NAGOYA UNIVERSITY

Lesson 7
Complex Numbers

7A

- **Solutions of a Quadratic Equation**
- **Complex Number**

Solutions of a Quadratic Equation

[Review]

The quadratic Eq. $ax^2 + bx + c = 0$

$$\Updownarrow$$

Solution

If $D = b^2 - 4ac > 0$

- there are two real distinct roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \alpha \\ \beta \end{cases}$

If $D = b^2 - 4ac = 0$

- there is one real double root are $x = -\dfrac{b}{2a}$

If $D = b^2 - 4ac < 0$

- there is no real root. (The equation is unsolvable.)

Complex Number

Introducing a New Number

We introduce an imaginary number

$$i = \sqrt{-1}$$

which is a theoretical number equal to the square root of -1.

Quadratic formula

The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $D = b^2 - 4ac > 0$, there are two distinct real roots.

If $D = b^2 - 4ac = 0$, there is a double root.

If $D = b^2 - 4ac < 0$, there are two distinct imaginary roots

Complex Number

Complex number

$$a + bi$$

(a : real part, b : imaginary part, i : imaginary unit)

Special Cases

a : real number (case of $a \neq 0, \quad b = 0$)

$a + bi$: imaginary number (case of $b \neq 0$)

bi : pure imaginary number (case of $a = 0, \quad b \neq 0$)

Complex number = real number + imaginary number

Operation on Complex Numbers

Equality

$a + bi = c + di$ equals $a = c$ and $b = d$

$a + bi = 0$ equals $a = 0$ and $b = 0$

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtraction

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Multiplication

$$(a + bi)(c + di) = (ac - bd) + (ad - bc)i$$

Division

$$\frac{(c + di)}{(a + bi)} = \frac{(ac + bd)}{a^2 + b^2} + \frac{(ad - bc)}{a^2 + b^2} i$$

(refer to next slide)

In the calculation, deal with i just as you would do with x ,
but replace i^2 by —1 .

Complex Conjugates

A complex number $a - bi$ is called a complex conjugate of $a + bi$.

the same real part

equal magnitude and opposite signs.

A notation $z(= a + bi)$ is sometimes used to express a complex number and $\bar{z}(= a - bi)$ is used to express its complex conjugate.

[Note] One merit of complex conjugates

Multiplication of complex conjugates produces a real number

$$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$$

Therefore, it is useful in the calculation of division

<Example>

$$\frac{(c + di)}{(a + bi)} = \frac{(c + di)(a - bi)}{(a + bi)(a - bi)} = \frac{(ac + bd)}{a^2 + b^2} + \frac{(ad - bc)}{a^2 + b^2} i$$

Example 1. Factor the following equation considering complex numbers.

$$x^2 - 2x + 4 = 0$$

Ans.

From the quadratic formula, we have

$$x = \frac{+2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2} = 1 \pm \sqrt{3}\, i$$

Therefore

Example

Example 2. Let the two roots of the quadratic equation $ax^2 + bx + c = 0$ be **α** and **β**. Here we admit the case for complex roots. Prove the following relationships.

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{b}{a}$$

Ans.

Using the quadratic formula, we have

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$$

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

Exercise

Exercise 1. Simplify the following expressions.

(1) $(\sqrt{3}+\sqrt{-2})(\sqrt{2}-\sqrt{-3})$ (2) $1+i+i^2+\cdots+i^{10}$ (3) $\dfrac{2+i}{2-i}+\dfrac{3-i}{3+i}$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 1. Simplify the following expressions.

(1) $(\sqrt{3}+\sqrt{-2})(\sqrt{2}-\sqrt{-3})$ (2) $1+i+i^2+\cdots+i^{10}$ (3) $\dfrac{2+i}{2-i}+\dfrac{3-i}{3+i}$

Ans.

(1) $$(\sqrt{3}+\sqrt{-2})(\sqrt{2}-\sqrt{-3}) = (\sqrt{3}+\sqrt{2}\,i)(\sqrt{2}-\sqrt{3}\,i)$$
$$= (\sqrt{3}\sqrt{2}+\sqrt{2}\sqrt{3}) + (\sqrt{2}\sqrt{2}-\sqrt{3}\sqrt{3})i = 2\sqrt{6}-i$$

(2) $$1+i+i^2+\cdots+i^{10} = (1+i-1-i)+(1+i-1-i)+1+i-1 = i$$

(3) $$\frac{2+i}{2-i}+\frac{3-i}{3+i} = \frac{(2+i)^2}{(2-i)(2+i)}+\frac{(3-i)^2}{(3+i)(3-i)} = \frac{3+4i}{5}+\frac{8-6i}{10} = \frac{7+i}{5}$$

NAGOYA UNIVERSITY

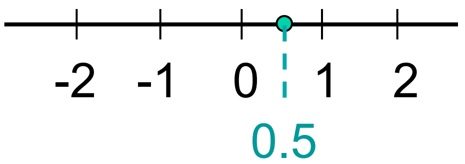
Lesson 7
Complex Numbers

7B

- Complex Plane
- Product of Complex Numbers
- De Moivre’s Formula

Complex Plane (Argand Diagram)

A ***real number*** can be visualized by a ***number line.***



Complex plane

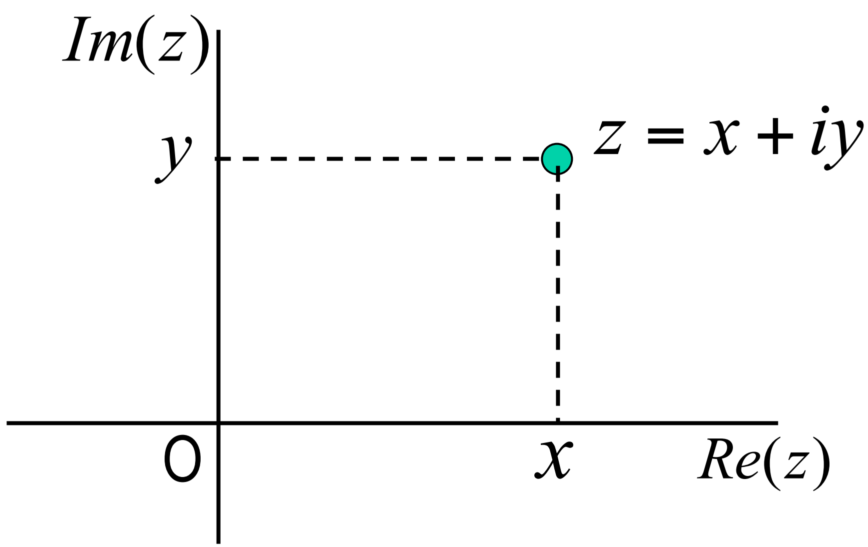
A ***complex number*** can be visualized by a ***complex plane***.

Complex plane is established by the real axis and the imaginary axis.

Complex number | **Complex plane**

$\boxed{z = x + iy}$ $\Rightarrow$

displacement in in the x-axis.
displacement in in the y-axis

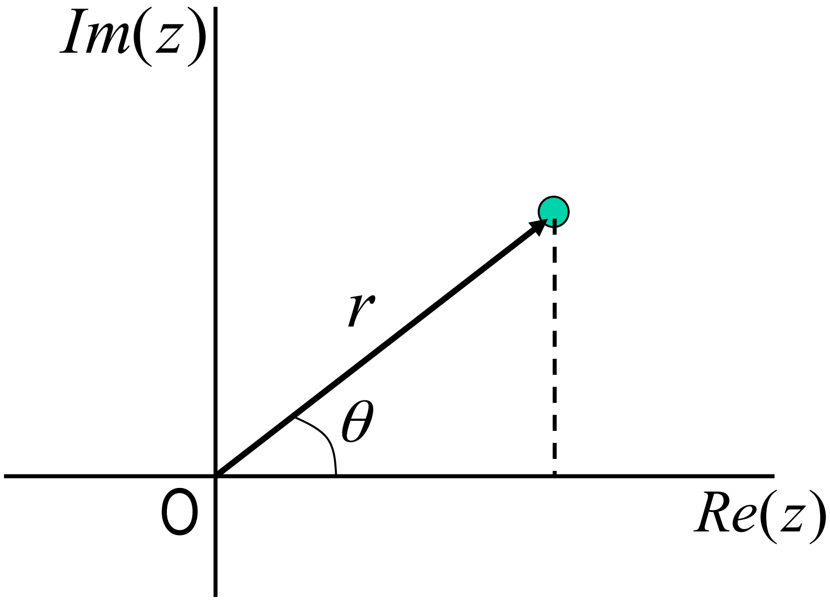


Polar Coordinate Expression

$(x,\ y) \quad z = x + iy$: Cartesian coordinate expression

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$

$(r,\ \theta) \quad z = r(\cos\theta + i\sin\theta)$: Polar coordinate expression



Modulus

$$r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$$

Argument

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Product of Complex Numbers

Two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Product

$$\boxed{z_1 z_2 = r_1 r_2\left\{\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right\}}$$

In the complex plane:

the modulus of the product is given by the product of two moduli

the argument of the product is given by the summation of two arguments

Example 4 Prove this product formula.

Ans.

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\left\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right\} \\ &= r_1 r_2\left\{\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right\} \end{aligned}$$

Multiplication of " i "

In the case $z_1 = z, \quad z_2 = i$

$$z = r(\cos\theta + i\sin\theta)$$

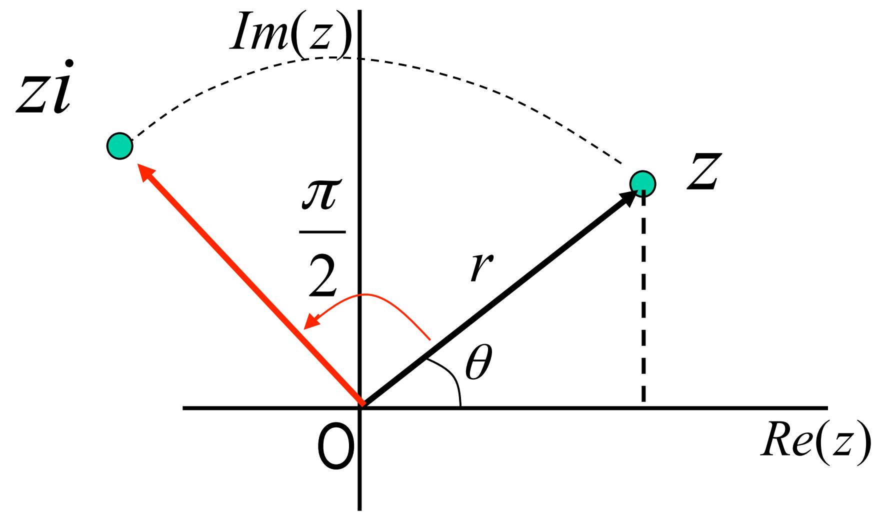
$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Product

$$\boxed{zi = r\left\{\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right\}}$$

That makes sense!

> Multiplying " i " to z means the rotation of z by 90° without changing its modulus



De Moivre's Formula

From the product rule, we have

$$z^2 = r(\cos\theta + i\sin\theta)\,r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

Similarly,

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta)\,r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)$$

$$\bullet\bullet\ \bullet\bullet\ \bullet\bullet$$

This suggests the following formula

De Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- This connects complex numbers and trigonometry
- This produces formulae for $\cos n\theta$ and $\sin n\theta$. (Compare after expanding the left hand side.)

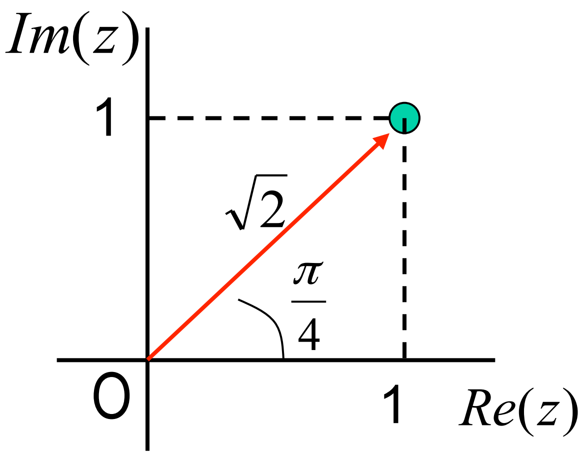
Example

Example 5. Find the following value. $(1+i)^8$

Ans.

From the position in the complex plane

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$



From De Moivre's Formula

$$(1+i)^8=\sqrt{2}^{\,8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^8=16\left(\cos\frac{8\pi}{4}+i\sin\frac{8\pi}{4}\right)$$
$$=16(\cos 2\pi+i\sin 2\pi)=16$$

Exercise

Exercise 2. Find the following value. $\left(\sqrt{3}+i\right)^{8}$

Ans.

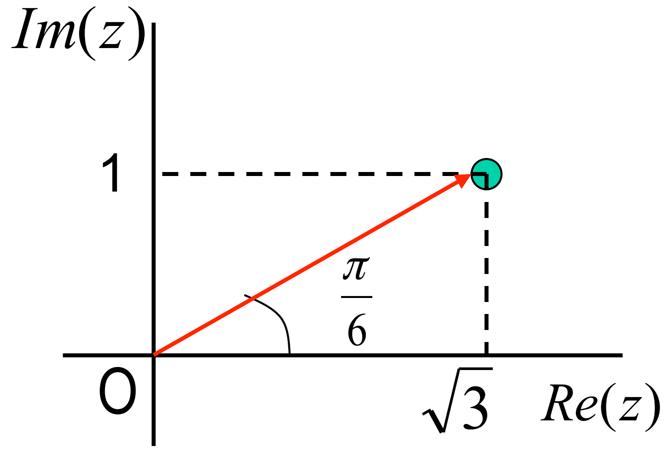
Pause the video and solve the problem.

Answer to the Exercise

Exercise 2. Find the following value. $\left(\sqrt{3}+i\right)^8$

Ans.

From the position in the complex plane



$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

From De Moivre's Formula

$$\left(\sqrt{3}+i\right)^8 = 2^8\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^8 = 256\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

$$= 256\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = -128-128\sqrt{3}i$$

Exercise

Exercise 3. Find the coefficients A, B, C in the following equalities.

$$\cos 5\theta = A\cos^5\theta + B\cos^3\theta + C\cos\theta$$

[Note] Utilize the expansion

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + x^5$$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 3. Find the coefficients A, B, C in the following equalities.

$$\cos 5\theta = A\cos^5\theta + B\cos^3\theta + C\cos\theta$$

[Note] Utilize the expansion

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + x^5$$

Ans. From De Moivre's Formula $(\cos\theta + i\sin\theta)^5 = (\underline{\cos 5\theta} + i\sin 5\theta)$

From the given expansion

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= (\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta) + 10(\cos\theta)^3(i\sin\theta)^2 \\ &\quad + 10(\cos\theta)^2(i\sin\theta)^3 + 5(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= \underline{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \\ &\quad + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

From comparing the real parts

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

$\therefore\ A = 16,\quad B = -20,\quad C = 5$