Lesson 9
Rational Function and Irrational Function

9A

- Linear Rational Function
- Simple Rational Function
- Standard Form
- Inequality

Linear Rational Function

Rational Function

$$y = \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials

Linear Rational Function

$$y = \frac{ax+b}{cx+d} \qquad (ad-bc \neq 0,\ c \neq 0)$$

In the case of $ad - bc = 0$

$$\therefore \frac{a}{c} = \frac{b}{d} (\equiv k),\ \therefore a = ck,\ b = dk,\ \therefore y = \frac{ckx+dk}{cx+d} = k$$: Constant

In the case of $c = 0$

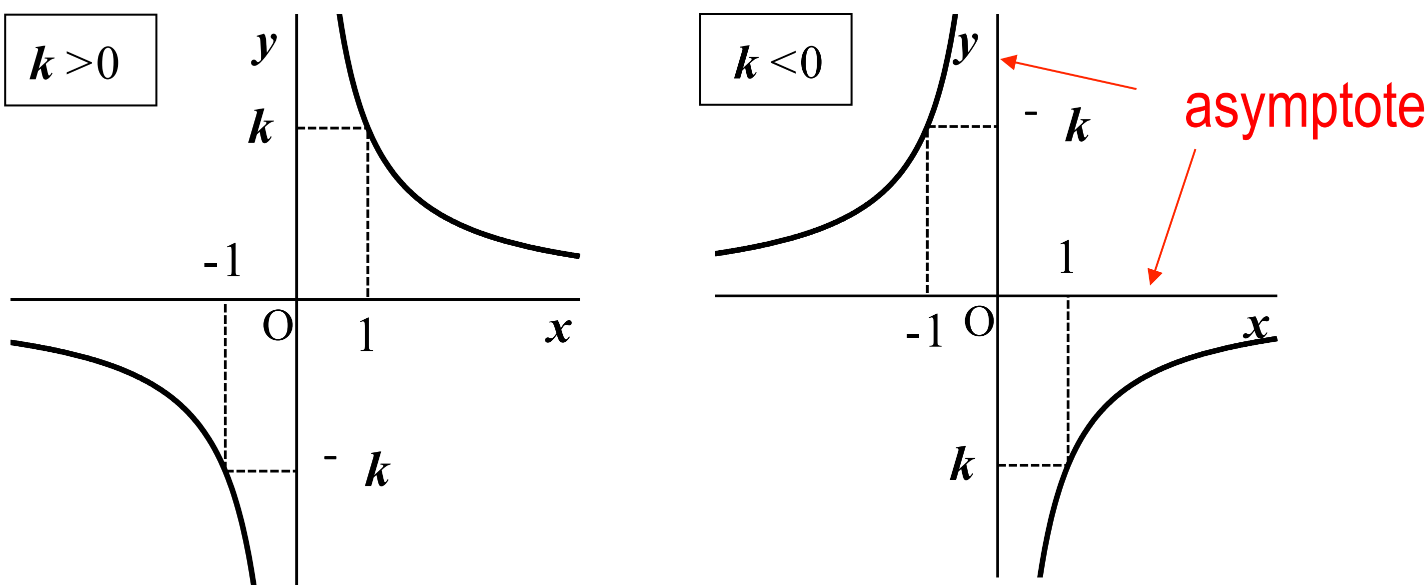
$$y = \frac{a}{d}x + \frac{b}{d}$$: Linear function

The Simplest Rational Function

$$y = \frac{k}{x}$$

- Two variables are inversely proportional.

- The horizontal asymptote is $x = 0$ / the vertical asymptote is $y = 0$
- The graph is an equilateral hyperbola.

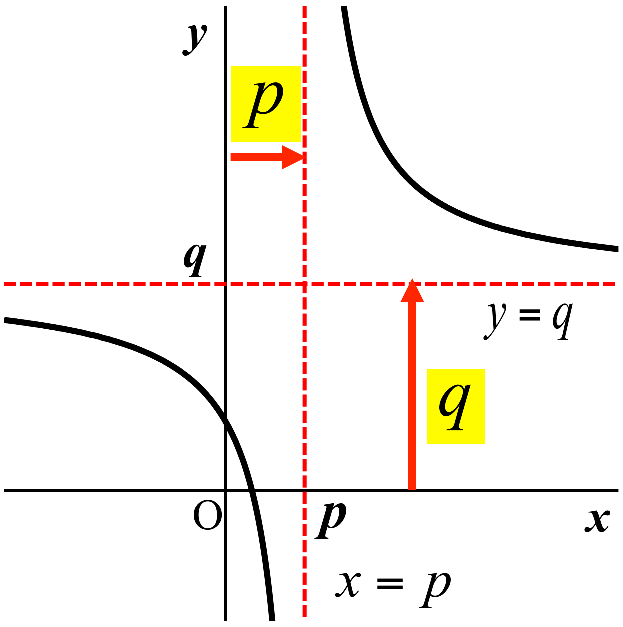


Function $y=(ax+b)/(cx+d)$

Change to the standard form

$$y=\frac{ax+b}{cx+d}$$

$$\Rightarrow \quad y=\frac{k}{x-p}+q$$



- The vertical asymptote is $x=p$
  the horizontal asymptote is $y=q$.

- The domain is $x\neq p$; the range is $y\neq q$.

Example

Example 1. Illustate the rational function $y = \dfrac{3x-6}{x-1}$, following each steps.

(1) [Step 1] Convert the function to the standard form.

(2) [Step 2] Find the horizontal and vertical asymptotes.

(3) [Step 3] Find the x-intercept and y-intercept.

(4) [Step 4] Illustrate the graph.

Ans.

(1) By rearranging, it becomes

$$y = \frac{3x-3-3}{x-1} = 3 - \frac{3}{x-1}$$

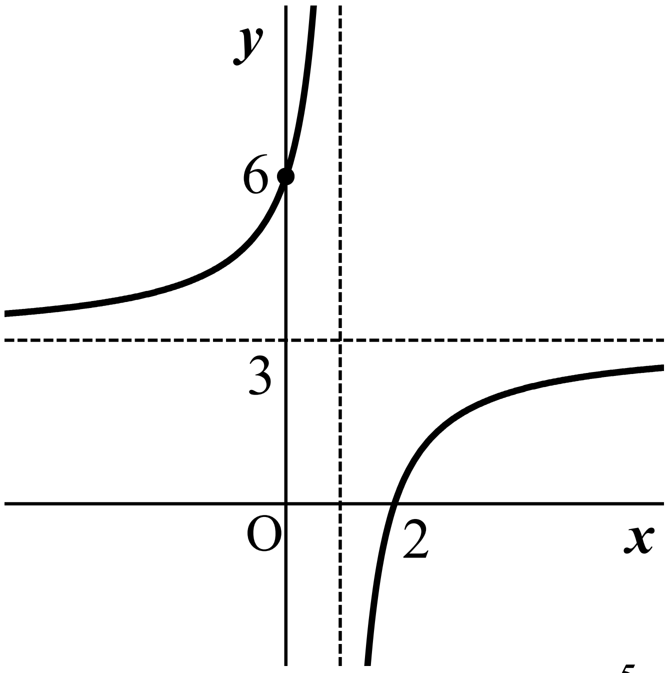
(2) The horizontal asymptote is $y = 3$

The vertical asymptote is $x = 1$

(3) By putting $y = 0$, the x-intercept is $x = 2$

By putting $x = 0$, the y-intercept is $y = 6$

(4) See the right figure



Example 2. Solve the following inequality $\dfrac{3x-6}{x-1} \geq -x+6$

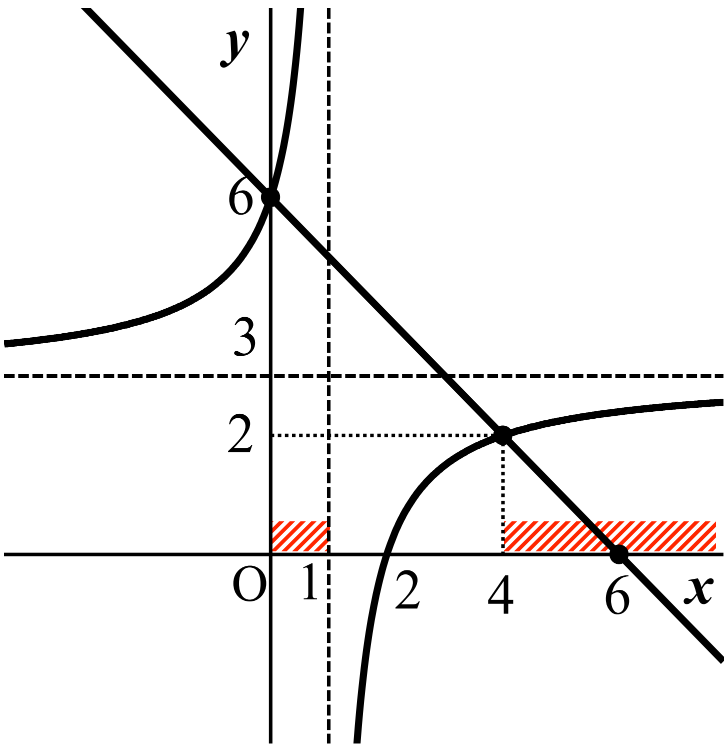
Ans.

We utilize the result of Example 1.

The cross points of $y = \dfrac{3x-6}{x-1}$ and $y = -x+6$ are

$$\frac{3x-6}{x-1} = -x+6 \quad \therefore 3x-6 = (x-1)(-x+6)$$

$$\therefore x(x-4) = 0 \quad \therefore \quad x = 0, \; x = 4$$



From the figure and the cross points.
we have $0 \leq x < 1, \quad 4 \leq x$

Exercise

Exercise 1. Solve the inequality $\dfrac{x+1}{x-1} > x+1$

Ans.

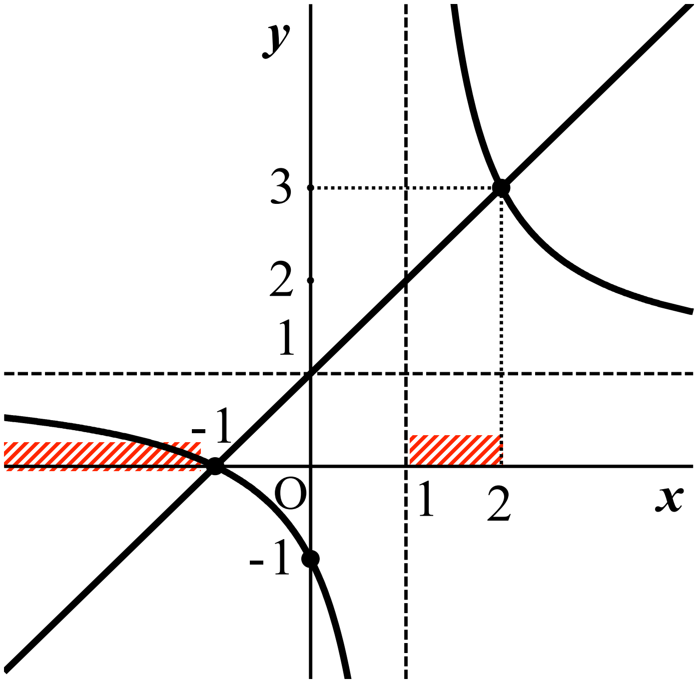
Pause the video and solve the problem.

Answer to the Exercise

Exercise 1. Solve the inequality $\dfrac{x+1}{x-1} > x+1$

Ans.

- We consider two functions $y = \dfrac{x+1}{x-1} = \dfrac{2}{x-1} + 1$ and $y = x+1$
- The horizontal asymptote and vertical asymptote of the former are $y = 1$ and $x = 1$, respectively.



- The x- and y-intercepts of the former are $x = -1$ and $y = -1$, respectively.
- The x- and the y-intercepts of the latter are $x = -1$ and $y = 1$, respectively.
- The cross-points are

$$\frac{x+1}{x-1} = x+1 \quad \therefore (x+1)(x-2) = 0 \quad \therefore \quad x = -1,\ 2$$

- From the figure, the solution is

$$x < -1,\ 1 < x < 2$$

Lesson 9
Exponential Functions

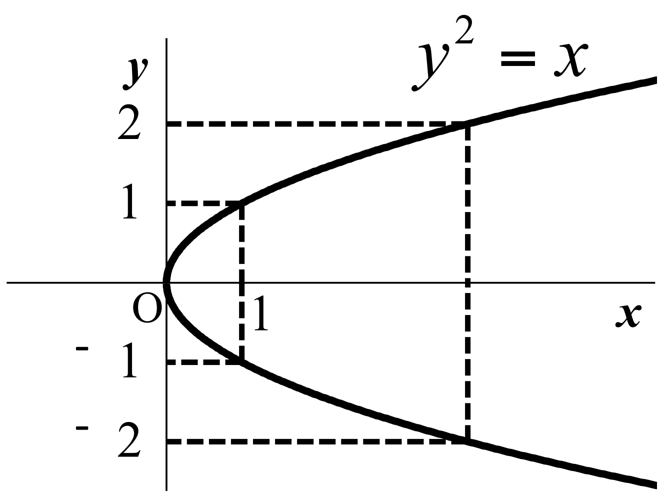
9B

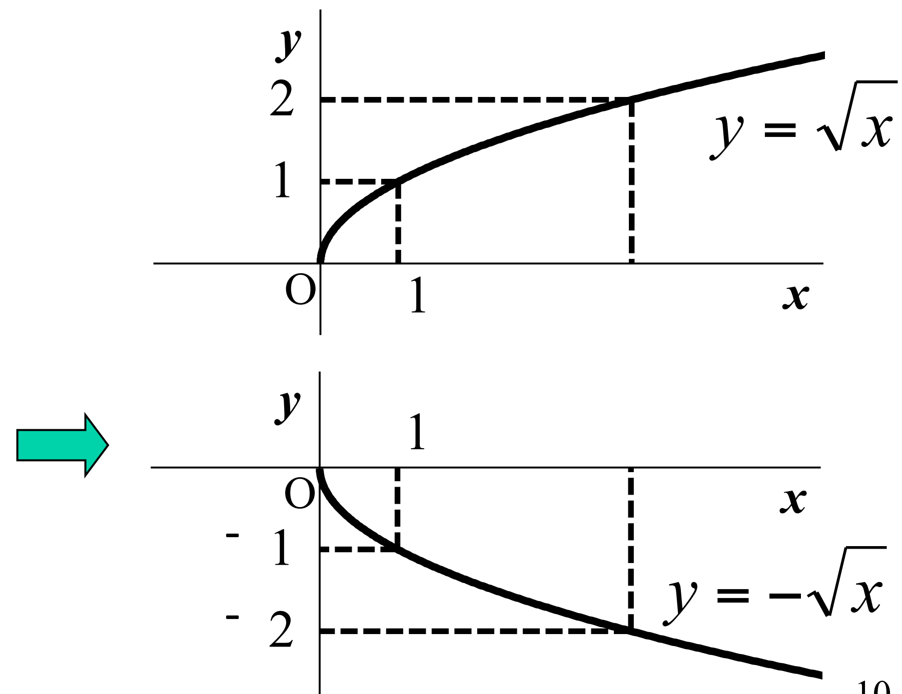
- Irrational Functions
- Standard Form
- Inequality

Irrational Function

- There is no rigorous definition of irrational function.
- We can say that irrational function is the one that cannot be written as the quotient of two polynomials (but this definition is not used.).
- Customary, a function which include variables in the root is called an irrational function.

Example

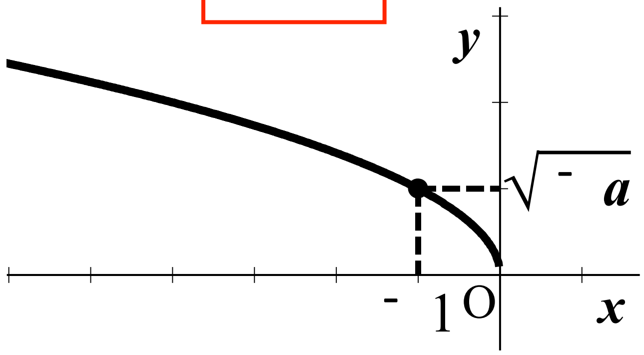




Fundamental Form

$$y = \sqrt{ax}$$

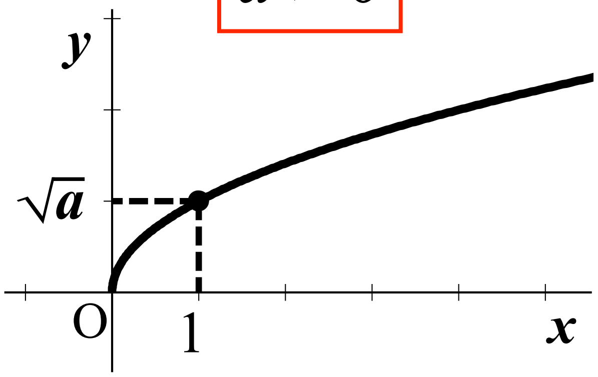
$a < 0$



Domain $x \leq 0$

Range $y \geq 0$

$a > 0$



Domain $x \geq 0$

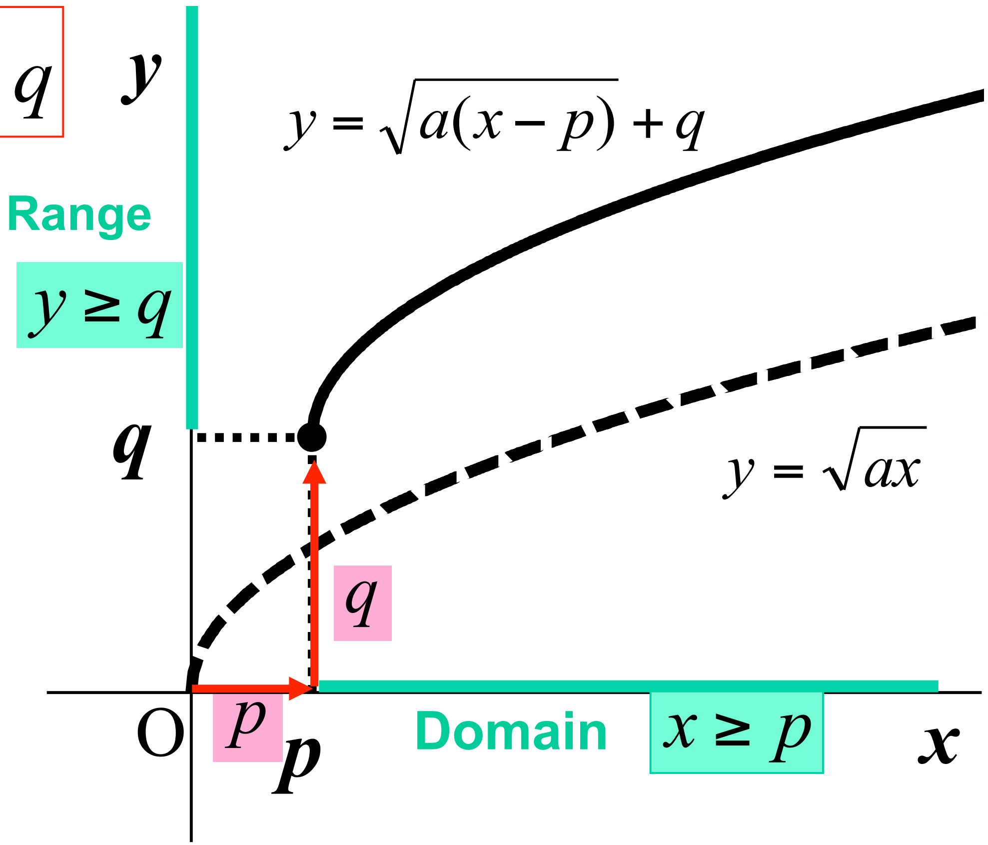
Range $y \geq 0$

Function $y=\sqrt{ax+b}+q$

Change to the standard form

$$y=\sqrt{ax+b}+q$$

$$\Rightarrow \quad \boxed{y=\sqrt{a(x-p)}+q}$$



That wasn't so hard!

Equation and Inequality

Example 3. Answer the following questions

(1) Illustrate the graphs of the functions $y = \sqrt{2-x}$ and $y = -x + \sqrt{2}$

(2) Solve the equation $\sqrt{2-x} = -x + \sqrt{2}$

(3) Solve the inequality $\sqrt{2-x} \geq -x + \sqrt{2}$

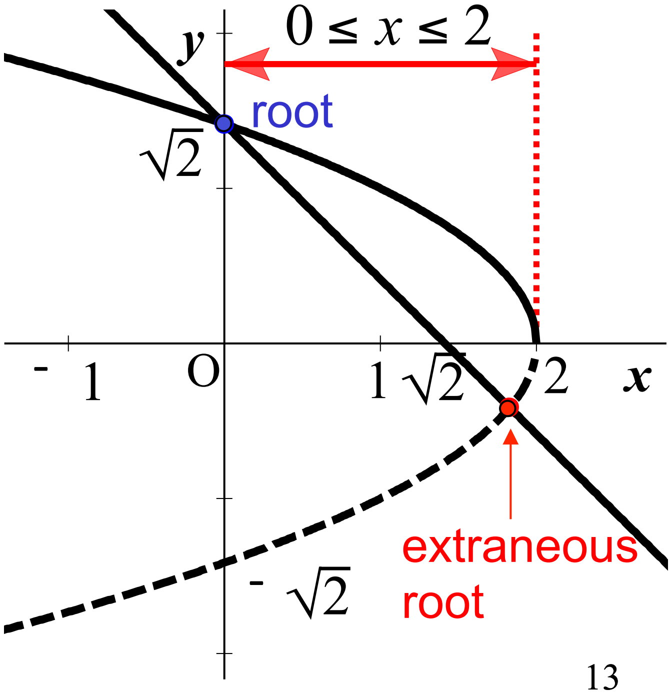
Ans. (1) See the right figure

(2) $\sqrt{2-x} = -x + \sqrt{2} \quad \therefore\ 2 - x = \left(-x + \sqrt{2}\right)^2$

$\therefore\ x(x - 2\sqrt{2} + 1) = 0 \quad \therefore\ x = 0,\ 2\sqrt{2} - 1$

From the figure, the root is only $x = 0$

(3) From (1) and (2), we have $0 \leq x \leq 2$



[Note] We must check that the obtained roots really satisfy the equation.

Exercise

Exercise 2. Solve the inequality $\sqrt{x+2} > x$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 2. Solve the inequality $\sqrt{x+2} > x$

Ans.

- We consider two functions $y = \sqrt{x+2}$ and $y = x$

- The x- and the y-intercepts of the former are $x = -2$ and $y = \sqrt{2}$, respectively.

- The cross-points of these functions are

$$\sqrt{x+2} = x \qquad \therefore\ x+2 = x^2$$

$$\therefore\ (x+1)(x-2) = 0 \qquad \therefore\ x = -1, \quad x = 2$$

Only $x = 2$ satisfies this condition.

- From the figure, the solution is

$$-2 \le x < 2$$

(Note the boundaries.)