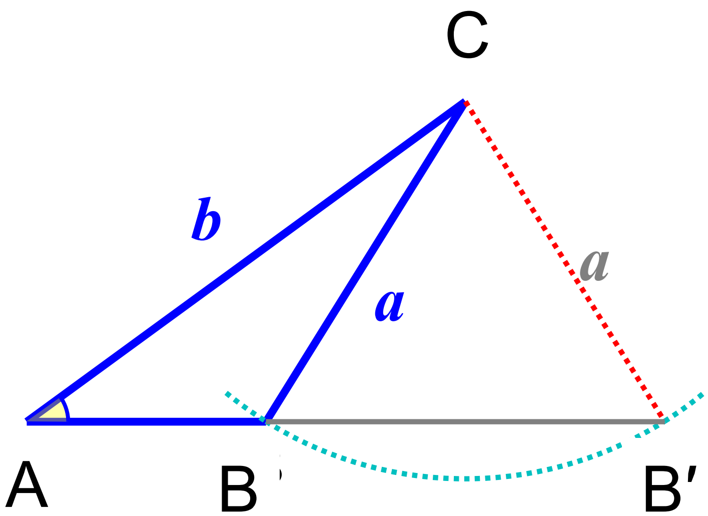
$$\sin 45° = \frac{\sqrt{2}}{2}, \quad \sin 60° = \frac{\sqrt{3}}{2}, \quad \sin 75° = \sin(30° + 45°) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Therefore

$$b = 10\frac{\sin 60°}{\sin 45°} = 5\sqrt{6}, \qquad c = 10\frac{\sin 75°}{\sin 45°} = 5(\sqrt{3}+1)$$

Ambiguous Case

Even if we know " two side and an angle not between ", we cannot determine the last side.



Angle A and sides a and b are given.

But

- Triangle ABC
- Triangle AB′C

are possible.

You can swing side a to left and right.

Huh?

Exercise

[Ex.12.1] In triangle ABC, $b = 12$, $C = 135°$ and the radius $R = 12$ of its circumscribed circle are given. Find angle B and side c.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.12.1] In triangle ABC, $b = 12,\ C = 135°$ and the radius $R = 12$ of its circumscribed circle are given. Find B and c.

Ans. From the law of sine and its relation to the radius of circumscribed circle

$$\frac{a}{\sin A} = \frac{12}{\sin B} = \frac{c}{\sin 135°} = 2 \times 12$$

Therefore

$$\sin B = \frac{12}{2 \times 12} = \frac{1}{2} \quad \therefore\ B = 30° \quad (\because\ B < 180° - 135°)$$

$$c = 2 \times 12 \times \sin 135° = 2 \times 12 \times \sin 135° = 24 \times \sin 45° = 12\sqrt{2}$$

NAGOYA UNIVERSITY

Lesson 12
The Law of Sines and the Law of Cosines

12B

- Law of Cosines

The Law of Cosines

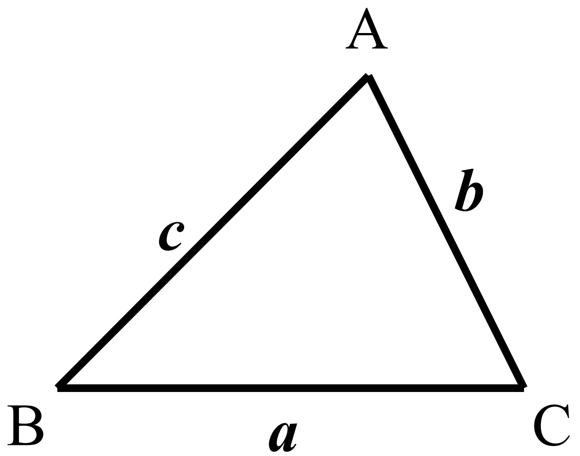
If we know two sides and the included angle, we can find the side which is opposite to this angle.

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = c^2 + a^2 - 2ca\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

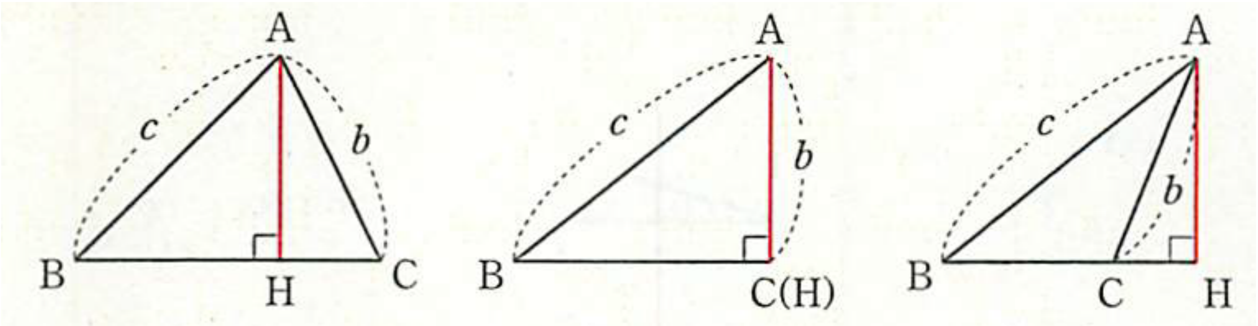


Another Expression

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We can find angles from three sides.

Proof of the Law of Cosines



Take projections of point A to side BC and name this point H.

$$a = \mathrm{CH} + \mathrm{BH} = b\cos C + c\cos B \quad (1)$$

Similarly

$$b = c\cos A + a\cos C \qquad (2)$$

$$c = a\cos B + b\cos A \qquad (3)$$

From $(2)\times b + (3)\times c - (1)\times a$

$$b^2 + c^2 - a^2 = 2bc\cos A$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$

Example

[Example 12.3] In triangle ABC, $b = 2,\ c = 1+\sqrt{3}$ and $A = 60°$ are given. Solve for $a,\ B,\ C$.

Ans.

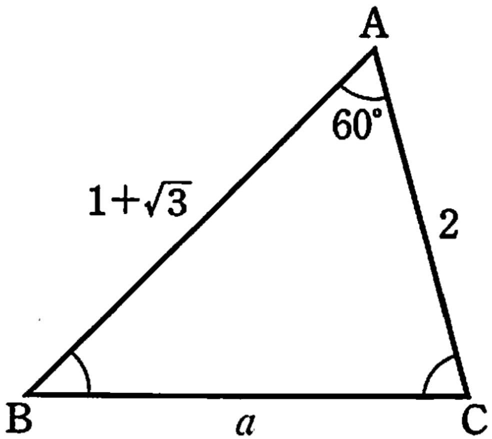
From the law of cosines

$$a^2 = 2^2 + (1+\sqrt{3})^2 - 2\cdot 2(1+\sqrt{3})\cos 60° = 6$$

From the law of sines

$$\frac{\sqrt{6}}{\sin 60°} = \frac{2}{\sin B} \qquad \therefore \sin B = \frac{2}{\sqrt{6}}\cdot\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad B = 45° \text{ or } 135°$$



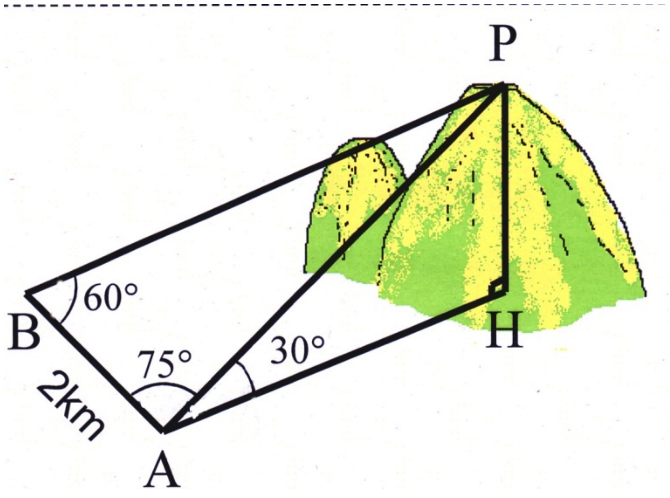
Since $a = \sqrt{6} > b = 2$, the angle $B < A = 60°$

Therefore, $B = 45°$

$$\therefore \quad C = 180° - (A+B) = 75°$$

Exercise

[Ex 12.2] When we see the top P of the mountain from two points A and B which are separated by 2km, the angles are $\angle PAB = 75°$ and $\angle PBA = 60°$. In addition, the angle of elevation of the mountain top P from the point A is 30°. What is the height of the mountain ?

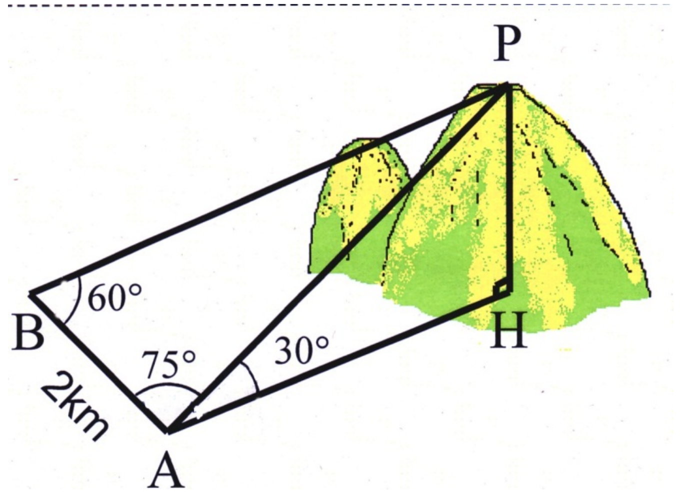


Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex 12.2] When we see the top P of the mountain from two points A and B which are separated by 2km, the angles are $\angle PAB = 75°$ and $\angle PBA = 60°$. In addition, the angle of elevation of the mountain top P from the point A is 30°. What is the height of the mountain ?



Ans. $\angle APB = 180° - (75° + 60°) = 45°$

From the law of sines

$$\frac{AP}{\sin 60°} = \frac{2}{\sin 45°} \qquad \therefore\ AP = \frac{2\sin 60°}{\sin 45°} = 2 \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{2} = \sqrt{6}$$

Since the triangle AHP is a right triangle,

$$PH = AP\sin 30° = \frac{\sqrt{6}}{2} \quad \text{km}$$