Chapter 7

Higher K-functors, Bott periodicity

In this chapter, we first show that $K_1(\mathcal{C})$ is isomorphic to $K_0\big(S(\mathcal{C})\big)$, where $S(\mathcal{C})$ is the suspension of a C^*-algebra $\mathcal{C}$ defined in (4.3). Higher K-groups are then defined iteratively, and various exact sequences are considered. The Bott map is constructed and Bott periodicity is stated. However, its full proof is not provided.

7.1 The isomorphism between $K_1(\mathcal{C})$ and $K_0\big(S(\mathcal{C})\big)$

Let us first recall that the suspension of an arbitrary C^*-algebra $\mathcal{C}$ is defined by

$$S(\mathcal{C}) := \big\{f \in C([0,1];\mathcal{C}) \mid f(0) = f(1) = 0\big\}$$

and observe that this C^*-algebra is equal to $C_0\big((0,1);\mathcal{C}\big)$. Clearly, the norm on $S(\mathcal{C})$ is defined by $\|f\| := \sup_{t\in[0,1]} \|f(t)\|_{\mathcal{C}}$, and $f^*(t) := f(t)^*$. With any $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$ one can associate a $*$-homomorphism $S(\varphi) : S(\mathcal{C}) \to S(\mathcal{Q})$ by $[S(\varphi)(f)](t) := \varphi\big(f(t)\big)$ for any $f \in S(\mathcal{C})$ and $t \in [0,1]$. In this way, S defines a functor from the category of C^*-algebras to itself, with $S(\{0\}) = \{0\}$ and $S(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{S(\mathcal{C})\to S(\mathcal{Q})}$.

The following lemma is a classical statement about density. Its proof is left to the reader, see also [RLL00, Lemma 10.1.1].

Lemma 7.1.1. *Let Ω be a locally compact Hausdorff space and let $\mathcal{C}$ be a C^*-algebra. For any $f \in C_0(\Omega)$ and any $a \in \mathcal{C}$ one writes fa for the element of $C_0(\Omega;\mathcal{C})$ defined by $[fa](x) = f(x)a$ for any $x \in \Omega$. Then the set*

$$\mathrm{span}\{fa \mid f \in C_0(\Omega), a \in \mathcal{C}\}$$

is dense in $C_0(\Omega;\mathcal{C})$.

We can now show the main result about the functor S:

Lemma 7.1.2 (Exactness of S)**.** *The functor S is exact.*

Proof. Given the short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

one has to show that

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0$$

is also a short exact sequence. In fact, the only non-trivial part is to show that $S(\psi)$ is surjective. However, this easily follows from the density of $\operatorname{span}\{fb \mid f \in C_0(\Omega), b \in \mathcal{Q}\}$ in $S(\mathcal{Q})$ and from the fact that any element of this dense set belongs to the range of $S(\mathcal{C})$ by $S(\psi)$, since $S(\psi)(fa) = f\psi(a)$ for any $a \in \mathcal{C}$ and any $f \in C_0\big((0,1)\big)$. □

Theorem 7.1.3. *For any C^*-algebra $\mathcal{C}$ there exists an isomorphism*

$$\theta_{\mathcal{C}} : K_1(\mathcal{C}) \to K_0\big(S(\mathcal{C})\big)$$

satisfying the following property: If φ is a $*$-homomorphism between two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$ then the following diagram is commutative:

$$\begin{tikzcd} K_1(\mathcal{C}) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_{\mathcal{C}}"] & K_1(\mathcal{Q}) \arrow[d, "\theta_{\mathcal{Q}}"] \\ K_0\big(S(\mathcal{C})\big) \arrow[r, "K_0(S(\varphi))"] & K_0\big(S(\mathcal{Q})\big) \end{tikzcd} \tag{7.1}$$

Proof. Let us first consider the short exact sequence

$$0 \longrightarrow S(\mathcal{C}) \overset{\iota}{\hookrightarrow} C(\mathcal{C}) \xrightarrow{\pi} \mathcal{C} \longrightarrow 0, \tag{7.2}$$

where $C(\mathcal{C})$ denotes the cone of $\mathcal{C}$. Since $C(\mathcal{C})$ is homotopy equivalent to $\{0\}$, as shown at the end of Section 4.1, it follows that $K_0\big(C(\mathcal{C})\big) = K_1\big(C(\mathcal{C})\big) = \{0\}$. By applying then the exact sequence of Abelian groups obtained in Proposition 6.3.3 to the above short exact sequence of C^*-algebras one infers that the map $\delta_1 : K_1(\mathcal{C}) \to K_0\big(S(\mathcal{C})\big)$ is an isomorphism. One can thus set $\theta_{\mathcal{C}} = \delta_1$.

Observe now that every $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S(\mathcal{C}) \arrow[r] \arrow[d, "S(\varphi)"] & C(\mathcal{C}) \arrow[r] \arrow[d, "C(\varphi)"] & \mathcal{C} \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & S(\mathcal{Q}) \arrow[r] & C(\mathcal{Q}) \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

where $[C(\varphi)(f)](t) := \varphi\big(f(t)\big)$ for any $f \in C(\mathcal{C})$ and $t \in [0,1]$. By applying then the naturality of the index map, see Proposition 6.1.5, one directly gets the commutative diagram (7.1). □

For later use, let us provide a more concrete description of the isomorphism $\theta_{\mathcal{C}}$. For that purpose, let $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $s(u) = \mathbf{1}_n$ be given. Let $v \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ be such that $v(0) = \mathbf{1}_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$, and $s\big(v(t)\big) = \mathbf{1}_{2n}$ for any $t \in [0,1]$, and set $p := v\,\mathrm{diag}(\mathbf{1}_n, 0)v^*$. Then $p \in \mathcal{P}_{2n}\big(\widetilde{S(\mathcal{C})}\big)$, $s(p) = \mathrm{diag}(\mathbf{1}_n, 0)$ and

$$\theta_{\mathcal{C}}([u]_1) := [p]_0 - [s(p)]_0 \ .$$

For the justification of this formula observe first that any $g \in K_1(\mathcal{C})$ can be represented by an element $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $s(u) = \mathbf{1}_n$. Indeed, for any $g \in K_1(\mathcal{C})$ there exists $n \in \mathbb{N}$ and $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [w]_1$. Then one can set $u := ws(w)^*$ and check that $s(u) = \mathbf{1}_n$ and $g = [u]_1$. Note that the latter equality holds since $s(w)^* \sim_h \mathbf{1}_n$, and this follows from Corollary 2.1.3 about the property that the unitary group in $M_n(\mathbb{C})$ is connected.

Now, for each $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $s(u) = \mathbf{1}_n$ we can find $v \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ with $v(0) = \mathbf{1}_{2n}$, $v(1) = \mathrm{diag}(u, u^*)$ and $s\big(v(t)\big) = \mathbf{1}_{2n}$ for every $t \in [0,1]$. Indeed, by Whitehead's Lemma (Lemma 2.1.4) one can find $z \in C\big([0,1];\mathcal{U}_{2n}(\widetilde{\mathcal{C}})\big)$ with $z(0) = \mathbf{1}_{2n}$ and $z(1) = \mathrm{diag}(u, u^*)$. The element v is then defined by $v(t) := s\big(z(t)\big)^* z(t)$ and has the desired properties.

Let us finally observe that an element $f \in C\big([0,1]; M_{2n}(\widetilde{\mathcal{C}})\big)$ belongs to $M_{2n}\big(\widetilde{C(\mathcal{C})}\big)$ if and only if $s\big(f(t)\big) = f(0)$ for each $t \in [0,1]$, while f belongs to $M_{2n}\big(\widetilde{S(\mathcal{C})}\big)$ if and only if $s\big(f(t)\big) = f(0) = f(1)$ for each $t \in [0,1]$. Note also that if π is defined as in (7.2), then $\tilde{\pi}(f) = f(1)$ for any $f \in M_{2n}\big(\widetilde{C(\mathcal{C})}\big)$. With these identifications, it follows that $v \in \mathcal{U}_{2n}\big(\widetilde{C(\mathcal{C})}\big)$, and

$$\tilde{\pi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad p = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v \ \in \mathcal{P}_{2n}\big(\widetilde{S(\mathcal{C})}\big).$$

By the definition of the index map, one infers that

$$\theta_{\mathcal{C}}([u]_1) = \delta_1([u]_1) = [p]_0 - [s(p)]_0,$$

as already mentioned.

7.2 The long exact sequence in K-theory

In this section we define the higher functor K_n for every integer $n \geq 2$. Part of the construction should be considered as a preliminary step for the six-term exact sequence which will be obtained later on.

Definition 7.2.1. *For each integer $n \geq 2$ one defines iteratively the functor K_n from the category of C^*-algebras to the category of Abelian groups by*

$$K_n := K_{n-1} \circ S$$

where the suspension S is seen as a functor from the category of C^*-algebras into itself.

More specifically, for any $n \geq 2$ and for any C^*-algebra $\mathcal{C}$ one sets

$$K_n(\mathcal{C}) := K_{n-1}\big(S(\mathcal{C})\big)$$

and for each $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between C^*-algebras one also sets

$$K_n(\varphi) := K_{n-1}\big(S(\varphi)\big).$$

Now, let us denote by $S^n(\mathcal{C})$ the n-th iterated suspension of the C^*-algebra $\mathcal{C}$. It is inductively defined by $S^n(\mathcal{C}) := S\big(S^{n-1}(\mathcal{C})\big)$. Similarly, if $\mathcal{Q}$ is another C^*-algebra and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, then one gets a $*$-homomorphism $S^n(\varphi) : S^n(\mathcal{C}) \to S^n(\mathcal{Q})$. This $*$-homomorphism is defined by induction by the relation $S^n(\varphi) = S\big(S^{n-1}(\varphi)\big)$. The higher K-groups are then given by

$$K_n(\mathcal{C}) = K_1\big(S^{n-1}(\mathcal{C})\big) \cong K_0\big(S^n(\mathcal{C})\big), \tag{7.3}$$

and

$$K_n(\varphi) = K_1\big(S^{n-1}(\varphi)\big). \tag{7.4}$$

We shall also apply the convention that $S^0(\mathcal{C}) = \mathcal{C}$ and $S^0(\varphi) = \varphi$.

Proposition 7.2.2. *For each integer $n \geq 2$, K_n is a half exact functor from the category of C^*-algebras to the category of Abelian groups.*

Proof. As already mentioned, the suspension S is an exact functor from the category of C^*-algebras to itself, see Lemma 7.1.2. On the other hand, K_1 is a half exact functor, as shown in Proposition 5.2.3. Since the composition of two functors is again a functor, we obtain by formulas (7.3) and (7.4) that K_n is a functor for each $n \geq 2$. The half exactness of K_n easily follows from the mentioned properties of S and of K_1. □

For the short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

let us now define the higher index maps. For that purpose and for $n \geq 1$ one defined inductively the index maps $\delta_{n+1} : K_{n+1}(\mathcal{Q}) \to K_n(\mathcal{J})$ as follows. By the exactness of S, the sequence

$$0 \longrightarrow S^n(\mathcal{J}) \xrightarrow{S^n(\varphi)} S^n(\mathcal{C}) \xrightarrow{S^n(\psi)} S^n(\mathcal{Q}) \longrightarrow 0 \tag{7.5}$$

is exact, and by Theorem 7.1.3 we have an isomorphism

$$\theta_{S^{n-1}(\mathcal{J})} : K_n(\mathcal{J}) = K_1\big(S^{n-1}(\mathcal{J})\big) \to K_0\big(S^n(\mathcal{J})\big).$$

As a consequence, there exists one and only one group homomorphism δ_{n+1} making the diagram

$$\begin{tikzcd}
K_{n+1}(\mathcal{Q}) \arrow[r, "\delta_{n+1}"] \arrow[d, "\mathrm{id}"] & K_n(\mathcal{J}) \\
K_1\big(S^n(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] & K_0\big(S^n(\mathcal{J})\big) \arrow[u, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"']
\end{tikzcd} \tag{7.6}$$

commutative, where $\bar{\delta}_1$ is the index map associated with the short exact sequence (7.5).

Note that the index maps $\delta_1, \delta_2, \ldots$ are natural in the following sense: Given a commutative diagram of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0 \end{tikzcd} \tag{7.7}$$

with $*$-homomorphisms α, β, γ, then the diagram

$$\begin{tikzcd} K_{n+1}(\mathcal{Q}) \arrow[r, "\delta_{n+1}"] \arrow[d, "K_{n+1}(\beta)"'] & K_n(\mathcal{J}) \arrow[d, "K_n(\gamma)"] \\ K_{n+1}(\mathcal{Q}') \arrow[r, "\delta'_{n+1}"] & K_n(\mathcal{J}')\ . \end{tikzcd} \tag{7.8}$$

is commutative. To see this, let us apply the exact functor S^n to the diagram (7.7), let $\bar{\delta}_1$ and $\bar{\delta}'_1$ be the index maps of the two resulting short exact sequences, and consider the diagram

$$\begin{tikzcd} K_{n+1}(\mathcal{Q}) \arrow[r, "\mathrm{id}"] \arrow[d, "K_{n+1}(\beta)"'] & K_1\big(S^n(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] \arrow[d, "K_1(S^n(\beta))"'] & K_0\big(S^n(\mathcal{J})\big) \arrow[r, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"] \arrow[d, "K_0(S^n(\gamma))"] & K_n(\mathcal{J}) \arrow[d, "K_n(\gamma)"] \\ K_{n+1}(\mathcal{Q}') \arrow[r, "\mathrm{id}"] & K_1\big(S^n(\mathcal{Q}')\big) \arrow[r, "\bar{\delta}'_1"] & K_0\big(S^n(\mathcal{J}')\big) \arrow[r, "\theta^{-1}_{S^{n-1}(\mathcal{J})}"] & K_n(\mathcal{J}')\ . \end{tikzcd} \tag{7.9}$$

The center square of this diagram commutes by naturality of the index map δ_1, see Proposition 6.1.5, and the right-hand square commutes by naturality of θ, as obtained in Theorem 7.1.3. Hence, (7.9) is a commutative diagram. Since δ_{n+1} corresponds to the composition of the three horizontal homomorphisms, this implies that (7.8) is commutative.

Proposition 7.2.3 (The long exact sequence in K-theory)**.** *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

induces an exact sequence of K-groups:

$$\begin{aligned} \cdots \xrightarrow{K_{n+1}(\psi)} K_{n+1}(\mathcal{Q}) \xrightarrow{\delta_{n+1}} K_n(\mathcal{J}) \xrightarrow{K_n(\varphi)} K_n(\mathcal{C}) \xrightarrow{K_n(\psi)} K_n(\mathcal{Q}) \xrightarrow{\delta_n} K_{n-1}(\mathcal{J}) \xrightarrow{K_{n-1}(\varphi)} \cdots \\ \cdots \xrightarrow{\delta_1} K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \xrightarrow{K_0(\psi)} K_0(\mathcal{Q}), \end{aligned}$$

where δ_1 is the index map and δ_n its higher analogues for $n \geq 2$.

Proof. Let $\{\bar{\delta}_n\}_{n=1}^{\infty}$ be the index maps associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0.$$

It follows directly from the definition of the index maps and Theorem 7.1.3 that the diagrams

$$\begin{tikzcd}
K_2(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_2(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"'] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] \arrow[d, "\mathrm{id}"] & K_1(\mathcal{J}) \arrow[r] \arrow[d, "\theta_{\mathcal{J}}"] & K_1(\mathcal{C}) \arrow[r] \arrow[d, "\theta_{\mathcal{C}}"] & K_1(\mathcal{Q}) \arrow[d, "\theta_{\mathcal{Q}}"] \\
K_1\big(S(\mathcal{J})\big) \arrow[r] & K_1\big(S(\mathcal{C})\big) \arrow[r] & K_1\big(S(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"'] & K_0\big(S(\mathcal{J})\big) \arrow[r] & K_0\big(S(\mathcal{C})\big) \arrow[r] & K_0\big(S(\mathcal{Q})\big)
\end{tikzcd}$$

and, for $n \geq 3$

$$\begin{tikzcd}
K_n(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_n(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"'] & K_n(\mathcal{Q}) \arrow[r, "\delta_n"] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{J}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & K_{n-1}(\mathcal{Q}) \arrow[d, "\mathrm{id}"] \\
K_{n-1}(S(\mathcal{J})) \arrow[r] & K_{n-1}(S(\mathcal{C})) \arrow[r] & K_{n-1}(S(\mathcal{Q})) \arrow[r, "\bar{\delta}_{n-1}"'] & K_{n-2}(S(\mathcal{J})) \arrow[r] & K_{n-2}(S(\mathcal{C})) \arrow[r] & K_{n-2}(S(\mathcal{Q}))
\end{tikzcd}$$

are commutative. The lower row in the first diagram is exact by Proposition 6.3.3, and for both diagrams the exactness of the lower row implies the exactness of the upper row. Exactness of the long exact sequence is then established by induction. □

Example 7.2.4. *The suspension $S(\mathcal{C}) = C_0\big((0,1);\mathcal{C}\big)$ of a C^*-algebra $\mathcal{C}$ is isomorphic to $C_0(\mathbb{R};\mathcal{C})$ since $\mathbb{R}$ is homeomorphic to $(0,1)$. Note also that $C_0\big(X;C_0(Y)\big)$ is isomorphic to $C_0(X \times Y)$ for any pair of locally compact Hausdorff spaces X and Y. As a consequence, $S^n(\mathbb{C})$ is isomorphic to $C_0(\mathbb{R}^n)$, from which one infers that*

$$K_n(\mathbb{C}) \cong K_0\big(C_0(\mathbb{R}^n)\big), \qquad K_{n+1}(\mathbb{C}) \cong K_1\big(C_0(\mathbb{R}^n)\big)$$

for any $n \geq 1$.

7.3 The Bott map

From now on, the following picture for $S(\mathcal{C})$ will be used:

$$S(\mathcal{C}) := \big\{f \in C(\mathbb{T};\mathcal{C}) \mid f(1) = 0\big\}$$

with $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. Although this definition does not corresponds to the previous one, the two algebras are clearly isomorphic.

Let us first consider a unital C^*-algebra $\mathcal{C}$. For any $n \in \mathbb{N}^*$ and $p \in \mathcal{P}_n(\mathcal{C})$ one defines *the projection loop* $f_p : \mathbb{T} \to \mathcal{U}_n(\mathcal{C})$ by

$$f_p(z) := zp + (\mathbf{1}_n - p), \qquad \forall z \in \mathbb{T}.$$

By identifying $M_n\big(\widetilde{S(\mathcal{C})}\big)$ with the set of elements f of $C(\mathbb{T};M_n(\mathcal{C}))$ such that $f(1)$ belongs to $M_n(\mathbb{C}\mathbf{1}_{\mathcal{C}})$, by considering $f-f(1)$ which belongs to $M_n\big(S(\mathcal{C})\big)$, we obtain that f_p belongs to $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$. In addition, observe that the maps $p\mapsto f_p$ and $f_p\mapsto p$ are continuous because of the equalities

$$\|f_p-f_q\|=\sup_{z\in\mathbb{T}}\|f_p(z)-f_q(z)\|=2\|p-q\|.$$

One then easily infers that the following properties hold:

(i) $f_{p\oplus q}=f_p\oplus f_q$ for any projections $p,q\in\mathcal{P}_\infty(\mathcal{C})$,

(ii) $f_0=\mathbf{1}$,

(iii) If $p\sim_h q$ in $\mathcal{P}_n(\mathcal{C})$ for some $n\in\mathbb{N}^*$, then $f_p\sim_h f_q$ in $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$.

Thus, from the universal property of K_0, one gets a unique group homomorphism

$$\beta_{\mathcal{C}}:K_0(\mathcal{C})\to K_1\big(S(\mathcal{C})\big)$$

such that $\beta_{\mathcal{C}}([p]_0)=[f_p]_1$ for any $p\in\mathcal{P}_\infty(\mathcal{C})$. The map $\beta_{\mathcal{C}}$ is called *the Bott map*.

If $\varphi:\mathcal{C}\to\mathcal{Q}$ is a unital $*$-homomorphism between unital C^*-algebras, then for any $z\in\mathbb{T}$ one has

$$[\widetilde{S(\varphi)}(f_p)](z)=\varphi\big(f_p(z)\big)=f_{\varphi(p)}(z)$$

since $[\widetilde{S(\varphi)}(f)](z)=\varphi\big(f(z)\big)$ for any $f\in M_n\big(\widetilde{S(\mathcal{C})}\big)$. This implies that the diagram

$$\begin{tikzcd} K_0(\mathcal{C}) \arrow[r, "K_0(\varphi)"] \arrow[d, "\beta_{\mathcal{C}}"] & K_0(\mathcal{Q}) \arrow[d, "\beta_{\mathcal{Q}}"] \\ K_1\big(S(\mathcal{C})\big) \arrow[r, "K_1(S(\varphi))"] & K_1\big(S(\mathcal{Q})\big) \end{tikzcd} \tag{7.10}$$

is commutative. This fact is referred to by saying that the Bott map is natural.

Suppose now that $\mathcal{C}$ is a non-unital C^*-algebra. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathcal{C}) \arrow[r] \arrow[d, dotted, "\beta_{\mathcal{C}}"] & K_0(\widetilde{\mathcal{C}}) \arrow[r, shift left] \arrow[d, "\beta_{\widetilde{\mathcal{C}}}"] & K_0(\mathbb{C}) \arrow[r] \arrow[l, shift left] \arrow[d, "\beta_{\mathbb{C}}"] & 0 \\ 0 \arrow[r] & K_1\big(S(\mathcal{C})\big) \arrow[r] & K_1\big(S(\widetilde{\mathcal{C}})\big) \arrow[r, shift left] & K_1\big(S(\mathbb{C})\big) \arrow[r] \arrow[l, shift left] & 0\,. \end{tikzcd} \tag{7.11}$$

The right square is commutative because of the commutativity of (7.10). It then follows that there is a unique group homomorphism $\beta_{\mathcal{C}}:K_0(\mathcal{C})\to K_1\big(S(\mathcal{C})\big)$ making the left square commutative. In addition, a direct computation leads to

$$\beta_{\mathcal{C}}\big([p]_0-[s(p)]_0\big)=[f_pf^*_{s(p)}]_1 \qquad p\in\mathcal{P}_\infty(\widetilde{\mathcal{C}}). \tag{7.12}$$

It then follows from (7.12) that (7.10) holds also in the non-unital case.

The main result of this section then reads:

Theorem 7.3.1 (Bott periodicity)**.** *The Bott map $\beta_{\mathcal{C}} : K_0(\mathcal{C}) \to K_1\big(S(\mathcal{C})\big)$ is an isomorphism for any C^*-algebra $\mathcal{C}$.*

Note that if $\mathcal{C}$ is non-unital, a diagram chase in (7.11) (or the five lemma) shows that $\beta_{\mathcal{C}}$ is an isomorphism if $\beta_{\widetilde{\mathcal{C}}}$ and $\beta_{\mathbb{C}}$ are isomorphisms. Hence it is sufficient to prove the above theorem for unital C^*-algebras. The proof is rather long and technical and will not be reported here. In fact, we shall only state a rather technical lemma from which the main result will be deduced. For more details, we refer to [RLL00, Sec. 11.2] or to [W-O93, Sec. 9.2].

In the following statements, the notation z^k means the map $\mathbb{T} \ni z \mapsto z^k \in \mathbb{T}$ for any natural number k.

Lemma 7.3.2 (Lemma 11.2.13 of [RLL00])**.** *Let n be a natural number.*

(i) For any $u \in \mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$ there are natural numbers $m \geq n$ and k and an element $p \in \mathcal{P}_m(\mathcal{C})$ such that $(z^k u) \oplus \mathbf{1}_{m-n} \sim_h f_p$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$,

(ii) If p, q belong to $\mathcal{P}_n(\mathcal{C})$ with $f_p \sim_h f_q$ in $\mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$, then there exist a natural number $m \geq n$ and $r \in \mathcal{P}_{m-n}(\mathcal{C})$ such that $p \oplus r \sim_h q \oplus r$ in $\mathcal{P}_m(\mathcal{C})$.

Proof of Theorem 7.3.1. We prove that the Bott map is both surjective and injective.

(i) For a given $g \in K_1\big(S(\mathcal{C})\big)$, let $n \in \mathbb{N}$ and $u \in \mathcal{U}_n\big(\widetilde{S(\mathcal{C})}\big)$ such that $g = [u]_1$. By Lemma 7.3.2.(i), there exist two natural numbers $m \geq n$ and k and an element $p \in \mathcal{P}_m(\mathcal{C})$ such that $(z^k u) \oplus \mathbf{1}_{m-n} \sim_h f_p$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$. By Whitehead's Lemma (Lemma 2.1.4) one also infers that

$$f_{\mathbf{1}_{nk}} = z\mathbf{1}_{nk} \sim_h z^k \mathbf{1}_n \oplus \mathbf{1}_{nk-1} \quad \text{in } \mathcal{U}_{nk}\big(\widetilde{S(\mathcal{C})}\big).$$

As a consequence, one deduces that

$$\begin{aligned} \beta_{\mathcal{C}}\big([p]_0 - [\mathbf{1}_{nk}]_0\big) &= [f_p]_1 - [f_{\mathbf{1}_{nk}}]_1 = [z^k u]_1 - [z^k \mathbf{1}_n]_1 \\ &= [u]_1 + [z^k \mathbf{1}_n]_1 - [z^k \mathbf{1}_n]_1 = [u]_1 = g, \end{aligned}$$

from which one infers the surjectivity of $\beta_{\mathcal{C}}$.

(ii) Let us now consider $g \in K_0(\mathcal{C})$ such that $\beta_{\mathcal{C}}(g) = 0$. Let $n \in \mathbb{N}^*$ and $p, q \in \mathcal{P}_n(\mathcal{C})$ such that $g = [p]_0 - [q]_0$, see Proposition 3.2.4. One then infers that $[f_p]_1 = [f_q]_1$, which implies that $f_p \oplus \mathbf{1}_{m-n} \sim_h f_1 \oplus \mathbf{1}_{m-n}$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$ for some $m \geq n$. Let us then set

$$p_1 := \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_m(\mathcal{C}), \qquad q_1 := \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{P}_m(\mathcal{C}).$$

Then $f_{p_1} = f_p \oplus \mathbf{1}_{m-n}$ and $f_{q_1} = f_q \oplus \mathbf{1}_{m-n}$, and consequently $f_{p_1} \sim_h f_{q_1}$ in $\mathcal{U}_m\big(\widetilde{S(\mathcal{C})}\big)$. It follows then from Lemma 7.3.2.(ii) that there exist a natural number $k \geq m$ and $r \in \mathcal{P}_{k-m}(\mathcal{C})$ such that $p_1 \oplus r \sim_h q_1 \oplus r$ in $\mathcal{P}_k(\mathcal{C})$. We then conclude that

$$g = [p]_0 - [q]_0 = [p_1 \oplus r]_0 - [q_1 \oplus r]_0 = 0,$$

from which one infers the injectivity of $\beta_{\mathcal{C}}$. □

7.4 Applications of Bott periodicity

Bott periodicity makes it possible to compute the K-groups of several algebras. First of all, let us state one of its corollary.

Corollary 7.4.1. *For any C^*-algebras $\mathcal{C}$ and any integer n one has*

$$K_{n+2}(\mathcal{C}) \cong K_n(\mathcal{C}).$$

Proof. The case $n = 0$ corresponds precisely to the content of Theorem 7.3.1. The general case follows then by induction on n because

$$K_{n+2}(\mathcal{C}) = K_{n+1}\big(S(\mathcal{C})\big) \cong K_{n-1}\big(S(\mathcal{C})\big) = K_n(\mathcal{C})$$

for any $n \geq 1$. □

Example 7.4.2. *We deduce from the previous corollary together with the content of Example 7.2.4 that for any natural number n*

$$K_0\big(C_0(\mathbb{R}^n)\big) \cong K_n(\mathbb{C}) \cong \begin{cases} K_0(\mathbb{C}) \cong \mathbb{Z} & n \text{ even} \\ K_1(\mathbb{C}) = \{0\} & n \text{ odd.} \end{cases}$$

Similarly we have

$$K_1\big(C_0(\mathbb{R}^n)\big) \cong \begin{cases} \{0\} & n \text{ even} \\ \mathbb{Z} & n \text{ odd.} \end{cases}$$

Example 7.4.3. *For any integer $n \geq 0$ consider the n-sphere defined by*

$$\mathbb{S}^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}.$$

Clearly, the one-point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$ for any $n \geq 1$, and therefore we have an isomorphism $\widetilde{C_0(\mathbb{R}^n)} \cong C(\mathbb{S}^n)$. In addition, observe from the split exactness of K_0, see Proposition 4.3.3, together with the equality $K_0(\mathbb{C}) \cong \mathbb{Z}$, see (3.12)*, that for any C^*-algebra $\mathcal{Q}$ one has*

$$K_0(\widetilde{\mathcal{Q}}) \cong K_0(\mathcal{Q}) \oplus \mathbb{Z}.$$

It then follows that

$$K_0\big(C(\mathbb{S}^n)\big) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases} \qquad K_1\big(C(\mathbb{S}^n)\big) \cong \begin{cases} \{0\} & n \text{ even} \\ \mathbb{Z} & n \text{ odd.} \end{cases}$$

Note that the equality $K_1(\mathcal{C}) \cong K_1(\widetilde{\mathcal{C}})$ of (5.2) *has been used for the computation of the K_1-group.*