Chapter 5

The functor K_1

In this chapter, we define the K_1-group of a C^*-algebra $\mathcal{C}$ as the set of homotopy equivalent classes of unitary elements in the matrix algebras over $\widetilde{\mathcal{C}}$. It will also be shown that the functor K_1 is half exact and homotopy invariant. Since we shall prove in the sequel that $K_1(\mathcal{C})$ is naturally isomorphic to $K_0\big(S(\mathcal{C})\big)$, some of the properties of K_1 will directly be inferred from equivalent properties of K_0. For that reason, their proofs will be provided only once this isomorphism has been exhibited.

5.1 Definition of the K_1-group

Let us first recall that the set of unitary elements of a unital C^*-algebra $\mathcal{C}$ is denoted by $\mathcal{U}(\mathcal{C})$. For any $n \in \mathbb{N}^*$ one sets

$$\mathcal{U}_n(\mathcal{C}) := \mathcal{U}\big(M_n(\mathcal{C})\big) \qquad \text{and} \qquad \mathcal{U}_\infty(\mathcal{C}) := \bigcup_{n \in \mathbb{N}^*} \mathcal{U}_n(\mathcal{C}).$$

We define a binary operation $\oplus$ on $\mathcal{U}_\infty(\mathcal{C})$: for $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$ one sets

$$u \oplus v := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(\mathcal{C}).$$

In addition, a relation $\sim_1$ on $\mathcal{U}_\infty(\mathcal{C})$ is defined as follows: for $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$ one writes $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus \mathbf{1}_{k-n} \sim_h v \oplus \mathbf{1}_{k-m}$ in $\mathcal{U}_k(\mathcal{C})$. With these definitions at hand one can show:

Lemma 5.1.1. *Let $\mathcal{C}$ be a unital C^*-algebra. Then:*

(i) $\sim_1$ *is an equivalence relation on* $\mathcal{U}_\infty(\mathcal{C})$,

(ii) $u \sim_1 u \oplus \mathbf{1}_n$ *for any* $u \in \mathcal{U}_\infty(\mathcal{C})$ *and* $n \in \mathbb{N}$,

(iii) $u \oplus v \sim_1 v \oplus u$ *for any* $u, v \in \mathcal{U}_\infty(\mathcal{C})$,

(iv) *If* $u, v, u', v' \in \mathcal{U}_\infty(\mathcal{C})$, $u \sim_1 u'$ *and* $v \sim_1 v'$ *then* $u \oplus v \sim_1 u' \oplus v'$,

(v) *If* $u, v \in \mathcal{U}_n(\mathcal{C})$, *then* $uv \sim_1 vu \sim_1 u \oplus v$,

(vi) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ *for any* $u, v, w \in \mathcal{U}_\infty(\mathcal{C})$.

Proof. The proofs of (i), (ii) and (vi) are trivial, and (v) follows from Lemma 2.1.4. For the proof of (iii), let us consider $u \in \mathcal{U}_n(\mathcal{C})$ and $v \in \mathcal{U}_m(\mathcal{C})$, and set

$$z = \begin{pmatrix} 0 & \mathbf{1}_m \\ \mathbf{1}_n & 0 \end{pmatrix} \in \mathcal{U}_{n+m}(\mathcal{C}).$$

Then by taking (v) into account, one gets

$$v \oplus u = z(u \oplus v)z^* \sim_1 z^* z(u \oplus v) = u \oplus v.$$

For the proof of (iv) it is sufficient to show that

(I) $(u \oplus \mathbf{1}_k) \oplus (v \oplus \mathbf{1}_\ell) \sim_1 u \oplus v$ for any $u, v \in \mathcal{U}_\infty(\mathcal{C})$ and any $k, \ell \in \mathbb{N}$,

(II) $u \sim_h u'$ and $v \sim_h v'$ imply that $u \oplus v \sim_h u' \oplus v'$ for all $u, u' \in \mathcal{U}_n(\mathcal{C})$ and $v, v' \in \mathcal{U}_m(\mathcal{C})$.

Now, statement (I) follows from (ii), (iii) and (vi). To see that (II) holds, let $t \mapsto u(t)$ and $t \mapsto v(t)$ be continuous paths of unitary elements with $u = u(0)$, $u' = u(1)$, $v = v(0)$ and $v' = v(1)$. Then $t \mapsto u(t) \oplus v(t)$ is a continuous path of unitary elements from $u \oplus v$ to $u' \oplus v'$. □

Definition 5.1.2. *For any* C^**-algebra* $\mathbb{C}$ *one defines*

$$K_1(\mathcal{C}) := \mathcal{U}_\infty(\widetilde{\mathcal{C}})/\sim_1 .$$

The equivalent class in $K_1(\mathcal{C})$ *containing* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ *is denoted by* $[u]_1$. *A binary operation on* $K_1(\mathcal{C})$ *is defined by* $[u]_1 + [v]_1 := [u \oplus v]_1$ *for any* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

It follows from Lemma 5.1.1 that $+$ is well-defined, commutative, associative, has zero element $[\mathbf{1}]_1 \equiv [\mathbf{1}_n]_1$ for any $n \in \mathbb{N}^*$, and that

$$0 = [\mathbf{1}]_1 = [uu^*]_1 = [u]_1 + [u^*]_1$$

for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$. All this shows that $\big(K_1(\mathcal{C}), +\big)$ is an Abelian group, and that $-[u]_1 = [u^*]_1$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

We now collect these information and provide the standard picture of K_1. The statements follow either directly from the definitions or from Lemma 5.1.1.

Proposition 5.1.3. *Let* $\mathcal{C}$ *be a* C^**-algebra. Then*

$$K_1(\mathcal{C}) = \{[u]_1 \mid u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})\},$$

and the map $[\cdot]_1 : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to K_1(\mathcal{C})$ *has the following properties:*

(i) $[u \oplus v]_1 = [u]_1 + [v]_1$ *for any* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$

(ii) $[\mathbf{1}]_1 = 0$,

(iii) If $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ *and* $u \sim_h v$*, then* $[u]_1 = [v]_1$,

(iv) If $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$*, then* $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$,

(v) For $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$*,* $[u]_1 = [v]_1$ *if and only if* $u \sim_1 v$.

We provide some additional information on the K_1-group. The first one corresponds to the universal property of K_1, which is the analogue of Proposition 3.2.5 for K_0.

Proposition 5.1.4 (Universal property of K_1)**.** *Let* $\mathcal{C}$ *be a* C^**-algebra and let* H *be an Abelian group. Suppose that there exists* $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to H$ *satisfying the three conditions:*

(i) $\nu(u \oplus v) = \nu(u) + \nu(v)$ *for any* $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

(ii) $\nu(\mathbf{1}) = 0$,

(iii) If $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ *for some* $n \in \mathbb{N}^*$ *and if* $u \sim_h v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$*, then* $\nu(u) = \nu(v)$.

Then there exists a unique group homomorphism $\alpha : K_1(\mathcal{C}) \to H$ *such that the diagram*

$$\begin{array}{ccc} \mathcal{U}_\infty(\widetilde{\mathcal{C}}) & & \\ {\scriptstyle [\cdot]_1}\downarrow & \searrow^{\nu} & \\ K_1(\mathcal{C}) & \xrightarrow[\alpha]{} & H \end{array} \tag{5.1}$$

is commutative.

Proof. We first show that if $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and $v \in \mathcal{U}_m(\widetilde{\mathcal{C}})$ satisfies $u \sim_1 v$, then $\nu(u) = \nu(v)$. For that purpose, let $k \in \mathbb{N}$ with $k \geq \max\{m, n\}$ such that $u \oplus \mathbf{1}_{k-n} \sim_h v \oplus \mathbf{1}_{k-m}$ in $\mathcal{U}_k(\widetilde{\mathcal{C}})$. By taking (i) and (ii) into accounts, one infers that $\nu(\mathbf{1}_r) = 0$ for any $r \in \mathbb{N}^*$. As a consequence, (i) and (iii) imply that

$$\nu(u) = \nu(u \oplus \mathbf{1}_{k-n}) = \nu(v \oplus \mathbf{1}_{k-m}) = \nu(v).$$

It follows from this equality that there exists a map $\alpha : K_1(A) \to H$ making the diagram (5.1) commutative. Then, the computation

$$\alpha\big([u]_1 + [v]_1\big) = \alpha([u \oplus v]_1) = \nu(u \oplus v) = \nu(u) + \nu(v) = \alpha([u]_1) + \alpha([v]_1)$$

shows that α is a group morphism. The uniqueness of α follows from the surjectivity of the map $[\cdot]_1$. $\square$

If $\mathcal{C}$ is a unital algebra, it would be natural to define directly the K_1-group of $\mathcal{C}$ by $\mathcal{U}_\infty(\mathcal{C})/\sim_1$ without using the algebra $\widetilde{\mathcal{C}}$. This is indeed possible, as shown in the following statement. For that purpose, recall from the proof of Lemma 2.2.4 that if $\tilde{\mathbf{1}}$ denotes the unit of $\widetilde{\mathcal{C}}$ and if $\mathbf{1}$ denotes the unit of $\mathcal{C}$, then $1 := \tilde{\mathbf{1}} - \mathbf{1}$ is a projection in $\widetilde{\mathcal{C}}$. In addition, $\widetilde{\mathcal{C}} = \mathcal{C} + \mathbb{C}1$, with $a1 = 1a = 0$ for any $a \in \mathcal{C}$. One also defines the $*$-homomorphism $\mu : \widetilde{\mathcal{C}} \to \mathcal{C}$ by $\mu(a+\alpha 1) := a$ and extends it to a unital $*$-homomorphism $M_n(\widetilde{\mathcal{C}}) \to M_n(\mathcal{C})$ for any $n \in \mathbb{N}^*$. In this way one obtains a map $\mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\mathcal{C})$.

Proposition 5.1.5. *Let $\mathcal{C}$ be a unital C^*-algebra. Then there exists an isomorphism $\rho : K_1(\mathcal{C}) \to \mathcal{U}_\infty(\mathcal{C})/\sim_1$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{U}_\infty(\widetilde{\mathcal{C}}) & \xrightarrow{\ \mu\ } & \mathcal{U}_\infty(\mathcal{C}) \\ \big\downarrow{\scriptstyle [\cdot]_1} & & \big\downarrow \\ K_1(\mathcal{C}) & \xrightarrow[\ \rho\ ]{} & \mathcal{U}_\infty(\mathcal{C})/\sim_1 . \end{array}$$

Proof. Observe first that the map $\mu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\mathcal{C})$ is surjective. Then, it is sufficient to show that

(I) $\mu(u) \sim_1 \mu(v)$ if and only if $u \sim_1 v$ for any $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

(II) $\mu(u \oplus v) = \mu(u) \oplus \mu(v)$ for any $u, v \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

Clearly, (II) is a direct consequence of the definition of the map μ. For (I) it is sufficient to show that

(I') $\mu(u) \sim_h \mu(v)$ in $\mathcal{U}_n(\mathcal{C})$ if and only if $u \sim_h v$ in $\mathcal{U}_n(\widetilde{\mathcal{C}})$, for any $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and any $n \in \mathbb{N}^*$.

For that purpose, observe that if $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ are such that $u \sim_h v$, then $\mu(u) \sim_h \mu(v)$. For the converse implication, assume that $u, v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ and that $\mu(u) \sim_h \mu(v)$ in $\mathcal{U}_n(\mathcal{C})$. By the definition of μ one can find u_0 and v_0 in $\mathcal{U}_n(\mathbb{C}1)$ such that $u = \mu(u) + u_0$ and $v = \mu(v) + v_0$. By Corollary 2.1.3 one infers that $u_0 \sim_h v_0$ in $M_n(\mathbb{C}1)$, which easily proves that $u \sim_h v$ in $M_n(\widetilde{\mathcal{C}})$. Indeed, one can consider the continuous path $t \mapsto a(t)$ and $t \mapsto b(t)$ of unitary elements in $M_n(\mathcal{C})$ and $M_n(\mathbb{C}1)$, respectively, with $\mu(u) = a(0)$, $\mu(v) = a(1)$, $u_0 = b(0)$ and $u_1 = b(1)$. Then $t \mapsto a(t) + b(t)$ is a continuous path in $\mathcal{U}_n(\widetilde{\mathcal{C}})$ with $u = a(0) + b(0)$ and $v = a(1) + b(1)$. □

When $\mathcal{C}$ is unital, we shall often identify $K_1(\mathcal{C})$ with $\mathcal{U}_\infty(\mathcal{C})/\sim_1$ through the isomorphism ρ of the previous proposition. If u is a unitary element of $\mathcal{U}_\infty(\mathcal{C})$, then $[u]_1$ will denote the element of $K_1(\mathcal{C})$ it represents under this identification. As a immediate consequence of the previous proposition, one also obtains that for any C^*-algebra:

$$K_1(\mathcal{C}) \cong K_1(\widetilde{\mathcal{C}}). \tag{5.2}$$

Let us finally conclude this section with the explicit computation of a K_1-group.

Lemma 5.1.6. *One has* $K_1(\mathbb{C}) = K_1\big(M_n(\mathbb{C})\big) = \{0\}$ *for any* $n \in \mathbb{N}^*$*. More generally one has* $K_1\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$ *for any separable Hilbert space* $\mathcal{H}$*.*

Proof. It has been proved in Corollary 2.1.3 that the unitary group of $M_k\big(M_n(\mathbb{C})\big) = M_{kn}(\mathbb{C})$ is connected for every n and k in $\mathbb{N}^*$. This implies that $\mathcal{U}_\infty\big(M_n(\mathbb{C})\big)/\sim_1$ is the trivial group with only one element. From the description of K_1 for a unital C^*-algebra provided by Proposition 5.1.5 one infers that $K_1\big(M_n(\mathbb{C})\big) = \{0\}$.

Let us now consider any separable Hilbert space $\mathcal{H}$ and first show that $u \sim_h \mathbf{1}_n$ for any unitary element $u \in M_n\big(\mathcal{B}(\mathcal{H})\big)$. Indeed, let us define $\varphi : \mathbb{T} \to [0, 2\pi)$ by

$$\varphi(e^{i\theta}) = \theta, \qquad 0 \leq \theta < 2\pi.$$

Then φ is a bounded Borel measurable map, and $z = e^{i\varphi(z)}$ for any $z \in \mathbb{T}$. As a consequence, for any $u \in \mathcal{U}_n\big(\mathcal{B}(\mathcal{H})\big) = \mathcal{U}\big(\mathcal{B}(\mathcal{H}^n)\big)$, one infers that $\varphi(u) = \varphi(u)^*$ in $\mathcal{B}(\mathcal{H}^n)$, and that $u = e^{i\varphi(u)}$. By Lemma 2.1.2.(i) it follows that $u \sim_h \mathbf{1}_n$. Consequently, one deduces that $u \sim_1 \mathbf{1}$, and then that $\mathcal{U}_\infty\big(\mathcal{B}(\mathcal{H})\big)/\sim_1 = \{0\}$. In other words, one concludes that $K_1\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$ as above. □

5.2 Functoriality of K_1

This section is partially analogue to Section 3.3. Let us first consider two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a $*$-homomorphism. Then φ induces a unital $*$-homomorphism $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ which itself extends to a unital $*$-homomorphism $\tilde{\varphi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ for any $n \in \mathbb{N}^*$. This gives rise to a map $\tilde{\varphi} : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$, and one can set $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{C}}) \to K_1(\mathcal{Q})$ by $\nu(u) := [\tilde{\varphi}(u)]_1$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$. It is straightforward to check that ν satisfies the three conditions of Proposition 5.1.4, and hence there exists precisely one group homomorphism $K_1(\varphi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$ with the property

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1 \tag{5.3}$$

for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$.

Note that if $\mathcal{C}$ and $\mathcal{Q}$ are unital C^*-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a unital $*$-homomorphism, then $K_1(\varphi)([u]_1) = [\varphi(u)]_1$ for any $u \in \mathcal{U}_\infty(\mathcal{C})$.

The following proposition shows that K_1 is a homotopy invariant functor which preserves the zero objects.

Proposition 5.2.1 (Functoriality and homotopy invariance of K_1)**.** *Let* $\mathcal{J}$*,* $\mathcal{C}$ *and* $\mathcal{Q}$ *be* C^**-algebras. Then*

(i) $K_1(\mathrm{id}_\mathcal{C}) = \mathrm{id}_{K_1(\mathcal{C})}$,

(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ *and* $\psi : \mathcal{C} \to \mathcal{Q}$ *are* $*$*-homomorphisms, then*

$$K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi),$$

(iii) $K_1(\{0\}) = \{0\}$,

(iv) $K_1(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_1(\mathcal{C})\to K_1(\mathcal{Q})}$,

(v) *If* $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ *are homotopic* $*$*-homomorphisms, then* $K_1(\varphi) = K_1(\psi)$,

(vi) *If* $\mathcal{C}$ *and* $\mathcal{Q}$ *are homotopy equivalent, then* $K_1(\mathcal{C})$ *is isomorphic to* $K_1(\mathcal{Q})$*. More specifically, if* (3.4) *is a homotopy between* $\mathcal{C}$ *and* $\mathcal{Q}$*, then* $K_1(\varphi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$ *and* $K_1(\psi) : K_1(\mathcal{Q}) \to K_1(\mathcal{C})$ *are isomorphisms, with* $K_1(\varphi)^{-1} = K_1(\psi)$.

Proof. The proof of (i) and (ii) can directly be inferred from (5.3) together with the equalities $\widetilde{\mathrm{id}_{\mathcal{C}}} = \mathrm{id}_{\widetilde{\mathcal{C}}}$ and $\widetilde{(\psi \circ \varphi)} = \tilde{\psi} \circ \tilde{\varphi}$.

As already mentioned in (5.2), the equality $K_1(\mathcal{C}) = K_1(\widetilde{\mathcal{C}})$ holds for any C^*-algebra. In particular, $K_1(\{0\})$ is isomorphic to $K_1(\mathbb{C})$, which is equal to $\{0\}$ by Lemma 5.1.6. This implies (iii).

The zero homomorphism $0_{\mathcal{C}\to\mathcal{Q}}$ can be seen as the composition of the maps $\mathcal{C} \to \{0\}$ and $\{0\} \to \mathcal{Q}$. Hence, (iv) follows from (iii) and (ii).

(v) Let us now consider a path $t \mapsto \varphi(t)$ of $*$-homomorphisms from $\mathcal{C}$ to $\mathcal{Q}$, with $\varphi(0) = \varphi$ and $\varphi(1) = \psi$, and such that the map $[0,1] \ni t \mapsto \varphi(t)(a) \in \mathcal{Q}$ is continuous, for any $a \in \mathcal{C}$. The induced $*$-homomorphism $\tilde{\varphi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ is unital, for any $n \in \mathbb{N}^*$, and the map $[0,1] \ni t \mapsto \varphi(t)(a) \in M_n(\widetilde{\mathcal{Q}})$ is continuous, for any $a \in M_n(\widetilde{\mathcal{C}})$. Hence for any $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ one has in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$:

$$\tilde{\varphi}(u) = \tilde{\varphi}(0)(u) \sim_h \tilde{\varphi}(1)(u) = \tilde{\psi}(u).$$

As a consequence, one infers that

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1 = [\tilde{\psi}(u)]_1 = K_1(\psi)([u]_1),$$

which proves (v).

Finally, statement (vi) is a consequence of (i), (ii) and (v). □

Let us also prove a short lemma which will be useful in the next proposition.

Lemma 5.2.2. *Let* $\mathcal{C}$ *and* $\mathcal{Q}$ *be* C^**-algebras, let* $\varphi : \mathcal{C} \to \mathcal{Q}$ *be a* $*$*-homomorphism, and let* $g \in \mathsf{Ker}\big(K_1(\varphi)\big)$*. Then*

(i) There exists an element $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ *such that* $g = [u]_1$ *and* $\tilde{\varphi}(u) \sim_h \mathbf{1}$,

(ii) If φ *is surjective, then there exists* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$ *such that* $g = [u]_1$ *and* $\tilde{\varphi}(u) = \mathbf{1}$.

Proof. (i) Choose $v \in \mathcal{U}_m(\widetilde{\mathcal{C}})$ such that $g = [v]_1$. Then $[\tilde{\varphi}(v)]_1 = 0 = [\mathbf{1}_m]_1$, and hence there exists an integer $n \geq m$ such that

$$\tilde{\varphi}(v) \oplus \mathbf{1}_{n-m} \sim_h \mathbf{1}_m \oplus \mathbf{1}_{n-m} = \mathbf{1}_n.$$

Set $u = v \oplus \mathbf{1}_{n-m}$, and then $[u]_1 = [v]_1 = g$ and $\tilde{\varphi}(u) = \tilde{\varphi}(v) \oplus \mathbf{1}_{n-m} \sim_h \mathbf{1}_n$.

(ii) Use (i) to find $v \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $g = [v]_1$ and $\tilde{\varphi}(v) \sim_h \mathbf{1}$. By Lemma 2.1.7.$(iii)$ and (i), there exists $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $\tilde{\varphi}(w) = \tilde{\varphi}(v)$ and $w \sim_h \mathbf{1}$. Then $u := w^* v$ has the desired properties. □

Proposition 5.2.3 (Half exactness of K_1)**.** *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

induces an exact sequence of Abelian groups

$$K_1(\mathcal{J}) \xrightarrow{K_1(\varphi)} K_1(\mathcal{C}) \xrightarrow{K_1(\psi)} K_1(\mathcal{Q}),$$

that is $\mathsf{Ran}\big(K_1(\varphi)\big) = \mathsf{Ker}\big(K_1(\psi)\big)$.

Proof. By functoriality of K_1 one already knows that

$$K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi) = K_1(0_{\mathcal{J}\to\mathcal{Q}}) = 0_{K_1(\mathcal{J})\to K_1(\mathcal{Q})},$$

which implies that $\mathsf{Ran}\big(K_1(\varphi)\big) \subset \mathsf{Ker}\big(K_1(\psi)\big)$.

Conversely, assume that $g \in \mathsf{Ker}\big(K_1(\psi)\big)$. According to Lemma 5.2.2.(ii) there exist $n \in \mathbb{N}^*$ and $u \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [u]_1$ and $\tilde{\psi}(u) = \mathbf{1}$. Then, by Lemma 4.3.1.$(ii)$ there exists $v \in M_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(v) = u$. Finally, $[v]_1$ belongs to $K_1(\mathcal{J})$, and $K_1(\varphi)([v]) = [\tilde{\varphi}(v)]_1 = [u]_1 = g$. □

Let us now mention that the functor K_1 is split exact and preserves direct sums of C^*-algebras. These statements can be proved in the same way as for the functor K_0 in Propositions 4.3.3 and 4.3.4. These statements also follow from the isomorphism $K_1(\mathcal{C}) \cong K_0\big(S(\mathcal{C})\big)$ which will be established later on. For this reason, we state these results without providing a proof.

Proposition 5.2.4 (Split exactness of K_1)**.** *Every split exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \mathcal{Q} \longrightarrow 0$$

induces a split exact sequence of Abelian groups

$$0 \longrightarrow K_1(\mathcal{J}) \xrightarrow{K_1(\varphi)} K_1(\mathcal{C}) \underset{K_1(\lambda)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(\mathcal{Q}) \longrightarrow 0.$$

Proposition 5.2.5. *For any C^*-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ the K_0-groups $K_1(\mathcal{C}_1 \oplus \mathcal{C}_2)$ and $K_1(\mathcal{C}_1) \oplus K_1(\mathcal{C}_2)$ are isomorphic. More precisely, if $\iota_i : \mathcal{C}_i \to \mathcal{C}_1 \oplus \mathcal{C}_2$ denotes the canonical inclusion $*$-homomorphism, then the group morphism is provided by the map*

$$K_1(\mathcal{C}_1) \oplus K_1(\mathcal{C}_2) \ni (g, h) \mapsto K_1(\iota_1)(g) + K_1(\iota_2)(h) \in K_1(\mathcal{C}_1 \oplus \mathcal{C}_2).$$

We close this section with an important result for the computation of K_1-groups, which is the analogue for K_1 of the content of Proposition 4.3.7 on the stability of K_0. Note that the proof of the following statement can be proved from its analogue for K_0 by taking the isomorphism $K_1(\mathcal{C}) \cong K_0\big(S(\mathcal{C})\big)$ into account.

Proposition 5.2.6 (Stability of K_1)**.** *Let $\mathcal{C}$ be a C^*-algebra and let $n \in \mathbb{N}^*$. Then $K_1(\mathcal{C})$ is isomorphic to $K_1\big(M_n(\mathcal{C})\big)$. In addition, for any separable Hilbert space $\mathcal{H}$ the following equality holds*

$$K_1\big(\mathcal{C} \otimes \mathcal{K}(\mathcal{H})\big) \cong K_1(\mathcal{C}). \tag{5.4}$$

Corollary 5.2.7. *For any separable Hilbert space $\mathcal{H}$ one has $K_1\big(\mathcal{K}(\mathcal{H})\big) = \{0\}$.*

Proof. From equation (5.4) one infers that $K_1\big(\mathcal{K}(\mathcal{H})\big) \cong K_1(\mathbb{C})$, but $K_1(\mathbb{C}) = \{0\}$ by Lemma 5.1.6. □

Extension 5.2.8. *Work on the relations between K_1-group and determinant for unital Abelian C^*-algebras, as presented in [RLL00, Sec. 8.3].*