Chapter 9

Cyclic cohomology

In Section 3.4 we have shown that any bounded trace τ on a C^*-algebra $\mathcal{C}$ naturally defines a group morphism $K_0(\tau) : K_0(\mathcal{C}) \to \mathbb{C}$ satisfying for any $p \in \mathcal{P}_\infty(\mathcal{C})$

$$K_0(\tau)([p]_0) = \tau(p).$$

However, many important C^*-algebras do not possess such a bounded trace, and one still would like to extract some numerical invariants from their K-groups. One solution to this problem can be obtained by using cyclic cohomology, which is the subject of this chapter. Our main references are the books [Con94, Chap. III], [Kha13, Chap. 3], [MN08, Chap. 5] and the paper [KS04].

9.1 Basic definitions

Let $\mathcal{A}$ be a complex associative algebra, and for any $n \in \mathbb{N}$ let $C^n(\mathcal{A})$ denote the set of $(n+1)$-linear functionals on $\mathcal{A}$. The elements $\eta \in C^n(\mathcal{A})$ are called *n-cochains*.

Definition 9.1.1. *An element $\eta \in C^n(\mathcal{A})$ is* cyclic *if it satisfies for each $a_0, \dots, a_n \in \mathcal{A}$:*

$$\eta(a_1, \dots, a_n, a_0) = (-1)^n \eta(a_0, \dots, a_n) \ .$$

The set of all cyclic n-cochains is denoted by $C^n_\lambda(\mathcal{A})$.

For any $\eta \in C^n(\mathcal{A})$ let us also define $\mathtt{b}\eta$ by

$$[\mathtt{b}\eta](a_0, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \eta(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \eta(a_{n+1}a_0, \dots, a_n). \tag{9.1}$$

Then one has:

Lemma 9.1.2. *The space of cyclic cochains is invariant under the action of* $\mathtt{b}$, i.e. *for any $n \in \mathbb{N}$ one has*

$$\mathtt{b}\, C^n_\lambda(\mathcal{A}) \subset C^{n+1}_\lambda(\mathcal{A}).$$

Proof. Define the operator $\lambda : C^n(\mathcal{A}) \to C^n(\mathcal{A})$ and $\mathtt{b}' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ by

$$\big(\lambda\eta\big)(a_0,\dots,a_n) := (-1)^n\eta(a_n,a_0,\dots,a_{n-1}), \tag{9.2}$$

$$\big(\mathtt{b}'\eta\big)(a_0,\dots,a_{n+1}) := \sum_{j=0}^{n}(-1)^j\eta(a_0,\dots,a_ja_{j+1},\dots,a_{n+1}). \tag{9.3}$$

One readily checks that $(\mathbf{1}-\lambda)\mathtt{b} = \mathtt{b}'(\mathbf{1}-\lambda)$ and that

$$C^n_\lambda(\mathcal{A}) = \mathsf{Ker}(\mathbf{1}-\lambda),$$

which imply the statement of the lemma. □

As a consequence of the previous lemma one can consider the complex

$$C^0_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} C^1_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} C^2_\lambda(\mathcal{A}) \xrightarrow{\mathtt{b}} \dots$$

which is called *the cyclic complex of* $\mathcal{A}$. Note that the property $\mathtt{b}^2 = 0$, necessary for the next definition, is quite standard in this framework and can be checked quite easily.

Definition 9.1.3. *(i) An element $\eta \in C^n_\lambda(\mathcal{A})$ satisfying $\mathtt{b}\eta = 0$ is called a* cyclic n-cocycle*, and the set of all cyclic n-cocycles is denoted by $Z^n_\lambda(\mathcal{A})$,*

(ii) An element $\eta \in C^n_\lambda(\mathcal{A})$ with $\eta \in \mathtt{b}\big(C^{n-1}_\lambda(\mathcal{A})\big)$ is called a cyclic n-coboundary*, and the set all cyclic n-coboundaries is denoted by $B^n_\lambda(\mathcal{A})$,*

(iii) The cohomology of the cyclic complex of $\mathcal{A}$ is called the cyclic cohomology of $\mathcal{A}$*, and more precisely $HC^n(\mathcal{A}) := Z^n_\lambda(\mathcal{A})/B^n_\lambda(\mathcal{A})$ for any $n \in \mathbb{N}$. The elements of $HC^n(\mathcal{A})$ are called* the classes of cohomology*.*

Note that a cyclic n-cocycle η is simply a $(n+1)$-linear functional on $\mathcal{A}$ which satisfies the two conditions

$$\mathtt{b}\eta = 0 \quad \text{and} \quad (\mathbf{1}-\lambda)\eta = 0, \tag{9.4}$$

where $\mathtt{b}$ has been introduced in (9.1) and λ in (9.2). Observe also that the equality

$$Z^0_\lambda(\mathcal{A}) = HC^0(\mathcal{A}) \tag{9.5}$$

holds, and that this space corresponds to the set of all traces on $\mathcal{A}$.

Example 9.1.4. *Let us consider the case $\mathcal{A} = \mathbb{C}$. Then, any $\eta \in C^n_\lambda(\mathbb{C})$ is completely determined by its value $\eta(1,\dots,1)$. In addition, by the cyclicity property one infers that $\eta = 0$ whenever n is odd. Hence the cyclic complex of $\mathbb{C}$ is simply given by*

$$\mathbb{C} \to 0 \to \mathbb{C} \to \dots$$

from which one deduces that for any $k \in \mathbb{N}$:

$$HC^{2k}(\mathbb{C}) \cong \mathbb{C} \quad \textit{and} \quad HC^{2k+1}(\mathbb{C}) = 0.$$

Let us now consider another convenient way of looking at cyclic n-cocycles in terms of characters of a graded differential algebra over $\mathcal{A}$. For that purpose, we first recall that a graded differential algebra $(\Omega, \mathtt{d})$ is a graded algebra

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \dots \ ,$$

with each Ω^j an associative algebra over $\mathbb{C}$, together with a map $\mathtt{d} : \Omega^j \to \Omega^{j+1}$ which satisfy

(i) $w_j w_k \in \Omega^{j+k}$ for any elements $w_j \in \Omega^j$ and $w_k \in \Omega^k$,

(ii) For any homogeneous elements w_1, w_2 and if $\deg(w)$ denotes the degree of a homogeneous element w:

$$\mathtt{d}(w_1 w_2) = (\mathtt{d}w_1)w_2 + (-1)^{\deg(w_1)} w_1(\mathtt{d}w_2), \tag{9.6}$$

(iii) $\mathtt{d}^2 = 0$.

Definition 9.1.5. *A* closed graded trace of dimension n *on the graded differential algebra* $(\Omega, \mathtt{d})$ *is a linear functional* $\int : \Omega^n \to \mathbb{C}$ *satisfying for any* $w \in \Omega^{n-1}$

$$\int \mathtt{d}w = 0, \tag{9.7}$$

and for any $w_j \in \Omega^j$, $w_k \in \Omega^k$ *such that* $j + k = n$*:*

$$\int w_j w_k = (-1)^{jk} \int w_k w_j \ . \tag{9.8}$$

With these definitions at hand, we can now set:

Definition 9.1.6. *A* n-cycle *is a triple* $(\Omega, \mathtt{d}, \int)$ *consisting in a graded differential algebra* $(\Omega, \mathtt{d})$ *together with a closed graded trace* $\int$ *of dimension* n*. This* n*-cycle is* over the algebra $\mathcal{A}$ *if in addition there exists an algebra homomorphism* $\rho : \mathcal{A} \to \Omega^0$.

Given a n-cycle over the algebra $\mathcal{A}$, one defines the corresponding *character* η by

$$\eta(a_0, \dots, a_n) := \int \rho(a_0)\, \mathtt{d}\rho(a_1) \dots \mathtt{d}\rho(a_n) \in \mathbb{C} \tag{9.9}$$

for any $a_0, \dots, a_n \in \mathcal{A}$. The link between such characters and cyclic n-cocycles can now be proved.

Proposition 9.1.7. *Any* n*-cycle* $(\Omega, \mathtt{d}, \int)$ *over the algebra* $\mathcal{A}$ *defines a cyclic* n*-cocycle through its character* η *given by* (9.9)*.*

For simplicity we shall drop the homomorphism $\rho : \mathcal{A} \to \Omega^0$ in the sequel, or equivalently identify $\mathcal{A}$ with $\rho(\mathcal{A})$ in Ω^0.

Proof. With the convention mentioned before, one can rewrite (9.9) as

$$\eta(a_0,\dots,a_n) := \int a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n.$$

By using Leibniz rule as recalled in (9.6) and the graded trace property as mentioned in (9.8) one then gets

$$\begin{aligned}
&\big(\mathtt{b}\eta\big)(a_0,\dots,a_{n+1})\\
&= \int a_0\,a_1\mathtt{d}a_2\dots\mathtt{d}a_{n+1} + \sum_{j=1}^{n}(-1)^j\int a_0\,\mathtt{d}a_1\dots\mathtt{d}(a_ja_{j+1})\dots\mathtt{d}a_{n+1}\\
&\quad + (-1)^{n+1}\int a_{n+1}a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n\\
&= (-1)^n\int a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n\,a_{n+1} + (-1)^{n+1}\int a_{n+1}a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n\\
&= 0.
\end{aligned}$$

On the other hand, by using the closeness property recalled in (9.7) one also gets

$$\begin{aligned}
\big((\mathbf{1}-\lambda)\eta\big)(a_0,\dots,a_n) &= \int a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n - (-1)^n\int a_n\,\mathtt{d}a_0\dots\mathtt{d}a_{n-1}\\
&= (-1)^{n-1}\int \mathtt{d}\big(a_na_0\,\mathtt{d}a_1\dots\mathtt{d}a_{n-1}\big)\\
&= 0.
\end{aligned}$$

Since the two conditions of (9.4) are satisfied, the statement follows. □

Let us end this section with some examples of cyclic cocycles.

Example 9.1.8. *Let $\mathcal{A}$ be a complex and associative algebra, and let τ be a linear functional on $\mathcal{A}$ with the tracial property,* i.e. *$\tau(ab)=\tau(ba)$ for any $a,b\in\mathcal{A}$. Assume also that $\delta:\mathcal{A}\to\mathcal{A}$ is a derivation on $\mathcal{A}$ such that $\tau\big(\delta(a)\big)=0$ for any $a\in\mathcal{A}$. Then the map*

$$\mathcal{A}\times\mathcal{A}\ni(a_0,a_1)\mapsto\tau\big(a_0\,\delta(a_1)\big)\in\mathbb{C}$$

is a cyclic 1-cocycle on $\mathcal{A}$.

More generally, if $\delta_1,\dots,\delta_n$ are mutually commuting derivations on $\mathcal{A}$ which satisfy $\tau\big(\delta_j(a)\big)=0$ for any $a\in\mathcal{A}$ and $j\in\{1,2,\dots,n\}$, then for any $a_0,\dots,a_n\in\mathcal{A}$

$$\eta(a_0,a_1,\dots,a_n) := \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\tau\big(a_0\,\delta_{\sigma(1)}(a_1)\dots\delta_{\sigma(n)}(a_n)\big)$$

defines a cyclic n-cocycle on $\mathcal{A}$. Here we have used the standard notation S_n for the permutation group of n elements.

Example 9.1.9. *Let $\mathcal{M}$ be an oriented compact and smooth manifold of dimension n, and set $\mathcal{A} := C^\infty(\mathcal{M})$. Then the map η defined for $f_0, f_1, \dots, f_n \in \mathcal{A}$ by*

$$\eta(f_0, f_1, \dots, f_n) := \int_{\mathcal{M}} f_0 \, \mathtt{d} f_1 \wedge \dots \wedge \mathtt{d} f_n$$

is a cyclic n-cocycle on $\mathcal{A}$.

Example 9.1.10. *In $\mathcal{H} := L^2(\mathbb{R}^n)$ consider the algebra $\mathcal{K}^\infty$ of smooth integral operators defined for any $u \in \mathcal{H}$ by*

$$[Au](x) = \int_{\mathbb{R}^n} a(x,y) u(y) \, \mathrm{d}y$$

for some $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ (the Schwartz space on $\mathbb{R}^n \times \mathbb{R}^n$). Define the derivation $\delta_1, \dots, \delta_{2n}$ on $\mathcal{K}^\infty$ by

$$\delta_{2j}(A) := [X_j, A], \quad \delta_{2j-1}(A) := [D_j, A], \qquad j \in \{1, 2, \dots, n\}$$

with X_j the operator of multiplication by the variable x_j and $D_j := -i\frac{\partial}{\partial x_j}$. Then one easily checks that $\delta_j \delta_k = \delta_k \delta_j$, and that $\mathrm{Tr}\big(\delta_j(A)\big) = 0$, where $\mathrm{Tr}(A) = \int_{\mathbb{R}^n} a(x,x) \, \mathrm{d}x$. Then the map η defined for any $a_0, a_1, \dots, a_{2n} \in \mathcal{K}^\infty$ by

$$\eta(a_0, a_1, \dots, a_{2n}) := \frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{Tr}\big(a_0 \, \delta_{\sigma(1)}(a_1) \dots \delta_{\sigma(2n)}(a_{2n})\big)$$

is a cyclic $2n$-cocycle on $\mathcal{K}^\infty$.

9.2 Cup product in cyclic cohomology

Starting from two classes of cohomology, our aim in this section is to construct a new class of cohomology. The key ingredient is the *cup product* which is going to be defined below.

First of all, let $(\Omega, \mathtt{d}, \int)$ be a n-cycle over $\mathcal{A}$ and let $(\Omega', \mathtt{d}', \int')$ be a n'-cycle over a second complex associative algebra $\mathcal{A}'$. The corresponding algebra homomorphisms are denoted by $\rho : \mathcal{A} \to \Omega^0$ and $\rho' : \mathcal{A}' \to \Omega'^0$. The graded differential tensor product algebra $\big(\Omega \otimes \Omega', \mathtt{d} \otimes \mathtt{d}'\big)$ is then defined by

$$(\Omega \otimes \Omega')^\ell := \oplus_{j+k=\ell} \, \Omega^j \otimes \Omega'^k$$

and

$$\mathtt{d} \otimes \mathtt{d}'(w \otimes w') := (\mathtt{d} w) \otimes w' + (-1)^{\deg(w)} w \otimes (\mathtt{d}' w').$$

We also set for $w \in \Omega^n$ and $w' \in \Omega'^{n'}$

$$\int^{\otimes} \omega \otimes \omega' := \int \omega \int' \omega'$$

which is a closed graded trace of dimension $n+n'$ on $\big(\Omega\otimes\Omega',\mathtt{d}\otimes\mathtt{d}'\big)$. Finally, the map

$$\rho\otimes\rho' : \mathcal{A}\otimes\mathcal{A}' \to (\Omega\otimes\Omega')^0 \equiv \Omega^0\otimes\Omega'^0$$

defines an algebra homomorphism which makes $\big(\Omega\otimes\Omega',\mathtt{d}\otimes\mathtt{d}',\int^{\otimes}\big)$ a $(n+n')$-cycle over $\mathcal{A}\otimes\mathcal{A}'$.

In a vague sense, the above construction associates with the two characters η and η', defined by (9.9), a new character obtained from the $(n+n')$-cycle over $\mathcal{A}\otimes\mathcal{A}'$. Since characters define cyclic n-cocyles, one has obtained a new $(n+n')$-cocycle in terms of a n-cycle and a n'-cycle. However, a deeper result can be obtained.

For that purpose, let us now denote by $\big(\Omega(\mathcal{A}),\mathtt{d}\big)$ the universal graded differential algebra over $\mathcal{A}$. We shall not recall its construction here, but refer to [Con85, p. 98-99], [Con94, p. 185-186] or [GVF01, Sec. 8.1]. For the time being, let us simply mention that any $(n+1)$-linear functional η on $\mathcal{A}$, *i.e.* any $\eta\in C^n(\mathcal{A})$, defines a linear functional $\hat\eta$ on $\Omega(\mathcal{A})^n$ by the formula

$$\hat\eta(a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n) := \eta(a_0,\dots,a_n) \qquad \forall a_0,\dots,a_n\in\mathcal{A}. \tag{9.10}$$

Also, in terms of $\Omega(\mathcal{A})$ a generalization of Proposition 9.1.7 reads:

Proposition 9.2.1. *Let η be a $(n+1)$-linear functional on $\mathcal{A}$. Then the following conditions are equivalent:*

(i) There is a n-cycle $(\Omega,\mathtt{d},\int)$ over $\mathcal{A}$, with $\rho:\mathcal{A}\to\Omega^0$ the corresponding algebra homomorphism, such that

$$\eta(a_0,\dots,a_n) := \int\rho(a_0)\,\mathtt{d}\rho(a_1)\dots\mathtt{d}\rho(a_n) \qquad \forall a_0,\dots,a_n,$$

(ii) There exists a closed graded trace $\hat\eta$ of dimension n on $\big(\Omega(\mathcal{A}),\mathtt{d}\big)$ such that

$$\eta(a_0,\dots,a_n) = \hat\eta(a_0\,\mathtt{d}a_1\dots\mathtt{d}a_n) \qquad \forall a_0,\dots,a_n,$$

(iii) η is a cyclic n-cocycle.

Note that the proof of this proposition is not very difficult, once a good description of $\big(\Omega(\mathcal{A}),\mathtt{d}\big)$ has been provided, see [Con94, Prop. III.1.4].

Let us now come to the cup product. In general, the graded differential algebra $\big(\Omega(\mathcal{A}\otimes\mathcal{A}'),\mathtt{d}\big)$ and the graded differential tensor product algebra $\big(\Omega(\mathcal{A})\otimes\Omega(\mathcal{A}'),\mathtt{d}\otimes\mathtt{d}'\big)$ are not equal. However, from the universal property of $\big(\Omega(\mathcal{A}\otimes\mathcal{A}'),\mathtt{d}\big)$ one infers that there exists a natural homomorphism

$$\pi:\Omega(\mathcal{A}\otimes\mathcal{A}')\to\Omega(\mathcal{A})\otimes\Omega(\mathcal{A}').$$

This homomorphism plays a role in the definition of the cup product.

For any $\eta \in C^n(\mathcal{A})$ and $\eta' \in C^{n'}(\mathcal{A}')$, recall that $\hat{\eta}$ and $\widehat{\eta'}$ are respectively linear functionals on $\Omega(\mathcal{A})^n$ and on $\Omega(\mathcal{A}')^{n'}$. One then defines the cup product $\eta \# \eta' \in C^{n+n'}(\mathcal{A} \otimes \mathcal{A}')$ by the equality

$$\widehat{(\eta \# \eta')} = \big(\hat{\eta} \otimes \widehat{\eta'}\big) \circ \pi.$$

Some properties of this product are gathered in the following statement.

Proposition 9.2.2. *(i) The cup product $\eta \otimes \eta' \mapsto \eta \# \eta'$ defines a homomorphism*

$$HC^n(\mathcal{A}) \otimes HC^{n'}(\mathcal{A}') \to HC^{n+n'}(\mathcal{A} \otimes \mathcal{A}'),$$

(ii) The character of the tensor product of two cycles is the cup product of their characters.

The proof is provided in [Con94, Thm. III.1.12]. Let us however mention that its main ingredients are the equivalences recalled Proposition 9.2.1 as well as the subsequent diagram (for the point *(ii)*). For it, let us consider again a n-cycle $\big(\Omega, \mathtt{d}, \int\big)$ over $\mathcal{A}$ and a n'-cycle $\big(\Omega', \mathtt{d}', \int'\big)$ over $\mathcal{A}'$, with respective algebra homomorphisms $\rho : \mathcal{A} \to \Omega^0$ and $\rho' : \mathcal{A}' \to \Omega'^0$. Since the universal graded differential algebra $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ is generated by $\mathcal{A}$, there exists a unique extension of ρ to a morphism of differential graded algebras

$$\tilde{\rho} : \Omega(\mathcal{A}) \to \Omega.$$

As a consequence, there also exist two additional algebra homomorphisms

$$\widetilde{\rho'} : \Omega(\mathcal{A}') \to \Omega' \quad \text{and} \quad \widetilde{\rho \otimes \rho'} : \Omega(\mathcal{A} \otimes \mathcal{A}') \to \Omega \otimes \Omega'.$$

Then, by the universal property of $\Omega(\mathcal{A} \otimes \mathcal{A}')$ the following diagram is commutative:

$$\begin{tikzcd}
\Omega(\mathcal{A} \otimes \mathcal{A}') \arrow[r, "\pi"] \arrow[dr, "\widetilde{\rho\otimes\rho'}"'] & \Omega(\mathcal{A}) \otimes \Omega(\mathcal{A}') \arrow[d, "\tilde{\rho}\otimes\widetilde{\rho'}"] \\
& \Omega \otimes \Omega' .
\end{tikzcd}$$

Example 9.2.3. *Let τ be a trace on an algebra $\mathcal{B}$,* i.e. *$\tau \in HC^0(\mathcal{B})$. Then the map*

$$HC^n(\mathcal{A}) \ni \eta \mapsto \eta \# \tau \ni HC^n(\mathcal{A} \otimes \mathcal{B})$$

is explicitly given on product elements by

$$[\eta \# \tau](a_0 \otimes b_0, \dots, a_n \otimes b_n) = \eta(a_0, \dots, a_n)\, \tau(b_0 b_1 \dots b_n).$$

As a special example, let $\mathcal{B} = M_k(\mathbb{C})$ and let $\tau = \mathrm{tr}$ be the usual traces on matrices. In this case, the cup product defines a map

$$HC^n(\mathcal{A}) \ni \eta \mapsto \eta \# \mathrm{tr} \in HC^n\big(M_k(\mathcal{A})\big).$$

On more general elements $A^0, \dots, A^n \in M_k(\mathcal{A})$ and if one denotes by A^ℓ_{ij} the component (i,j) of A^ℓ for $\ell \in \{0, 1, \dots, n\}$, then one has

$$[\eta \# \mathrm{tr}](A^0, A^1, \dots, A^n) = \sum_{j_0, \dots, j_n = 1}^{k} \eta\big(A^0_{j_0 j_1}, A^1_{j_1 j_2}, \dots, A^n_{j_n j_0}\big).$$

We end this section with the introduction of the periodicity operator S. This operator is obtained by considering the special case $\mathcal{A}' = \mathbb{C}$ in the above construction. As mentioned in Example 9.1.4, any cyclic 2-cocycle η' on $\mathbb{C}$ is uniquely defined by $\eta'(1,1,1)$ which we chose to be equal to 1[1]. Since $\mathcal{A} \otimes \mathbb{C} = \mathcal{A}$ one gets a map of degree 2 in cyclic cohomology, commonly denoted by S:

$$S : HC^n(\mathcal{A}) \ni \eta \mapsto S\eta := \eta \# \eta' \ \in HC^{n+2}(\mathcal{A}).$$

A computation involving the universal graded differential algebra $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ as presented in [Con94, Corol. III.1.13] leads then to the explicit formula

$$\begin{aligned}
[S\eta](a_0, a_1, \dots, a_{n+2}) = {} & \hat{\eta}\big(a_0\, a_1\, a_2\, \mathtt{d}a_3 \dots \mathtt{d}a_{n+2}\big) + \hat{\eta}\big(a_0\, \mathtt{d}a_1\, (a_2\, a_3)\, \mathtt{d}a_4 \dots \mathtt{d}a_{n+2}\big) \\
& + \sum_{i=3}^{n} \hat{\eta}\big(a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_{i-1}\, (a_i\, a_{i+1})\, \mathtt{d}a_{i+2} \dots \mathtt{d}a_{n+2}\big) \\
& + \hat{\eta}\big(a_0\, \mathtt{d}a_1 \dots \mathtt{d}a_n\, (a_{n+1}\, a_{n+2})\big),
\end{aligned}$$

where $\hat{\eta}$ is the linear functional on $\Omega(\mathcal{A})^n$ defined by η.

Finally, the following result can be checked by a direct computation.

Lemma 9.2.4. *For any cocycle $\eta \in Z^n_\lambda(\mathcal{A})$, $S\eta$ is a coboundary in $C^{n+2}_\lambda(\mathcal{A})$,* i.e. $S\eta = \mathtt{b}\psi$ *for $\psi \in C^{n+1}_\lambda(\mathcal{A})$ given by*

$$\psi(a_0, \dots, a_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j-1} \hat{\eta}\big(a_0 (\mathtt{d}a_1 \dots \mathtt{d}a_{j-1})\, a_j (\mathtt{d}a_{j+1} \dots \mathtt{d}a_{n+1})\big).$$

9.3 Unbounded derivations

In the previous two sections, only algebraic manipulations were considered, and it was not necessary for $\mathcal{A}$ to have a topology. We shall now consider topological vector spaces and topological algebras, the C^*-condition will appear only at the end of the game. However, interesting cocycles are often only defined on dense subalgebras, and this naturally leads us to the definition of unbounded operators/derivations and unbounded traces. In this section we focus on the former ones.

[1] The conventions about the normalization differ from one reference to another one. For example, in [Con94] it is assumed that $\eta'(1,1,1) = 1$ but in [Con85] the convention $\eta'(1,1,1) = 2\pi i$ is taken.

Given two topological vector spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ over $\mathbb{C}$ [2], a *linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$* consists in a pair $\big(T, \mathsf{Dom}(T)\big)$, where $\mathsf{Dom}(T) \subset \mathcal{B}_1$ is a linear subspace and $T : \mathsf{Dom}(T) \to \mathcal{B}_2$ is a linear map. Note that most of the time one simply speaks about the operator T, but a domain $\mathsf{Dom}(T)$ is always attached to it. This operator is *densely defined* if $\mathsf{Dom}(T)$ is dense in $\mathcal{B}_1$. One also says that this operator is *closable* if the closure of its graph

$$\mathsf{Graph}(T) := \big\{(\xi, T\xi) \in \mathcal{B}_1 \times \mathcal{B}_2 \mid \xi \in \mathsf{Dom}(T)\big\}$$

does not contain an element of the form $(0, \xi')$ with $\xi' \neq 0$. In this case, the closure $\overline{\mathsf{Graph}(T)}$ is equal to $\mathsf{Graph}(\bar{T})$ for a unique linear operator $\bar{T}$ called *the closure of T*.

For the definition of the dual operator, let us assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are locally convex topological vector spaces[3], and let $\mathcal{B}_j^*$ denote the strong dual of $\mathcal{B}_j$[4], see [Yos65, Sec. IV.7] for the details. Then the *dual operator* $\big(T^*, \mathsf{Dom}(T^*)\big)$ of a densely defined unbounded operator $\big(T, \mathsf{Dom}(T)\big)$ is defined as the unbounded operator from $\mathcal{B}_2^*$ to $\mathcal{B}_1^*$ such that

$$\begin{aligned} \mathsf{Dom}(T^*) &:= \big\{\zeta^* \in \mathcal{B}_2^* \mid \exists \xi^* \in \mathcal{B}_1^* \text{ s.t. } \langle T\xi, \zeta^*\rangle = \langle \xi, \xi^*\rangle \quad \forall \xi \in \mathsf{Dom}(T)\big\} \\ T^*\zeta^* &:= \xi^*. \end{aligned}$$

Note that we use the same notation $\langle\cdot,\cdot\rangle$ for various duality relations. Observe also that the equality $\langle \xi, T^*\zeta^*\rangle := \langle T\xi, \zeta^*\rangle$ holds for all $\xi \in \mathsf{Dom}(T)$ and $\zeta^* \in \mathsf{Dom}(T^*)$. We finally mention that the dual operator of a densely defined operator T is always closed. In the context of Banach spaces, the dual operator is often called *the adjoint operator*.

Let us now define the general notion of a derivation, and then consider more precisely unbounded derivations.

Definition 9.3.1. *Let $\mathcal{B}_1$ be a Banach algebra and let $\mathcal{B}_2$ be a topological vector space which is also a $\mathcal{B}_1$-bimodule*[5]*. Then, an unbounded operator δ from $\mathcal{B}_1$ to $\mathcal{B}_2$ is called* a derivation *if $\mathsf{Dom}(\delta)$ is a subalgebra of $\mathcal{B}_1$ and if for any $a, b \in \mathsf{Dom}(\delta)$:*

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The following result is then standard, see [Con94, Lem. III.6.2].

[2] A topological vector space X over $\mathbb{C}$ is a vector space which is endowed with a topology such that the addition $X \times X \to X$ and the scalar multiplication $\mathbb{C} \times X \to X$ are continuous.

[3] A locally convex topological vector space is a vector space together with a family of semi-norms $\{p_\mathfrak{a}\}$ which defines its topology. Any Banach space is a locally convex topological vector space.

[4] The strong dual of a locally convex topological vector space X consists in the set of all continuous linear functionals on X endowed with the bounded convergence topology. This topology is defined by the family of semi-norms of the form $p^*(T) \equiv p_B^*(T) := sup_{x\in B}|T(x)|$ with T a continuous linear functional on X and B any bounded set of X.

[5] A $\mathcal{B}$-bimodule is an Abelian group X together with a multiplication on the right and on the left by elements of a ring $\mathcal{B}$ (satisfying some natural conditions) and for which the equality $(a\xi)b = a(\xi b)$ holds for any $a, b \in \mathcal{B}$ and any $\xi \in X$.

Lemma 9.3.2. *Let $\mathcal{B}_1$ be a unital Banach algebra and let $\mathcal{B}_2$ be a Banach $\mathcal{B}_1$-bimodule*[6] *satisfying for any $b, b' \in \mathcal{B}_1$ and $\xi \in \mathcal{B}_2$*

$$\|b\xi b'\| \leq \|b\|\,\|\xi\|\,\|b'\|. \tag{9.11}$$

Let δ be a densely defined and closable derivation from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then its closure $\bar{\delta}$ is still a derivation, and its domain $\mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}_1$ stable under holomorphic functional calculus[7].

Let us illustrate the use of the previous definition with two examples. For that purpose, observe that for any Banach algebra $\mathcal{B}$ its dual space $\mathcal{B}^*$ can be viewed as a $\mathcal{B}$-bimodule by the relation

$$\langle b\xi b', a\rangle := \langle \xi, b'ab\rangle \quad \forall a, b, b' \in \mathcal{B} \text{ and } \xi \in \mathcal{B}^*.$$

Moreover the relation (9.11) holds, namely $\|b\xi b'\| \leq \|b\|\,\|\xi\|\,\|b'\|$.

Lemma 9.3.3. *Let $\mathcal{B}$ be a unital C^*-algebra, and let δ be a densely defined derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule $\mathcal{B}^*$. Assume in addition that $\mathbf{1}_\mathcal{B}$ belongs to $\mathsf{Dom}(\delta^*)$ (in $\mathcal{B}^{**}$). Then,*

- *(i) $\tau := \delta^*(\mathbf{1}_\mathcal{B}) \in \mathcal{B}^*$ defines a trace on $\mathcal{B}$,*
- *(ii) The map $K_0(\tau) : K_0(\mathcal{B}) \to \mathbb{C}$ corresponds to the 0-map.*

Let us mention that this statement holds for more general Banach algebras and the C^*-property does not play any role. However, since the K-groups have been introduced only for C^*-algebras, we restrict the statement to this framework. Unfortunately, the proof below uses the K-groups for more general algebras, and therefore is only partially understandable in our setting.

Proof. i) For any $a, b \in \mathsf{Dom}(\delta)$ one first infers that:

$$\begin{aligned}
\tau(ab) &:= \langle \delta^*(\mathbf{1}_\mathcal{B}), ab\rangle = \langle \mathbf{1}_\mathcal{B}, \delta(ab)\rangle = \langle \mathbf{1}_\mathcal{B}, \delta(a)b\rangle + \langle \mathbf{1}_\mathcal{B}, a\delta(b)\rangle \\
&= \langle b, \delta(a)\rangle + \langle a, \delta(b)\rangle = \langle a, \delta(b)\rangle + \langle b, \delta(a)\rangle \\
&= \langle \mathbf{1}_\mathcal{B}, \delta(b)a\rangle + \langle \mathbf{1}_\mathcal{B}, b\delta(a)\rangle = \langle \mathbf{1}_\mathcal{B}, \delta(ba)\rangle = \langle \delta^*(\mathbf{1}_\mathcal{B}), ba\rangle = \tau(ba).
\end{aligned}$$

[6] A Banach $\mathcal{B}$-bimodule is a Banach space X which is also a $\mathcal{B}$-bimodule with $\mathcal{B}$ a Banach algebra $\mathcal{B}$, and for which the multiplication on the left and on the right also satisfy $\|a\xi\| \leq c\|a\|\|\xi\|$ and $\|\xi b\| \leq c\|b\|\|\xi\|$ for some $c > 0$, all $\xi \in X$ and any $a, b \in \mathcal{B}$.

[7] In order to define the notion of *stable under holomorphic functional calculus*, let us consider a unital Banach algebra $\mathcal{B}$, an element $a \in \mathcal{B}$ and let f be a holomorphic function defined on a neighbourhood O of $\sigma(a)$. We then define

$$f(a) := \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}\,\mathrm{d}z \ \in \mathcal{B}$$

with γ a closed curve of finite length and without self-intersection in O encircling $\sigma(a)$ only once and counterclockwise. By holomorphy of f this integral is independent of the choice of γ, and for any fixed a, the map $f \mapsto f(a)$ defines a functional calculus, called *the holomorphic functional calculus of a*. In this setting, if $\mathcal{A}$ is a unital and dense subalgebra of a unital Banach algebra $\mathcal{B}$ one says that $\mathcal{A}$ is stable under holomorphic functional calculus if $f(a) \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and f is a holomorphic function in a neighbourhood of $\sigma(a)$.

Since τ is a bounded functional on $\mathcal{B}$, one deduces from the density of $\mathsf{Dom}(\delta)$ in $\mathcal{B}$ that the equality $\tau(ab) = \tau(ba)$ holds for any $a, b \in \mathcal{B}$.

ii) Let $a \in \mathsf{Dom}(\delta)$. The equality

$$\langle a, \delta(b)\rangle = \tau(ab) - \langle b, \delta(a)\rangle$$

valid for any $b \in \mathsf{Dom}(\delta)$ shows that $\mathsf{Dom}(\delta) \subset \mathsf{Dom}(\delta^*)$, with $\delta^*(a) = \tau(a\,\cdot) - \delta(a)$ for any $a \in \mathsf{Dom}(\delta)$. Since δ^* is closed operator and $\tau(a\,\cdot)$ is bounded operator, one infers that δ is a closable operator from $\mathcal{B}$ to $\mathcal{B}^*$, whose extension is denoted by $\bar{\delta}$. By Lemma 9.3.2 one deduces that its domain $\mathcal{A} := \mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}$ which is stable under holomorphic functional calculus. A consequence of this stability is that there exists an isomorphism between $K_0(\mathcal{A})$ (which has not been defined in these lecture notes) and $K_0(\mathcal{B})$, see [Con94, III.App.C] and the references mentioned there.

The next step consists in showing that the homomorphism $K_0(\tau)$ corresponds to the 0-map on all elements of $p \in M_n(\mathcal{A})$ satisfying $p^2 = p$. For simplicity, let us choose $n = 1$ (in the general case, use $\tau \otimes \mathrm{tr}$ on $\mathcal{A} \otimes M_n(\mathbb{C})$ instead of τ on $\mathcal{A}$). Then, for any $p \in \mathcal{A}$ satisfying $p^2 = p$ one has

$$\tau(p) = \tau(p^2) = \langle \delta^*(\mathbf{1}_{\mathcal{B}}), p^2\rangle = 2\langle p, \delta(p)\rangle.$$

But one also has

$$\langle p, \delta(p)\rangle = \langle p, \delta(p^2)\rangle = 2\langle p, \delta(p)\rangle$$

from which one infers that $\tau(p) = 0$. Since $p \in \mathcal{A}$ is an arbitrary idempotent, the statement then follows once a suitable description of $K_0(\mathcal{A})$ has been provided. □

The next statement has a similar flavor. It deals with the notion of a 1-trace, and motivates the introduction of more general n-traces in the next section. Again, let us mention that the following statement holds for more general Banach algebras.

Proposition 9.3.4. *Let $\mathcal{B}$ be a unital C^*-algebra, and let δ be a densely defined derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule $\mathcal{B}^*$ satisfying for all $a, b \in \mathsf{Dom}(\delta)$*

$$\langle \delta(a), b\rangle = -\langle \delta(b), a\rangle. \tag{9.12}$$

Then,

(i) δ is closable, with its closure denoted by $\bar{\delta}$,

(ii) There exists a unique map $\varphi : K_1(\mathcal{B}) \to \mathbb{C}$ such that for any $v \in \mathcal{GL}_n\big(\mathsf{Dom}(\bar{\delta})\big)$ one has

$$\varphi([w(v)]_1) = \langle \bar{\delta}(v), v^{-1}\rangle$$

where $w(v) \in \mathcal{U}_n(\mathcal{B})$ is defined by $w(v) = v|v|^{-1}$, as explained in Proposition 2.1.8.

Proof. The result about the closability of δ is standard. It follows from the closeness of δ^* and from the fact that $\delta^* = -\delta$ on $\mathsf{Dom}(\delta)$, which means that δ is a skew-symmetric operator and $-\delta^*$ is a closed extension of it. As a consequence, δ is closable with closure denoted by $\bar{\delta}$.

Since δ is closable, it follows from Lemma 9.3.2 that $\bar{\delta}$ is also a derivation, and its domain $\mathsf{Dom}(\bar{\delta})$ is a subalgebra of $\mathcal{B}$ stable under holomorphic functional calculus. Let us denote by $\mathcal{A}$ this subalgebra. Let us mention two important properties of this stability, and refer to [CMR07, Prop. 2.58 & Thm. 2.60] for a proof of these properties in a more general context: Firstly, if $v_1, v_2 \in \mathcal{GL}_n(\mathcal{A})$ are homotopic in $\mathcal{GL}_n(\mathcal{B})$ then they are also homotopic in $\mathcal{GL}_n(\mathcal{A})$. Secondly, there exists an isomorphism between $K_1(\mathcal{A})$ (which has not been defined in these lecture notes) and $K_1(\mathcal{B})$.

Let us now consider a piecewise affine path $t \mapsto v(t)$ in $\mathcal{GL}_n(\mathcal{A})$ and observe that the function $f(t) := \big\langle \bar{\delta}\big(v(t)\big), v(t)^{-1}\big\rangle$ is constant. Indeed, its derivative with respect to t satisfies

$$\begin{aligned}
f'(t) &= \big\langle \bar{\delta}\big(v'(t)\big), v(t)^{-1}\big\rangle - \big\langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\big\rangle \\
&= -\big\langle \bar{\delta}\big(v(t)^{-1}\big), v'(t)\big\rangle - \big\langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\big\rangle \\
&= \big\langle v(t)^{-1}\bar{\delta}\big(v(t)\big)v(t)^{-1}, v'(t)\big\rangle - \big\langle \bar{\delta}\big(v(t)\big), v(t)^{-1}v'(t)v(t)^{-1}\big\rangle \\
&= 0.
\end{aligned}$$

In other words, the expression $\langle \bar{\delta}(v), v^{-1}\rangle$ is constant on piecewise affine paths in $\mathcal{GL}_n(\mathcal{A})$, and the statement follows once a suitable description of $K_1(\mathcal{A})$ has been provided. □

9.4 Higher traces

We shall now insert the results obtained in the previous section in a more general framework. Before stating the main definition of this section, let us extend slightly the validity of (9.10). First of all, recall that if $\mathcal{A}$ is any associative and unital[8] algebra over $\mathbb{C}$ we set $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ for the corresponding universal graded differential algebra over $\mathcal{A}$. In this setting, the following equality holds for any $a_j, a_k \in \mathcal{A} \subset \Omega(\mathcal{A})^0$

$$(\mathtt{d}a_j)a_k = \mathtt{d}(a_j\, a_k) - a_j\,\mathtt{d}(a_k),$$

from which one infers the equality valid for any $a_0, \dots, a_k, a \in \mathcal{A}$:

$$\begin{aligned}
& a_0\,\mathtt{d}a_1 \dots \mathtt{d}a_k\, a \\
&= a_0\,\mathtt{d}a_1 \dots \mathtt{d}(a_k\, a) - a_0\,\mathtt{d}a_1 \dots \mathtt{d}a_{k-1}\, a_k\,\mathtt{d}a \\
&= (-1)^k a_0\, a_1\,\mathtt{d}a_2 \dots \mathtt{d}a_k\,\mathtt{d}a + \sum_{j=1}^{k-1}(-1)^{k-j} a_0\,\mathtt{d}a_1 \dots \mathtt{d}(a_j\, a_{j+1}) \dots \mathtt{d}a_k\,\mathtt{d}a \\
&\quad + a_0\,\mathtt{d}a_1 \dots \mathtt{d}a_{k-1}\,\mathtt{d}(a_k\, a).
\end{aligned}$$

[8] For simplicity we assume $\mathcal{A}$ to be unital but the following construction also holds in the non-unital case.

For any $(n+1)$-linear functional η on $\mathcal{A}$, the previous equalities allow us to give a meaning to expressions of the form

$$\hat\eta\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)$$

for any $a_0,\dots,a_n,x_1\dots,x_n \in \mathcal{A}$. Indeed, if one first replaces $x_j\,\mathtt{d}a_j$ with $\mathtt{d}(x_j\,a_j) - (\mathtt{d}x_j)a_j$ and then moves a_j to the left with the above equalities, one gets an equality of the form

$$x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n) = \sum_\ell (-1)^{m_\ell} b_{0\ell}\,\mathtt{d}b_{1\ell}\,\mathtt{d}b_{2\ell}\dots\mathtt{d}b_{n\ell}$$

for some $b_{j\ell}\in\mathcal{A}$ and $m_\ell\in\mathbb{N}$. By the linearity of η one can finally set

$$\hat\eta\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) = \sum_\ell (-1)^{m_\ell}\eta(b_{0\ell},b_{1\ell},\dots,b_{n\ell}).$$

For example, for $n=2$ one has

$$\hat\eta\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\big) = \eta(x_1,a_1x_1,a_2) - \eta(x_1a_1,x_2,a_2).$$

With these notations, the main definition of this section then reads:

Definition 9.4.1. *Let $\mathcal{B}$ be a unital Banach algebra and $n\in\mathbb{N}$. A* n-trace *on $\mathcal{B}$ is a cyclic n-cocycle η on a dense subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that for any $a_1,\dots,a_n\in\mathcal{A}$ there exists $c=c(a_1,\dots,a_n)>0$ with*

$$\big|\hat\eta\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)\big| \le c\,\|x_1\|\,\|x_2\|\dots\|x_n\| \qquad \forall x_1,\dots,x_n\in\mathcal{A}. \tag{9.13}$$

Our aim will be to show that when $\mathcal{B}$ is a unital C^*-algebra, any such n-traces on $\mathcal{B}$ determines a map from $K_i(\mathcal{B})$ to $\mathbb{C}$, for $i\in\{0,1\}$. For that purpose, lots of preliminary works are necessary.

Remark 9.4.2. *For fixed $a_1,\dots,a_n$, condition* (9.13) *allows one to extend the multi-linear functional defined on $\mathcal{A}^n$ to a bounded multi-linear functional on $\mathcal{B}^n$. The values taken by this functional on elements of $\mathcal{B}^n$ are obtained by a limiting process. For this extension, we shall freely write $\hat\eta\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)$ with $x_j\in\mathcal{B}$.*

Remark 9.4.3. *Let us observe that the two examples of the previous section fit into the definition of a* 0 *and of a* 1*-trace. Indeed, in the framework of Lemma 9.3.3 one already inferred that $\tau:=\delta^*(\mathbf{1}_\mathcal{B})$ is a trace on $\mathcal{B}$, and thus a cyclic* 0*-cocycle. In addition, the estimate $|\tau(x_1)|\le\|\tau\|\,\|x_1\|$ holds for any $x_1\in\mathcal{B}$. In the framework of Proposition 9.3.4 one can set $\eta(a_0,a_1):=\langle\delta(a_0),a_1\rangle$. By taking the property* (9.12) *into account, as well as the definition of the $\mathcal{B}$-bimodule $\mathcal{B}^*$, one gets for any $a_0,a_1,a_2\in\mathsf{Dom}(\delta)$*

$$\begin{aligned}
[\mathsf{b}\eta](a_0,a_1,a_2) &= \eta(a_0a_1,a_2) - \eta(a_0,a_1a_2) + \eta(a_2a_0,a_1)\\
&= \langle\delta(a_0a_1),a_2\rangle - \langle\delta(a_0),a_1a_2\rangle + \langle\delta(a_2a_0),a_1\rangle\\
&= -\langle\delta(a_2),a_0a_1\rangle - \langle\delta(a_0),a_1a_2\rangle + \langle\delta(a_2)a_0,a_1\rangle + \langle a_2\delta(a_0),a_1\rangle\\
&\quad - \langle\delta(a_2),a_0a_1\rangle - \langle\delta(a_0),a_1a_2\rangle + \langle\delta(a_2),a_0a_1\rangle + \langle\delta(a_0),a_1a_2\rangle\\
&= 0
\end{aligned}$$

from which one easily infers that η is a cyclic 1*-cocycle on $\mathcal{A} := \mathsf{Dom}(\delta)$. In addition, one has for any $a_1, x_1 \in \mathcal{A}$*

$$|\hat{\eta}(x_1 \mathsf{d} a_1)| = |\eta(x_1, a_1)| = |\langle \delta(x_1), a_1 \rangle| = |\langle \delta(a_1), x_1 \rangle| \leq \|\delta(a_1)\| \, \|x_1\|$$

which corresponds to (9.13) *in this special case.*

Let us now denote by E the vector space of all $(2n-1)$-linear functionals

$$\varphi : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n \text{ times}} \times \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{(n-1) \text{ times}} \to \mathbb{C}$$

which are continuous in the $\mathcal{B}$-variables. We endow E with the family of semi-norms $(p_{\mathfrak{a}})_{\mathfrak{a} \in \mathcal{A}^{n-1}}$ defined by

$$p_{\mathfrak{a}}(\varphi) \equiv p_{(a_1, \dots, a_{n-1})}(\varphi) := \sup_{x_j \in \mathcal{B}, \|x_j\| \leq 1} |\varphi(x_1, \dots, x_n, a_1, \dots, a_{n-1})|.$$

With this family of semi-norms, E becomes a locally convex topological vector space. In addition, for any $\varphi \in E$ and any $x \in \mathcal{B}$ we also set

$$\begin{aligned} [x\varphi](x_1, \dots, x_n, a_1, \dots, a_{n-1}) &:= \varphi(x_1 x, x_2, \dots, x_n, a_1, \dots, a_{n-1}), \\ [\varphi x](x_1, \dots, x_n, a_1, \dots, a_{n-1}) &:= \varphi(x_1, x x_2, \dots, x_n, a_1, \dots, a_{n-1}). \end{aligned}$$

Endowed with this additional structure, one has obtained a $\mathcal{B}$-bimodule E. Note finally that for any $\mathfrak{a} \in \mathcal{A}^{n-1}$ one has

$$p_{\mathfrak{a}}(x\varphi) \leq p_{\mathfrak{a}}(\varphi)\|x\| \quad \text{and} \quad p_{\mathfrak{a}}(\varphi x) \leq p_{\mathfrak{a}}(\varphi)\|x\|. \tag{9.14}$$

Based on the previous definition, the following lemma can now be proved. Recall that the notion of a derivation has been introduced in Definition 9.3.1.

Lemma 9.4.4. *Let $\mathcal{B}$ be a unital Banach algebra, $n \in \mathbb{N}$ and let η be a n-trace on $\mathcal{B}$ defined on a dense subalgebra denoted by $\mathcal{A}$. For any $a \in \mathcal{A}$ let*

$$\delta(a) : \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{n \textit{ times}} \times \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{(n-1) \textit{ times}} \to \mathbb{C}$$

be defined by

$$[\delta(a)](x_1, \dots, x_n, a_2, \dots, a_n) := \hat{\eta}\big((x_1 \, \mathsf{d}a)(x_2 \, \mathsf{d}a_2) \dots (x_n \, \mathsf{d}a_n)\big).$$

Then:

(i) δ is a derivation from $\mathcal{B}$ to the $\mathcal{B}$-bimodule E, with $\mathsf{Dom}(\delta) = \mathcal{A}$,

(ii) δ is closable, when $\mathcal{B}$ is endowed with its norm topology and E with the topology of simple convergence.

Proof. i) It clearly follows from the definition of the n-trace that $\delta(a)$ is an element of E. In addition, for any $a, b \in \mathcal{A}$ and since $\mathtt{d}(ab) = (\mathtt{d}a)b + a(\mathtt{d}b)$ one has

$$\begin{aligned}
&[\delta(ab)](x_1, \dots, x_n, a_2, \dots, a_n) \\
&= \hat{\eta}\big((x_1 \, \mathtt{d}(ab))(x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n)\big) \\
&= \hat{\eta}\big((x_1 \, \mathtt{d}a)(b \, x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n)\big) + \hat{\eta}\big((x_1 \, a \, \mathtt{d}b)(x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n)\big) \\
&= [\delta(a)](x_1, b \, x_2, x_3, \dots, x_n, a_2, \dots, a_n) + [\delta(b)](x_1 \, a, x_2, \dots, x_n, a_2, \dots, a_n) \\
&= [\delta(a)b](x_1, \dots, x_n, a_2, \dots, a_n) + [a\delta(b)](x_1, \dots, x_n, a_2, \dots, a_n).
\end{aligned}$$

By setting $\mathsf{Dom}(\delta) := \mathcal{A}$, one infers that δ is a derivation from $\mathcal{B}$ to E.

ii) By the definition of the topologies on $\mathcal{B}$ and E, we have to show that for any $\{a_\nu\} \subset \mathcal{A}$ with $\|a_\nu\| \to 0$ and $\delta(a_\nu) \to \varphi \in E$ in the weak sense[9] as $\nu \to \infty$, then $\varphi = 0$. By density of $\mathcal{A}$ in $\mathcal{B}$, it is sufficient to show that $\varphi(x_1, \dots, x_n, a_2, \dots, a_n) = 0$ for any $x_j, a_j \in \mathcal{A}$.

By assumption on φ and by taking into account the properties of the closed graded trace $\hat{\eta}$ of dimension n on $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ one gets

$$\begin{aligned}
&\big|\varphi(x_1, \dots, x_n, a_2, \dots, a_n)\big| \\
&= \lim_{\nu \to \infty} \big|\hat{\eta}\big((x_1 \, \mathtt{d}a_\nu)(x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n)\big)\big| \\
&= \lim_{\nu \to \infty} \big|\hat{\eta}\big(\mathtt{d}a_\nu (x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n) x_1\big)\big| \\
&= \lim_{\nu \to \infty} \Big|\hat{\eta}\Big(\mathtt{d}\big(a_\nu (x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n) x_1\big)\Big) - \hat{\eta}\Big(a_\nu \, \mathtt{d}\big((x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n) x_1\big)\Big)\Big| \\
&= \lim_{\nu \to \infty} \Big|\hat{\eta}\Big(a_\nu \, \mathtt{d}\big((x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n) x_1\big)\Big)\Big| \\
&\leq c \lim_{\nu \to \infty} \|a_\nu\| \\
&= 0.
\end{aligned}$$

Note that we had to chose $x_j \in \mathcal{A}$ instead of $x_j \in \mathcal{B}$ in order to give a meaning to the expression $\mathtt{d}\big((x_2 \, \mathtt{d}a_2) \dots (x_n \, \mathtt{d}a_n) x_1\big)$. $\square$

Our aim is now to extend the domain of the derivation δ, in a way similar to the one obtained in the less general framework of Lemma 9.3.2. For that purpose, let us call δ-*bounded* a subset X of $\mathcal{A}$ whenever there exist finitely many $c_1, \dots, c_k \in \mathcal{A}$ such that for all $a \in X$:

$$p_{\mathfrak{a}}\big(\delta(a)\big) \leq \sup_{j \in \{1, \dots, k\}} p_{\mathfrak{a}}\big(\delta(c_j)\big) \qquad \forall \mathfrak{a} \in \mathcal{A}^{n-1}.$$

We can then consider the subset $\mathscr{B}$ of $\mathcal{B}$ defined as follows: $a \in \mathscr{B}$ if there exists a δ-bounded sequence $\{a_m\}_m \subset \mathcal{A}$ converging to a in $\mathcal{B}$. In other words, $a \in \mathscr{B}$ if there exist $\{a_m\}_m \subset \mathcal{A}$ and finitely many $c_1, \dots, c_k \in \mathcal{A}$ such that $a_m \to a$ in $\mathcal{B}$ and

$$\sup_m p_{\mathfrak{a}}\big(\delta(a_m)\big) \leq \sup_{j \in \{1, \dots, k\}} p_{\mathfrak{a}}\big(\delta(c_k)\big) \qquad \forall \mathfrak{a} \in \mathcal{A}^{n-1}.$$

[9] Let us emphasize that ν belongs to a directed set, a sequence might not be general enough.

By this assumption the sequence $\delta(a_m)$ is then bounded in E, and for any $x_j, a_j \in \mathcal{A}$ one has

$$\begin{aligned}
&\Big|[\delta(a_m)](x_1,\dots,x_n,a_2,\dots,a_n) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&= \Big|\hat{\eta}\big((x_1\,\mathtt{d}a_m)(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&= \Big| -\hat{\eta}\Big(a_m\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big) + \hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \\
&\le c\,\|a_m - a\|
\end{aligned}$$

which converges to 0 as $m \to \infty$. As a consequence, one gets that $\delta(a_m)$ weakly converges to $\delta(a)$, independently of the choice of the sequence $\{a_m\}_m$, with

$$[\delta(a)](x_1,\dots,x_n,a_2,\dots,a_n) = -\hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big) \qquad \forall x_j, a_j \in \mathcal{A}.$$

Note that by construction one has $p_{\mathfrak{a}}\big(\delta(a)\big) \le \sup_k p_{\mathfrak{a}}\big(\delta(c_k)\big)$ for all $\mathfrak{a} \in \mathcal{A}^{n-1}$.

The following result then holds:

Lemma 9.4.5. *(i) $\mathscr{B}$ is a dense subalgebra of $\mathcal{B}$ containing $\mathcal{A}$,*

(ii) For any $q \in \mathbb{N}^*$, $M_q(\mathscr{B})$ is stable under holomorphic functional calculus.

We only provide the proof of the first statement. For the second one, we refer to [Con86, Lem. 2.3].

Proof. Let $a, a' \in \mathscr{B}$ with two sequences $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{A}$ satisfying $a_n \to a$ and $a'_n \to a'$. Clearly, $a_n a'_n \to aa'$ in $\mathcal{B}$, and in addition one has for any $\mathfrak{a} \in \mathcal{A}^{n-1}$

$$p_{\mathfrak{a}}\big(\delta(a_n a'_n)\big) = p_{\mathfrak{a}}\big(\delta(a_n)a'_n + a_n\delta(a'_n)\big) \le c\Big(p_{\mathfrak{a}}\big(\delta(a_n)\big) + p_{\mathfrak{a}}\big(\delta(a'_n)\big)\Big) \le \sup_{j\in\{1,\dots,k\}} p_{\mathfrak{a}}\big(\delta(c_j)\big)$$

for some $c < \infty$ independent of n and a finite family of elements $c_1, \dots, c_k \in \mathcal{A}$. Note that (9.14) has been used for the first inequality. By definition of $\mathscr{B}$, the previous computation means that $aa' \in \mathscr{B}$, which is thus an algebra.

For the density, it is sufficient to observe that $\mathcal{A} \subset \mathscr{B}$, and to recall that $\mathcal{A}$ is dense in $\mathcal{B}$. $\square$

In the next statement we mention an argument which has already been used in the proofs of Lemma 9.3.3 and of Proposition 9.3.4. Unfortunately, its content is not fully understandable in the context of these lecture notes since $K_0(\mathscr{B})$ and $K_1(\mathscr{B})$ have not been defined ($\mathscr{B}$ is not a C^*-algebra but a dense subalgebra which is closed under holomorphic functional calculus).

Proposition 9.4.6. *Let $\mathcal{B}$ be a unital C^*-algebra endowed with a n-trace, and let $\mathscr{B}$ be the dense subalgebra defined above.*

(i) The inclusion $\mathscr{B} \subset \mathcal{B}$ defines an isomorphism of $K_0(\mathscr{B})$ with $K_0(\mathcal{B})$,

(ii) The inclusion $\mathscr{B} \subset \mathcal{B}$ defines an isomorphism of $K_1(\mathscr{B})$ with $K_1(\mathcal{B})$.

Remark 9.4.7. *With this remark, we would like to emphasize that in the definition of $\mathscr{B}$ the n-trace plays a significant role. In that respect, the existence of a cyclic cocycle precedes the construction of a suitably dense subalgebra of $\mathcal{B}$. In other words in order to extract information from the K-groups, one should not first define any specific dense subalgebra but look for a n-trace, and then define a suitable subalgebra stable under holomorphic functional calculus. Note that a similar observation is also made on page 113 of [MN08].*

We end up this section with two technical lemmas. All this material will be useful in the next section and for the main theorem of this chapter. For that purpose and for any finite family $a_1, \dots, a_n \in \mathcal{A}$ we set

$$C(a_1, \dots, a_n) := p_{\mathfrak{a}}\big(\delta(a_1)\big) \quad \text{with} \quad \mathfrak{a} = (a_2, \dots, a_n) \in \mathcal{A}^{n-1}.$$

Lemma 9.4.8. *(i) For any $a_1, \dots, a_n \in \mathcal{A}$ the value $C(a_1, \dots, a_n)$ is invariant under cyclic permutations of its arguments,*

(ii) If X is a δ-bounded subset of $\mathcal{A}$ then

$$\sup_{a_1,\dots,a_n \in X} C(a_1, \dots, a_n) < \infty.$$

Proof. i) By definition one has

$$C(a_1, \dots, a_n) = \sup_{x_j \in \mathcal{B}, \|x_j\| \le 1} \big|\hat{\eta}\big((x_1\, \mathtt{d}a_1)(x_2\, \mathtt{d}a_2) \dots (x_n\, \mathtt{d}a_n)\big)\big|.$$

Since $\hat{\eta}$ is a closed graded trace of dimension n on $\big(\Omega(\mathcal{A}), \mathtt{d}\big)$ it follows from the permutation property (9.8) that $C(a_1, \dots, a_n)$ is invariant under cyclic permutations of its arguments.

ii) Let $c_1, \dots, c_k \in \mathcal{A}$ such that $p_{\mathfrak{a}}\big(\delta(a)\big) \le \sup_j p_{\mathfrak{a}}\big(\delta(c_j)\big)$ for all $\mathfrak{a} \in \mathcal{A}^{n-1}$ and $a \in X$. Now, observe that

$$C(a_1, \dots, a_n) = p_{\mathfrak{a}}\big(\delta(a_1)\big) \le \sup_j p_{\mathfrak{a}}\big(\delta(c_j)\big) = \sup_j C(c_j, a_2, \dots, a_n).$$

By the same argument and by taking point (i) into account one gets that for $a_2 \in X$

$$\begin{aligned} C(c_j, a_2, \dots, a_n) &= C(a_2, \dots, a_n, c_1) \\ &= p_{(a_3,\dots,a_n,c_1)}\big(\delta(a_2)\big) \\ &\le \sup_{j'} p_{(a_3,\dots,a_n,c_1)}\big(\delta(c_{j'})\big) \\ &\le \sup_{j'} C(c_{j'}, a_3, \dots, a_n, c_j). \end{aligned}$$

By iteration one finally infers that if $a_i \in X$ for any $i \in \{1, \dots, n\}$ then

$$C(a_1, \dots, a_n) \le \sup_{j_1,\dots,j_n} C(c_{j_1}, \dots, c_{j_n}) < \infty$$

as claimed. $\square$

Lemma 9.4.9. *Let $x_1,\dots,x_n \in \mathcal{B}$ and consider n δ-bounded convergent sequences $\{a_{j,m}\}_m \subset \mathcal{A}$ with $\lim_m a_{j,m} = a_{j,\infty}$ in $\mathcal{B}$, for any $j \in \{1,\dots,n\}$. Then the sequence*

$$m \mapsto \hat{\eta}\big((x_1\,\mathtt{d}a_{1,m})(x_2\,\mathtt{d}a_{2,m})\dots(x_n\,\mathtt{d}a_{n,m})\big)$$

converges to a limit which depends only on $a_{1,\infty},\dots,a_{n,\infty},x_1,\dots,x_n$.

Observe that by definition of $\mathscr{B}$, the above assumptions imply that $a_{1,\infty},\dots,a_{n,\infty}$ belong to $\mathscr{B}$.

Proof. Let us denote by X the set of all $a_{j,m}$ which is δ-bounded by assumption. By Lemma 9.4.8 the family of all multi-linear functionals φ on $\mathcal{B}^n$ defined by

$$\varphi(x_1,\dots,x_n) := \hat{\eta}\big((x_1\,\mathtt{d}a_1)(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)$$

with $a_j \in X$ is a bounded set. Let us then define

$$\varphi_m(x_1,\dots,x_n) := \hat{\eta}\big((x_1\,\mathtt{d}a_{1,m})(x_2\,\mathtt{d}a_{2,m})\dots(x_n\,\mathtt{d}a_{n,m})\big).$$

In order to show the simple convergence of the sequence φ_m we can assume that $x_1,\dots,x_n$ belong to the dense subset $\mathcal{A}$ of $\mathcal{B}$. We also slightly enlarge X by defining $X' = X \cup \{x_1,\dots,x_n\}$. By applying again Lemma 9.4.8.(ii) with X' instead of X, and by taking the equality

$$\hat{\eta}\big(x_1\,\mathtt{d}a\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) = -\hat{\eta}\Big(a\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)$$

into account, one gets that there exists a constant $c > 0$ (which depends on $x_1,\dots,x_n$ but not on $a, a_2,\dots,a_n$) such that

$$\Big|\hat{\eta}\Big(a\,\mathtt{d}\big((x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)x_1\big)\Big)\Big| \le c\,\|a\| \quad \forall a_j \in X.$$

Similarly, we also get that

$$\big|\hat{\eta}\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) - \hat{\eta}\big(x_1\,\mathtt{d}a_1'(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big)\big| \le c\,\|a_1 - a_1'\|$$

for any $a_1,\dots,a_n,a_1' \in X$. By using cyclic permutations one also infers that

$$\big|\hat{\eta}\big(x_1\,\mathtt{d}a_1\,(x_2\,\mathtt{d}a_2)\dots(x_n\,\mathtt{d}a_n)\big) - \hat{\eta}\big(x_1\,\mathtt{d}a_1'(x_2\,\mathtt{d}a_2')\dots(x_n\,\mathtt{d}a_n')\big)\big| \le c\sum_{j=1}^{n}\|a_j - a_j'\|$$

for any $a_1,\dots,a_n,a_1',\dots,a_j' \in X$.

We deduce from the previous estimates that the sequence $\varphi_m(x_1,\dots,x_n)$ is a Cauchy sequence, and that the limit of this sequence does not depend upon the choice of the δ-bounded sequences $\{a_{j,m}\}$ converging to $a_{j,\infty}$. ☐

9.5 Pairing of cyclic cohomology with K-theory

In this section we finally pair cyclic cohomology with K-theory. We shall first deal with even cyclic cocycle.

For any $n \in \mathbb{N}$ let us consider a $2n$-cyclic cocycle η on an associative complex unital algebra $\mathcal{A}$, *i.e.* $\eta \in Z^{2n}_\lambda(\mathcal{A})$. As seen in section 9.2 on the cup product, one can define an element $\eta\#\mathrm{tr} \in Z^{2n}_\lambda\big(M_q(\mathcal{A})\big)$ for any $q \in \mathbb{N}^*$. Let also p be an idempotent in $M_q(\mathcal{A})$, *i.e.* an element $p \in M_q(\mathcal{A})$ satisfying $p^2 = p$. With these two ingredients one can define and study the expression

$$\frac{1}{n!}\,[\eta\#\mathrm{tr}](\underbrace{p, p, \dots, p}_{2n+1 \text{ terms}}). \tag{9.15}$$

Let us first observe that this expression depends only on the cyclic cohomology class of η in $HC^{2n}(\mathcal{A})$. By considering directly the algebra $M_q(\mathcal{A})$ instead of $\mathcal{A}$, it is sufficient to concentrate on the special case $q = 1$. Thus, let us assume that $\eta = \mathtt{b}\psi$ for some $\psi \in C^{2n-1}_\lambda(\mathcal{A})$, or in another words let us assume that η is a coboundary. Then by the definition of $\mathtt{b}$ one has

$$\begin{aligned}
\eta(p, \dots, p) &= \mathtt{b}\psi(p, \dots, p) \\
&= \sum_{j=0}^{2n-1} (-1)^j \psi(p, \dots, p, \dots, p) + \psi(p, \dots, p) \\
&= \psi(p, \dots, p) \\
&= 0,
\end{aligned}$$

where the cyclicity of ψ has been used for the last equality.

Our next aim is to state and prove a result related to the periodicity operator S introduced at the end of Section 9.2. Let us stress once again that for the next statement we choose $\eta'(1,1,1) = 1$ for the normalization of $\eta' \in Z^2_\lambda(\mathbb{C})$, other conventions are also used in the literature. As in the computation made before, we can restrict our attention of the case $q = 1$.

Lemma 9.5.1. *For any $\eta \in Z^{2n}_\lambda(\mathcal{A})$ and for any idempotent $p \in \mathcal{A}$ one has*

$$[S\eta](\underbrace{p, p, \dots, p}_{2n+3 \ \textit{terms}}) = (n+1)\,\eta(\underbrace{p, p, \dots, p}_{2n+1 \ \textit{terms}}).$$

Proof. By taking the expression mentioned just before Lemma 9.2.4 into account one gets

$$\begin{aligned}
[S\eta](p, \dots, p) = {}& \hat{\eta}\big(p\,\mathtt{d}p \dots \mathtt{d}p\big) + \hat{\eta}\big(p\,\mathtt{d}p\,p\,\mathtt{d}p \dots \mathtt{d}p\big) \\
&+ \sum_{i=3}^{2n} \hat{\eta}\big(p\,\mathtt{d}p \dots \mathtt{d}p\,p\,\mathtt{d}p \dots \mathtt{d}p\big) + \hat{\eta}\big(p\,\mathtt{d}p \dots \mathtt{d}p\,p\big).
\end{aligned}$$

Then, since $\mathtt{d}p\, p = \mathtt{d}p - p\, \mathtt{d}p$, one easily infers that nearly half of the terms in the previous sum simply cancel out and one obtains that

$$[S\eta](p,\dots,p) = (n+1)\hat{\eta}\big(p\,\mathtt{d}p\dots\mathtt{d}p\big) = (n+1)\eta(p,\dots,p),$$

which corresponds to the statement. □

Our last aim with respect to the expression (9.15) is to show that it is invariant under the conjugation of p by invertible elements. As before, we can restrict our attention to the case $q = 1$.

Lemma 9.5.2. *For any $\eta \in Z^{2n}_\lambda(\mathcal{A})$, any idempotent $p \in \mathcal{A}$ and any $v \in \mathcal{GL}(\mathcal{A})$ one has*

$$\eta(p,\dots,p) = \eta(vpv^{-1},\dots,vpv^{-1}).$$

The proof of this statement is based on the following observation and result. If $\mathcal{A}, \mathcal{B}$ are associative complex algebras and if $\rho : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism, then it induces a morphism $\rho^* : C^n_\lambda(\mathcal{B}) \to C^n_\lambda(\mathcal{A})$ defined for any $a_0,\dots,a_n \in \mathcal{A}$ by

$$[\rho^*\eta](a_0,\dots,a_n) = \eta\big(\rho(a_0),\dots,\rho(a_n)\big).$$

As a consequence, it also induces a map $\rho^* : HC^n(\mathcal{B}) \to HC^n(\mathcal{A})$. This map depends only on the equivalence class of ρ modulo inner automorphisms, as mentioned in the next statement (see [Con94, Prop. III.1.8] for its proof):

Proposition 9.5.3. *Let $v \in \mathcal{GL}(\mathcal{A})$, and set $\theta(a) := vav^{-1}$ for any $a \in \mathcal{A}$ for the corresponding inner automorphism. Then the induced map $\theta^* : HC^n(\mathcal{A}) \to HC^n(\mathcal{A})$ is the identity for any $n \in \mathbb{N}$.*

Proof of Lemma 9.5.2. With the notations of the previous statement, it is enough to observe that

$$\eta(vpv^{-1},\dots,vpv^{-1}) = \eta\big(\theta(p),\dots,\theta(p)\big) = [\theta^*\eta](p,\dots,p) = \eta(p,\dots,p).$$

Note that for the last equality, the result of the previous proposition has been used together with the fact that (9.15) depends only on the cohomology class of η. □

Remark 9.5.4. *An alternative approach for the proof of Lemma 9.5.2 is provided in [Kha13, p. 169-170]. However, even if this alternative approach might look more intuitive (the derivative of a certain expression computed along a smooth path of idempotents is zero), it works only in some suitable topological algebras. In the previous proof, no topological structure is involved.*

All the information obtained so far on the expression (9.15) can now be gathered in a single statement. As before, the only missing bit of information is about the definition of $K_0(\mathcal{A})$ in this framework.

Proposition 9.5.5. *For any $q \in \mathbb{N}^*$ and $n \in \mathbb{N}$, the following map is bilinear*

$$K_0(\mathcal{A}) \times HC^{2n}(\mathcal{A}) \ni \ ([p]_0, [\eta]) \mapsto \langle [p]_0, [\eta] \rangle := \frac{1}{n!} [\eta \# \mathrm{tr}](p, \dots, p) \ \in \mathbb{C}, \tag{9.16}$$

with $p \in M_q(\mathcal{A})$ an idempotent and with $\eta \in Z^{2n}_\lambda(\mathcal{A})$. In addition, the following properties hold:

(i) $\langle [p]_0, [S\eta] \rangle = \langle [p]_0, [\eta] \rangle$,

(ii) If η, η' are even cyclic cocycles on the algebras $\mathcal{A}, \mathcal{A}'$, then for any idempotents $p \in \mathcal{A}$ and $p' \in \mathcal{A}'$ one has

$$\big\langle [p \otimes p']_0, [\eta \# \eta'] \big\rangle = \langle [p]_0, [\eta] \rangle \, \langle [p']_0, [\eta'] \rangle$$

and a similar formula holds in a matricial version.

For odd cyclic cocycles, a similar construction and similar statements can be proved. We proceed in a similar way. For $n \in \mathbb{N}$ let η be a $(2n+1)$-cyclic cocycle over $\mathcal{A}$, *i.e.* $\eta \in Z^{2n+1}_\lambda(\mathcal{A})$, with $\mathcal{A}$ an associative complex unital algebra. For any $q \in \mathbb{N}^*$ and $v \in \mathcal{GL}_q(\mathcal{A})$ we shall study the expression

$$[\eta \# \mathrm{tr}]\big(\underbrace{v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \dots, v^{-1} - \mathbf{1}, v - \mathbf{1}}_{2n+2 \text{ terms}}\big). \tag{9.17}$$

Before studying in details this expression, let us recall that the K_1-theory of a C^*-algebra $\mathcal{C}$ was computed from the equivalence classes of $\mathcal{U}_\infty(\widetilde{\mathcal{C}})$, see Definition 5.1.2. For the K_1-theory of $\mathcal{A}$, the addition of a unit is also part of the construction. Note also that at the price of changing the algebra $\mathcal{A}$ to $M_q(\mathcal{A})$, we can restrict our attention to the case $q = 1$ in the following developments.

As a preliminary step, let $\widetilde{\mathcal{A}}$ be the algebra obtained from $\mathcal{A}$ by adding a unit to it. Since $\mathcal{A}$ was already assumed to be unital, then the map

$$\rho : \widetilde{\mathcal{A}} \ni (a, \lambda) \mapsto (a + \lambda \mathbf{1}, \lambda) \in \mathcal{A} \times \mathbb{C}$$

is an isomorphism, with $\mathbf{1}$ the unit of $\mathcal{A}$. Less explicitly, this isomorphism had already been used in the proof of Lemmas 2.2.4 and 3.3.5.

For $\eta \in Z^{2n+1}_\lambda(\mathcal{A})$ let us now define $\tilde{\eta} \in C^{2n+1}(\widetilde{\mathcal{A}})$ by

$$\tilde{\eta}\big((a_0, \lambda_0), \dots, (a_{2n+1}, \lambda_{2n+1})\big) = \eta(a_0, \dots, a_{2n+1}), \qquad \forall (a_j, \lambda_j) \in \widetilde{\mathcal{A}}.$$

By cyclicity of η, $\tilde{\eta}$ clearly belongs to $C^{2n+1}_\lambda(\widetilde{\mathcal{A}})$. Let us now check that $\mathtt{b}\tilde{\eta} = 0$, which means that $\tilde{\eta} \in Z^{2n+1}_\lambda(\widetilde{\mathcal{A}})$. Indeed, for any $(a_j, \lambda_j) \in \widetilde{\mathcal{A}}$ observe first that

$$\begin{aligned}
&\tilde{\eta}\big((a_0, \lambda_0), \dots, (a_j, \lambda_j)(a_{j+1}, \lambda_{j+1}), \dots, (a_{2n+2}, \lambda_{2n+2})\big) \\
&= \eta(a_0, \dots, a_j a_{j+1}, \dots, a_{2n+2}) + \lambda_j \, \eta(a_0, \dots, a_{j-1}, a_{j+1}, \dots, a_{2n+2}) \\
&\quad + \lambda_{j+1} \, \eta(a_0, \dots, a_j, a_{j+2}, \dots, a_{2n+2}).
\end{aligned}$$

As a consequence, one infers from the above equality and from the equality $\mathtt{b}\eta = 0$ that

$$\begin{aligned}
&\mathtt{b}\tilde{\eta}\big((a_0,\lambda_0),\dots,(a_{2n+2},\lambda_{2n+2})\big)\\
&= \lambda_0\,\eta(a_1,\dots,a_{2n+2}) + (-1)^{2n+2}\lambda_0\,\eta(a_{2n+2},a_1,\dots,a_{2n+1})\\
&= \lambda_0\big[\eta(a_1,\dots,a_{2n+2}) + (-1)^{2n+1}\eta(a_1,\dots,a_{2n+2})\big]\\
&= 0.
\end{aligned}$$

Note that the cyclicity of η has been used for the last equality.

With the definition of $\tilde{\eta}$ at hand let us define

$$\boldsymbol{\eta} := \tilde{\eta}\circ \operatorname{diag}\big(\rho^{-1}(\cdot,1),\dots,\rho^{-1}(\cdot,1)\big) : \mathcal{A}^{2n+1}\to\mathbb{C},$$

and observe that for any $v\in\mathcal{GL}(\mathcal{A})$ the equality

$$\boldsymbol{\eta}(v^{-1},v,\dots,v^{-1},v) = \eta(v^{-1}-\mathbf{1},v-\mathbf{1},v^{-1}-\mathbf{1},\dots,v^{-1}-\mathbf{1},v-\mathbf{1})$$

holds. Observe also that the newly defined cyclic cocycle $\boldsymbol{\eta}$ satisfies the equality

$$\boldsymbol{\eta}(\mathbf{1},a_1,\dots,a_{2n+1}) = 0.$$

Such a cocycle is called a *normalized* cocycle on $\mathcal{A}$.

Let us now show that (9.17) depends only on the cyclic cohomology class of η in $HC^{2n+1}(\mathcal{A})$. For that purpose, let us assume that $\boldsymbol{\eta} = \mathtt{b}\psi$ for some $\psi\in C^{2n}_\lambda(\mathcal{A})$ with $\psi(\mathbf{1},a_1,\dots,a_{2n}) = 0$. Note that this normalization is indeed possible thanks to the equality $\mathtt{b}\tilde{\psi} = \widetilde{\mathtt{b}\psi}$. Then by taking the normalization and the cyclicity of ψ into account, one directly infers that

$$\begin{aligned}
&[\mathtt{b}\psi](v^{-1},v,\dots,v^{-1},v)\\
&= \sum_{j=0}^{2n}(-1)^j\psi(v^{-1},\dots,\mathbf{1},\dots,v) + (-1)^{2n+1}\psi(\mathbf{1},\dots,v)\\
&= 0,
\end{aligned}$$

as expected.

In the next statement, we consider the action of the periodicity operator, as in Lemma 9.5.2 for even cyclic cocycles. As before, without loss of generality it is sufficient to consider the case $q=1$. Let us stress that the following statement holds for the periodicity operator introduced at the end of Section 9.2, any other convention would lead to other constants[10].

Lemma 9.5.6. *For any normalized $\boldsymbol{\eta}\in Z^{2n+1}_\lambda(\mathcal{A})$ and any $v\in\mathcal{GL}(\mathcal{A})$ one has*

$$[S\boldsymbol{\eta}](\underbrace{v^{-1},v,\dots,v^{-1},v}_{2n+4\ \textit{terms}}) = (2n+2)\,[\boldsymbol{\eta}](\underbrace{v^{-1},v,\dots,v^{-1},v}_{2n+2\ \textit{terms}}).$$

[10] Be aware that the constants appearing in the literature are not consistent between different authors, or even in different papers of the same author.

Proof. For that purpose, recall that the periodicity operator S has been defined at the end of Section 9.2. Then, by using its explicit expression and the fact that $v^{-1}v = \mathbf{1} = vv^{-1}$ one gets

$$\begin{aligned}
[S\boldsymbol{\eta}](\underbrace{v^{-1}, v, v^{-1} \dots, v}_{2n+4 \text{ terms}}) &= \hat{\boldsymbol{\eta}}(v^{-1}\underbrace{\mathtt{d}v\,\mathtt{d}v^{-1} \dots \mathtt{d}v}_{2n+1 \text{ terms}}) + \hat{\boldsymbol{\eta}}(v^{-1}\,\mathtt{d}v\,\mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&\quad + \sum_{i=3}^{2n+1} \hat{\boldsymbol{\eta}}(v^{-1}\,\mathtt{d}v\,\mathtt{d}v^{-1} \dots \mathtt{d}v) + \hat{\boldsymbol{\eta}}(v^{-1}\,\mathtt{d}v\,\mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&= (2n+2)\,\hat{\boldsymbol{\eta}}(v^{-1}\,\mathtt{d}v\,\mathtt{d}v^{-1} \dots \mathtt{d}v) \\
&= (2n+2)\,\boldsymbol{\eta}(\underbrace{v^{-1}, v, v^{-1} \dots, v}_{2n+2 \text{ terms}}),
\end{aligned}$$

which leads directly to the result. □

By summing up the results obtained above, one can now state the main result for odd cocycles:

Proposition 9.5.7. *For any $q \in \mathbb{N}^*$ and $n \in \mathbb{N}$, the following map is bilinear*

$$K_1(\mathcal{A}) \times HC^{2n+1}(\mathcal{A}) \ni \ ([v]_1, [\eta]) \mapsto \langle [v]_1, [\eta] \rangle \ \in \mathbb{C}$$

with

$$\langle [v]_1, [\eta] \rangle := C_{2n+1}\,[\eta\#\mathrm{tr}](\underbrace{v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \dots, v^{-1} - \mathbf{1}, v - \mathbf{1}}_{2n+2 \ \textit{terms}})$$

with $v \in \mathcal{GL}_q(\mathcal{A})$ and $\eta \in Z_\lambda^{2n+1}(\mathcal{A})$. In addition, the constants C_{2n+1} can be chosen iteratively such that the following property holds :

$$\langle [v]_1, [S\eta] \rangle = \langle [v]_1, [\eta] \rangle. \tag{9.18}$$

Part of the proof of this statement has already been provided above. For example, its has been shown that the pairing depends only on the cohomology class of η, and that the relation (9.18) holds if and only if

$$C_{2n+3}\,(2n+2) = C_{2n+1}.$$

We emphasize once more that this relation is a by-product of the choice of the normalization for the periodicity operator S. In addition, the value of the constant C_1 can still be chosen arbitrarily.

On the other hand and quite unfortunately, the proof of the independence on the choice of a representative in the K_1-class of v is out of reach with the material introduced in these lecture notes. At first, a precise description of $K_1(\mathcal{A})$ should be provided, and more information on the Connes-Chern character are also necessary. Note also that our construction of the cyclic cohomology of $\mathcal{A}$ was based only on one boundary map, namely $\mathtt{b}$. However, a more efficient construction involving a second map B and the

Hochschild cohomology $H^n(\mathcal{A}, \mathcal{A}^*)$ is introduced in [Con94, Chap. III] and this approach leads more easily to some stronger results, see also [Kha13, Chap. 3].

Based on the this construction and by taking the result of Section 9.4 into account, the main result of this chapter now reads:

Theorem 9.5.8. *Let $\mathcal{B}$ be a unital C^*-algebra endowed with a n-trace η defined on a dense subalgebra denoted by $\mathcal{A}$.*

(i) If n is even, there exists a map $\varphi : K_0(\mathcal{B}) \to \mathbb{C}$ such that for any $p \in \mathcal{P}_q(\mathcal{A})$ one has

$$\varphi([p]_0) = \frac{1}{(n/2)!}[\eta\#\mathrm{tr}](p, \ldots, p), \tag{9.19}$$

(ii) If n is odd, there exists a map $\varphi : K_1(\mathcal{B}) \to \mathbb{C}$ such that for any $v \in \mathcal{GL}_q(\mathcal{A})$ one has

$$\varphi([w(v)]_1) = C_n[\eta\#\mathrm{tr}](v^{-1} - \mathbf{1}, v - \mathbf{1}, v^{-1} - \mathbf{1}, \ldots, v^{-1} - \mathbf{1}, v - \mathbf{1}), \tag{9.20}$$

where $w(v)$ is defined by $w(v) = v|v|^{-1} \in \mathcal{U}_q(\mathcal{B})$. With our normalization for the periodicity operator S, the constants C_n satisfy the relations $C_{n+2}(n+1) = C_n$.

Proof. This statement is a direct transcription of the content of Propositions 9.5.5 and 9.5.7 once there exist isomorphisms between the K-groups $K_i(\mathcal{A})$ and $K_i(\mathcal{B})$. In fact, instead of $\mathcal{A}$ one has to consider the slightly larger algebra $\mathscr{B}$ introduced in Section 9.4. By Proposition 9.4.6 one only has to extend the cocycle η to a cyclic cocycle on $\mathscr{B}$. For that purpose, it is sufficient to show that for any δ-bounded sequences $\{a_{j,m}\}_m \subset \mathcal{A}$ with $a_{j,m} \to a_{j,\infty}$ the sequence $\eta(a_{0,m}, \ldots, a_{n,m})$ converges to a limit as $m \to \infty$. However, with the notations of the Lemma 9.4.9 and of its proof one has

$$\eta(a_{0,m}, \ldots, a_{n,m}) = \varphi_m(a_{0,m}, \mathbf{1}, \ldots, \mathbf{1}),$$

and it is precisely proved in Lemma 9.4.9 that this expression has a limit as m goes to infinity. □