NAGOYA UNIVERSITY

Lesson 3
Linear and Quadratic Inequalities

3A

- Inequalities of numbers
- Linear inequalities

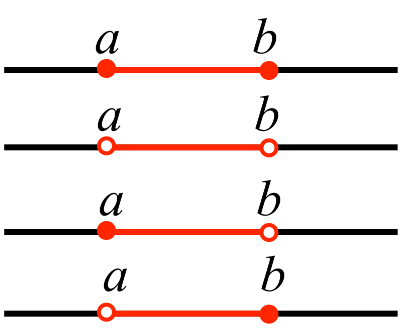
Inequality signs

$a < b$	means	a is less than b
$a \le b$		a is less than or equal to b
$a > b$		a is greater than b
$a \ge b$		a is greater than or equal to b

Intervals

A (real) interval is a set of real number that lies between two numbers .

Closed interval	$[a, b]$	$\{x \in R : a \le x \le b\}$
Open interval	(a, b)	$\{x \in R : a < x < b\}$
Half-open interval	$[a, b)$	$\{x \in R : a \le x < b\}$
Half-open interval	$(a, b]$	$\{x \in R : a < x \le b\}$



number line

Some Properties of Inequalities

1. Transitivity If $a > b$ and $b > c$, then $a > c$.

2. Addition If $a > b$, then $a + c > b + c$.

3. Subtraction If $a > b$, then $a - c > b - c$.

4. Multiplication and Division

If $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

If $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

From the third property, we can derive the following by putting $b = c$.

If $a > c$, then $a - c > 0$.

Arithmetic Mean and Geometric Mean

Example 1. Prove the following inequality $\dfrac{a+b}{2} \geq \sqrt{ab} \quad (\ a \geq 0,\ b \geq 0)$

Ans.

$$\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{\sqrt{a}^{2} + \sqrt{b}^{2} - 2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a} - \sqrt{b}\right)^{2}}{2} \geq 0$$

Therefore $\dfrac{a+b}{2} \geq \sqrt{ab}$

Equality holds when $a = b$.

[Note]

$\dfrac{a+b}{2}$: Arithmetic mean

$\sqrt{ab}$: Geometric mean

Example

$\dfrac{12+3}{2} = 7.5 \qquad \sqrt{12 \times 3} = 6$

$\dfrac{8+7}{2} = 7.5 \qquad \sqrt{8 \times 7} = 7.48$

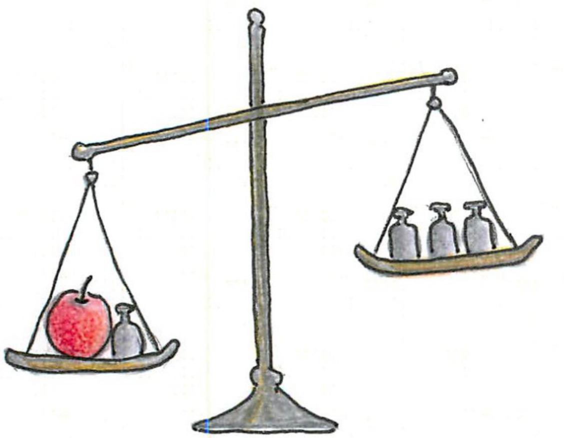
$\dfrac{7.5+7.5}{2} = 7.5 \qquad \sqrt{7.5 \times 7.5} = 7.5$

Linear Inequality

Linear Inequality

One balance weight has 100g. Let the weight of the apple be x . Then we have

$$x + 100 > 3 \times 100 \quad \therefore x > 200$$



Linear inequality $\boxed{ax + b > cx + d}$

$\rightarrow \; ax - cx > d - b$

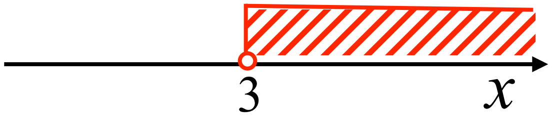
$\rightarrow$ Divide by $(a - c)$ but be careful of its sign.

Example 1. Solve the following inequality $4x - 2 > 10$

Ans. $4x - 2 > 10$

$\therefore \quad 4x > 12 \qquad \leftarrow$ add 2 to both sides

$\therefore \quad x > 3 \qquad \leftarrow$ divide by 4



The inequality in Example 1

$$4x - 12 > 0$$

Corresponding to this, we consider

$$y = 4x - 12$$

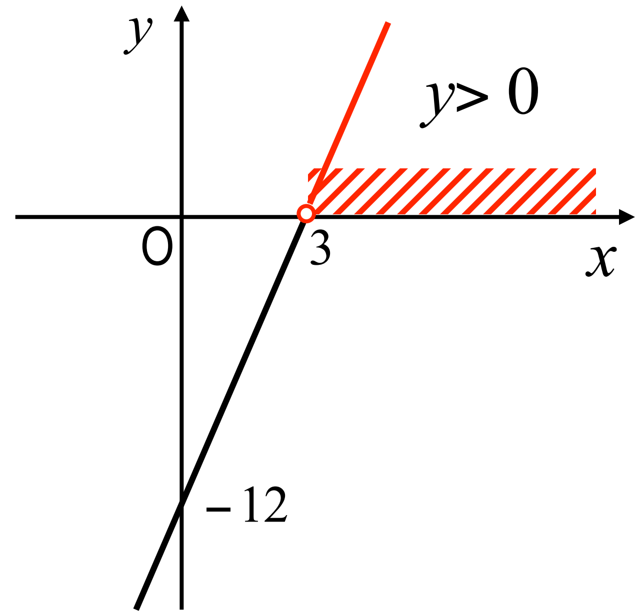
and illustrate this in the $x - y$ plane.

The x -intercept is $x = 3$.

The domain corresponding to $y > 0$ is $x > 3$

Therefore,

the solution. $x > 3$



Simple!

Exercise 1 Solve the following double inequality $\begin{cases} 7x - 1 \geq 4x - 7 \\ x + 5 > 3(1 + x) \end{cases}$

Ans.

Pause the video and solve the problem.

Exercise 1 Solve the following double inequality

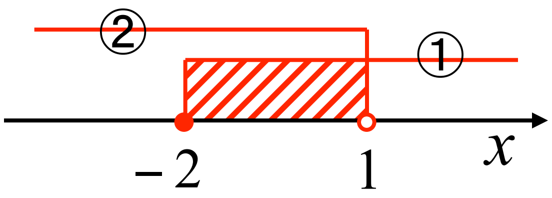
$$7x - 1 \geq 4x - 7$$
$$x + 5 > 3(1 + x)$$

Ans. The first inequality

$$7x - 1 \geq 4x - 7 \quad \therefore\ 3x \geq -6 \quad \therefore\ x \geq -2 \quad \cdot\cdot(1)$$

The second equation

$$x + 5 > 3(1 + x) \quad \therefore -2x > -2 \quad \therefore\ x < 1 \quad \cdot\cdot(2)$$



The intersection of the two solutions

$$-2 \leq x < 1$$

Lesson 3
Linear and Quadratic Inequalities

3B

- Quadratic Functions and Roots
- Quadratic Inequalities

Equations and Graphs of Functions

Quadratic Inequality

After rearrangement, quadratic inequality has the following standard form

$$ax^2 + bx + c > 0$$

$\geq, \quad <, \quad \leq$

[Review] Quadratic Functions and Roots

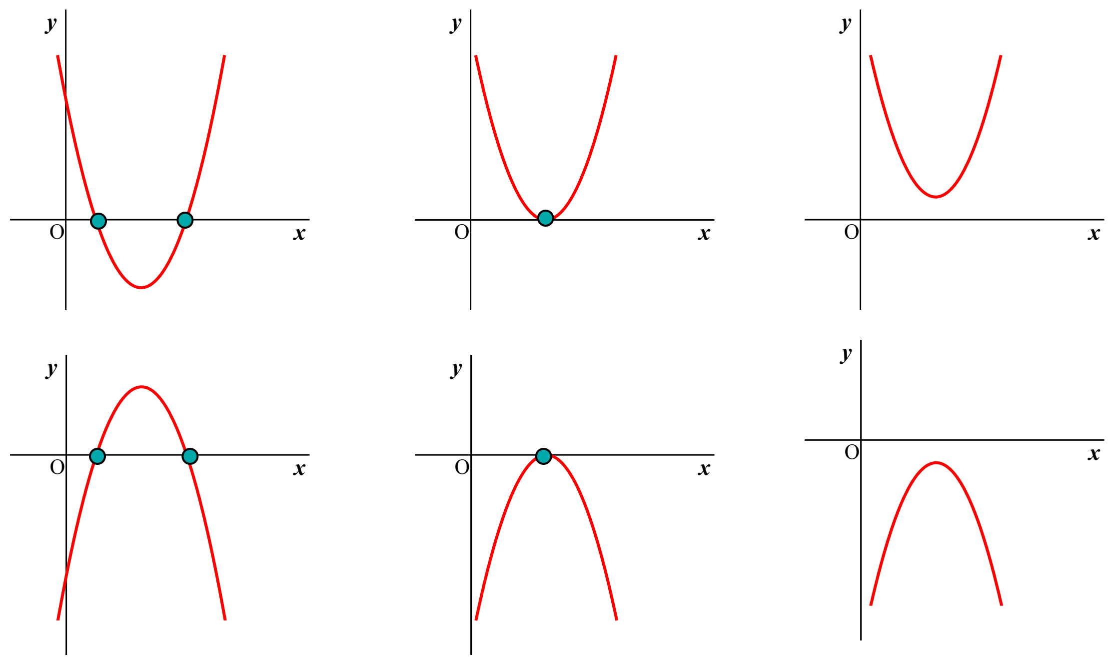
$D = b^2 - 4ac > 0$ — Two real roots

$D = b^2 - 4ac = 0$ — Double root

$D = b^2 - 4ac < 0$ — No real root

Case of $a > 0$

Case of $a < 0$



Steps to Solve Quadratic Inequalities

Step 1. Rearrange the inequality to the standard form

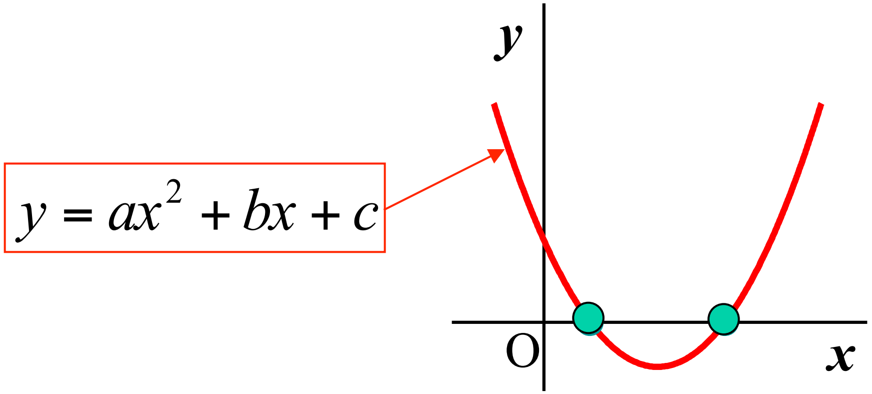
$$ax^2 + bx + c > 0$$

Step 2. Illustrate the corresponding quadratic function

$$y = ax^2 + bx + c = a(x - p)^2 + q$$

Step 3. Solve the quadratic equation $ax^2 + bx + c = 0$ and find its roots α and β .

Step 4. Find the sign of y in each interval divided by α and β, and select the intervals which satisfy the inequality $ax^2 + bx + c > 0$.



Case of $a > 0$ and $D = b^2 - 4ac > 0$

Example 2 Solve the inequality $x^2 - 4x + 3 > 0$

Ans.

The standard form $y = (x-2)^2 - 1$

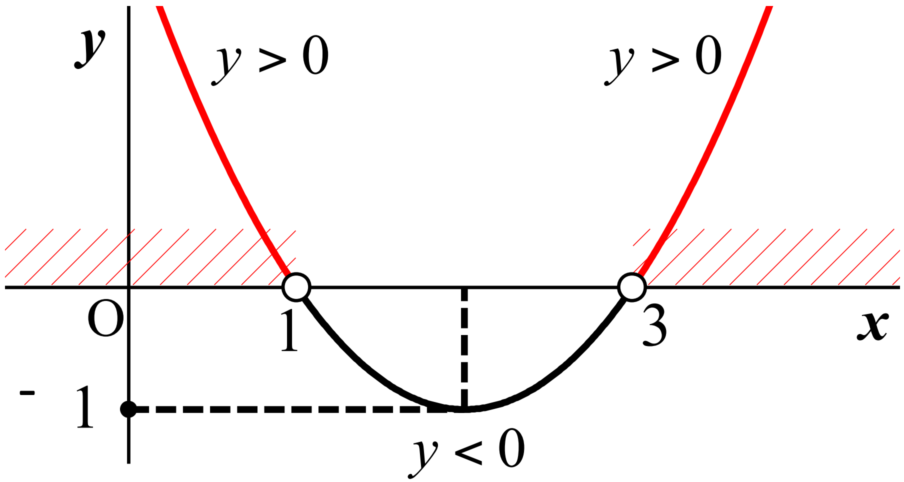
By factoring, we have $y = (x-1)(x-3)$

therefore, the roots are $x = 1,\ x = 3$

The inequality is satisfied in the shaded domain.

The solution is

$$x < 1, \quad x > 3$$



Case of $a > 0$ and $D = b^2 - 4ac = 0$

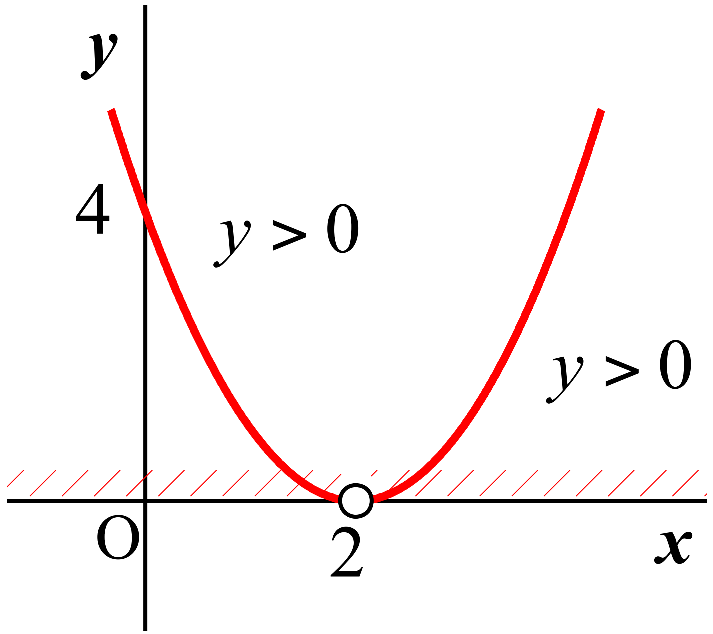
Example 3 Solve the inequality $x^2 - 4x + 4 > 0$.

Ans.

The standard form $y = (x-2)^2$

The graph has one contact point at $x = 2$.

Therefore, the answer is all real number except $x = 2$



Case of $a > 0$ and $D = b^2 - 4ac < 0$

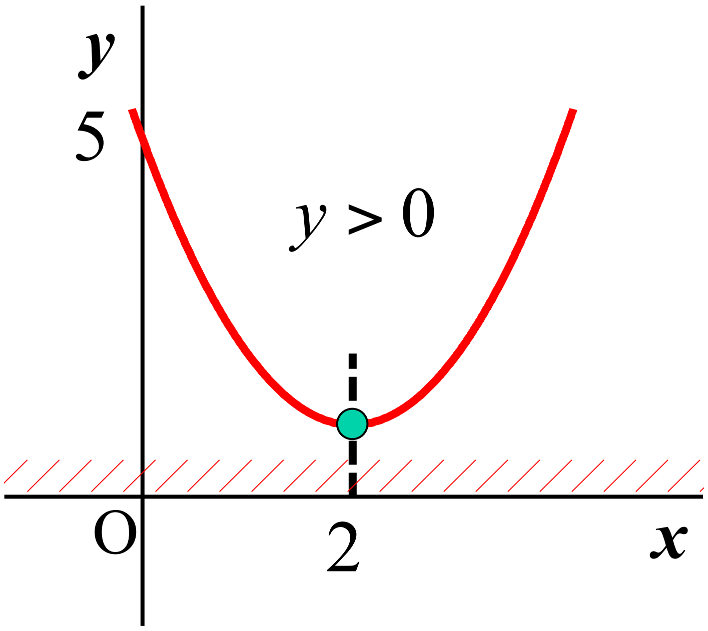
Example 4 Solve the inequality $x^2 - 4x + 5 > 0$

Ans.

The standard form $y = (x-2)^2 + 1$

The graph has no contact point

The solution of this inequality is all real numbers.



Exercise

Exercise 2. Solve the following inequalities.

(1) $-x^2 - 2x + 2 < 0$ (2) $x^2 + x + 2 < 0$

Pause the video and solve the problem.

Answer to the Exercise

Exercise 2. Solve the following inequalities.

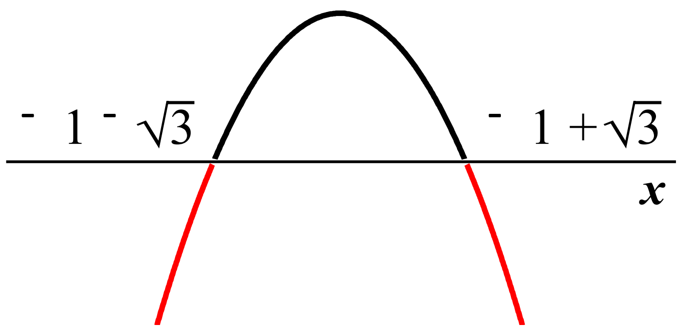
(1) $-x^2 - 2x + 2 < 0$ (2) $x^2 + x + 2 < 0$

Ans. (1) The corresponding quadratic equation :

$$-x^2 - 2x + 2 = 0$$

The roots :

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(2)}}{2(-1)} = -1 \pm \sqrt{3}$$



From the figure

$$x < -1 - \sqrt{3}, \quad x > -1 + \sqrt{3}$$

(2) $D = 1^2 - 4 \times 1 \times 2 = -7 < 0$

The graph $y = x^2 + x + 2$ does not cross with the x -axis.

A parabola opening upward.

Therefore, there is no solution.

Exercise

Exercise 3. Solve the following simultaneous inequalities.

$$x^2 + 4x + 3 > 0$$

$$2x^2 + x - 6 \leq x^2 + 2x$$

Pause the video and solve the problem.

Answer to the Exercise

Exercise 3. Solve the following simultaneous inequalities.

$$x^2 + 4x + 3 > 0$$

$$2x^2 + x - 6 \le x^2 + 2x$$

Ans. The first equation is

$$x^2 + 4x + 3 = (x+3)(x+1) > 0$$

The solutions are $x < -3, \quad x > -1$

The second equation is

$$(2x^2 + x - 6) - (x^2 + 2x) = x^2 - x - 6 = (x+2)(x-3) \le 0$$

Whose solution lies in the interval $-2 \le x \le 3$

From the figure, we have

$$-1 < x \le 3$$