Chapter 2

Projections and unitary elements

The K-theory of a C^*-algebra is constructed from equivalence classes of its projections and from equivalence classes of its unitary elements. For that reason, we shall consider several equivalence relations and look at the relations between them. The K-groups will be defined only in the following chapters. This chapter is mainly based on Chapter 2 of the book [RLL00].

2.1 Homotopy classes of unitary elements

Definition 2.1.1. *For any topological space Ω, one says that $a, b \in \Omega$ are* homotopic in Ω *if there exists a continuous map $v : [0,1] \ni t \mapsto v(t) \in \Omega$ with $v(0) = a$ and $v(1) = b$. In such a case one writes $a \sim_h b$ in Ω.*

Clearly, the relation $\sim_h$ defines an equivalence relation on Ω, and one says that v is a continuous path from a to b in Ω. Note that if Ω' is another topological space with $a, b \in \Omega'$ as well, then a, b could be homotopic in Ω without being homotopic in Ω'. Thus, mentioning the ambient space Ω is crucial for the definition of the homotopy relation. On the other hand, we shall often just write $t \mapsto v(t)$ for the continuous path, without specifying $t \in [0,1]$.

In the next statement, we consider this equivalence relation in the set $\mathcal{U}(\mathcal{C})$ of all unitary elements of a unital C^*-algebra $\mathcal{C}$. Clearly, this set is a group (for the multiplication) but not a vector space. Note also that if $u_0, u_1, v_0, v_1 \in \mathcal{U}(\mathcal{C})$ satisfy $u_0 \sim_h u_1$ and $v_0 \sim_h v_1$, then $u_0v_0 \sim_h u_1v_1$. Indeed, if $t \mapsto u(t)$ and $t \mapsto v(t)$ denote the corresponding continuous paths, then $t \mapsto u(t)v(t)$ is a continuous map between u_0v_0 and u_1v_1. In the sequel we shall denote by $\mathcal{U}_0(\mathcal{C})$ the set of elements in $\mathcal{U}(\mathcal{C})$ which are homotopic to $\mathbf{1} \in \mathcal{C}$. Let us also recall from Lemma 1.2.8 that for any unitary element u, one has $\sigma(u) \in \mathbb{T}$.

Lemma 2.1.2. *Let $\mathcal{C}$ be a unital C^*-algebra. Then:*

(i) If $a \in \mathcal{C}$ is self-adjoint, then e^{ia} belongs to $\mathcal{U}_0(\mathcal{C})$,

(ii) If $u \in \mathcal{C}$ is unitary and $\sigma(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(\mathcal{C})$,

(iii) If $u, v \in \mathcal{C}$ are unitary with $\|u - v\| < 2$, then $u \sim_h v$.

Proof. i) In the proof of Lemma 1.2.8 it has already been observed that if a is self-adjoint, then e^{ia} is unitary. By considering now the map $[0,1] \ni t \mapsto v(t) := e^{ita} \in \mathcal{U}(\mathcal{C})$, one easily observes that this map is continuous, and that $v(0) = \mathbf{1}$ and $v(1) = e^{ia}$. As a consequence, $e^{ia} \sim_h \mathbf{1}$, or equivalently $e^{ia} \in \mathcal{U}_0(\mathcal{C})$.

ii) Since $\sigma(u) \neq \mathbb{T}$, there exists $\theta \in \mathbb{R}$ such that $e^{i\theta} \notin \sigma(u)$. Let us then define $v : \sigma(u) \to \mathbb{R}$ by $v\big(e^{i(\theta+t)}\big) = \theta + t$ for any $t \in (0, 2\pi)$ such that $e^{i(\theta+t)} \in \sigma(u)$. Since $\sigma(u)$ is a closed set in $\mathbb{T}$, it follows that v is continuous. In addition, one has that $e^{iv(z)} = z$ for any $z \in \sigma(u)$. Thus, if one sets $a := v(u)$, one infers that a is a self-adjoint element of $\mathcal{C}$ and that $u = e^{ia}$. As a consequence of (i), one deduces that $u \in \mathcal{U}_0(\mathcal{C})$.

iii) If $\|u - v\| < 2$, it follows that $\|v^*u - \mathbf{1}\| = \|v^*(u - v)\| < 2$. Then, from the estimates $|z| \leq r(a) \leq \|a\|$ valid for any $z \in \sigma(a)$ and any $a \in \mathcal{C}$, one infers that $-2 \notin \sigma(v^*u - \mathbf{1})$, or equivalently $-1 \notin \sigma(v^*u)$. Since v^*u is a unitary element of $\mathcal{C}$, one infers then from (ii) that $v^*u \sim_h \mathbf{1}$. Finally, by multiplying the corresponding continuous path on the left by v (or by using the remark made just before the statement of the lemma), one infers that $u \sim_h v$, as expected. □

Let us stress that the previous lemma states that for any self-adjoint $a \in \mathcal{C}$, e^{ia} is a unitary element of $\mathcal{U}_0(\mathcal{C})$. However, not all unitary elements of $\mathcal{C}$ are of this form, and the point (ii) has only provided a sufficient condition for being of this form. Later on, we shall construct unitary elements which are not obtained from a self-adjoint element.

Let us observe that since unitary elements of $M_n(\mathbb{C})$ have only a finite spectrum, one can directly infer from the previous statement (ii) the following corollary:

Corollary 2.1.3. *The unitary group in $M_n(\mathbb{C})$ is connected, or in other words*

$$\mathcal{U}_0\big(M_n(\mathbb{C})\big) = \mathcal{U}\big(M_n(\mathbb{C})\big).$$

By considering matrix algebras, the following statement can easily be proved:

Lemma 2.1.4 (Whitehead)**.** *Let $\mathcal{C}$ be a unital C^*-algebra, and let $u, v \in \mathcal{U}(\mathcal{C})$. Then one has in $\mathcal{U}\big(M_2(\mathcal{C})\big)$*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & \mathbf{1} \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

In particular, one infers that

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{2.1}$$

in $\mathcal{U}\big(M_2(\mathcal{C})\big)$.

Proof. Since $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$ is a unitary element of $M_2(\mathcal{C})$, one infers from the previous corollary that $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$. Then, by observing that

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

one readily infers that the r.h.s. is homotopic to $\left(\begin{smallmatrix} u & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} v & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} uv & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$. The other relations can be proved similarly. □

Let us add some more information on $\mathcal{U}_0(\mathcal{C})$.

Proposition 2.1.5. *Let $\mathcal{C}$ be a unital C^*-algebra. Then,*

(i) $\mathcal{U}_0(\mathcal{C})$ *is a normal subgroup of* $\mathcal{U}(\mathcal{C})$, i.e. $vuv^* \in \mathcal{U}_0(\mathcal{C})$ *whenever* $u \in \mathcal{U}_0(\mathcal{C})$ *and* $v \in \mathcal{U}(\mathcal{C})$,

(ii) $\mathcal{U}_0(\mathcal{C})$ *is open and closed relative to* $\mathcal{U}(\mathcal{C})$,

(iii) An element $u \in \mathcal{C}$ *belongs to* $\mathcal{U}_0(\mathcal{C})$ *if and only if* $u = e^{ia_1} e^{ia_2} \dots e^{ia_n}$ *for some self-adjoint elements* $a_1, \dots, a_n \in \mathcal{C}$.

Exercise 2.1.6. *Provide the proof of this statement, see also [RLL00, Prop. 2.1.6].*

Based on the content of this proposition, the following lemma can be proved:

Lemma 2.1.7. *Let $\mathcal{C}$, $\mathcal{Q}$ be unital C^*-algebras, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective (and hence unit preserving) $*$-homomorphism. Then:*

(i) $\varphi\big(\mathcal{U}_0(\mathcal{C})\big) = \mathcal{U}_0(\mathcal{Q})$,

(ii) For each $u \in \mathcal{U}(\mathcal{Q})$ *there exists* $v \in \mathcal{U}_0\big(M_2(\mathcal{C})\big)$ *such that* $\varphi(v) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$,

(iii) If $u \in \mathcal{U}(\mathcal{Q})$, *and if there exists* $v \in \mathcal{U}(\mathcal{C})$ *such that* $u \sim_h \varphi(v)$, *then* u *belongs to* $\varphi\big(\mathcal{U}(\mathcal{C})\big)$.

Proof. i) Since a unital $*$-homomorphism is continuous and maps unitary elements on unitary elements, it follows that $\varphi\big(\mathcal{U}_0(\mathcal{C})\big)$ is contained in $\mathcal{U}_0(\mathcal{Q})$. Conversely, if u belongs to $\mathcal{U}_0(\mathcal{Q})$, then $u = e^{ib_1} e^{ib_2} \dots e^{ib_n}$ for some self-adjoint elements $b_1, \dots, b_n \in \mathcal{Q}$ by Proposition 2.1.5.(iii). Since φ is surjective, there exists $a_j \in \mathcal{C}$ such that $b_j = \varphi(a_j)$ for any $j \in \{1, \dots, n\}$. Note that a_j can be chosen self-adjoint since otherwise the element $(a_j + a_j^*)/2$ is self-adjoint and satisfies $\varphi\big((a_j + a_j^*)/2\big) = (b_j + b_j^*)/2 = b_j$. Then, by setting $v = e^{ia_1} e^{ia_2} \dots e^{ia_n}$ one gets, again by Proposition 2.1.5, that $v \in \mathcal{U}_0(\mathcal{C})$ and that $\varphi(v) = u$.

ii) For any $u \in \mathcal{U}(\mathcal{Q})$ consider the element $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$ which belongs to $\mathcal{U}_0\big(M_2(\mathcal{Q})\big)$ by (2.1). By applying then the point (i) to $\mathcal{U}_0\big(M_2(\mathcal{C})\big)$ and $\mathcal{U}_0\big(M_2(\mathcal{Q})\big)$ instead of $\mathcal{U}_0(\mathcal{C})$ and $\mathcal{U}_0(\mathcal{Q})$, one immediately deduces the second statement.

iii) If $u \sim_h \varphi(v)$, then $u\varphi(v)^* = u\varphi(v^*)$ is homotopic to $\mathbf{1} \in \mathcal{Q}$, *i.e.* $u\varphi(v^*) \in \mathcal{U}_0(\mathcal{Q})$. By (i) it follows that $u\varphi(v^*) = \varphi(w)$ for some $w \in \mathcal{U}_0(\mathcal{C})$. Consequently, one infers that $u = \varphi(wv)$, or in other words $u \in \varphi\big(\mathcal{U}(\mathcal{C})\big)$. □

Recall that for any unital C^*-algebra $\mathcal{C}$, one denotes by $\mathcal{GL}(\mathcal{C})$ the group of its invertible elements. The set of elements of $\mathcal{GL}(\mathcal{C})$ which are homotopic to $\mathbf{1}$ is denoted by $\mathcal{GL}_0(\mathcal{C})$. Clearly, $\mathcal{U}(\mathcal{C})$ is a subgroup of $\mathcal{GL}(\mathcal{C})$. The following statement establishes a more precise link between these two groups. Before this, observe that for any $a \in \mathcal{C}$, the element a^*a is positive, as recalled in Proposition 1.2.7. Thus, one can define $|a| := (a^*a)^{1/2}$ which is also a positive element of $\mathcal{C}$, and call it *the absolute value of* a.

Proposition 2.1.8. *Let $\mathcal{C}$ be a unital C^*-algebra.*

(i) If a belongs to $\mathcal{GL}(\mathcal{C})$, then $|a|$ belongs to $\mathcal{GL}(\mathcal{C})$ as well, and $w(a) := a|a|^{-1}$ belongs to $\mathcal{U}(\mathcal{C})$. In addition, the equality $a = w(a)|a|$ holds.

(ii) The map

$$w : \mathcal{GL}(\mathcal{C}) \ni a \mapsto w(a) \in \mathcal{U}(\mathcal{C})$$

is continuous, satisfies $w(u) = u$ for any $u \in \mathcal{U}(\mathcal{C})$, and verifies $w(a) \sim_h a$ in $\mathcal{GL}(\mathcal{C})$ for any $a \in \mathcal{GL}(\mathcal{C})$,

(iii) If $v_0, v_1 \in \mathcal{U}(\mathcal{C})$ satisfies $v_0 \sim_h v_1$ in $\mathcal{GL}(\mathcal{C})$, then $v_0 \sim_h v_1$ in $\mathcal{U}(\mathcal{C})$.

Proof. i) If a is invertible, it follows that a^* and a^*a are invertible as well. As a consequence, the element $|a| = (a^*a)^{1/2}$ is also invertible, with inverse $\big((a^*a)^{-1}\big)^{1/2}$. For simplicity, let us set $w := a|a|^{-1}$ which verifies $a = w|a|$. Since w is the product of two invertible elements, w is invertible as well, and it satisfies $w^* = w^{-1}$ since

$$w^*w = |a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = \mathbf{1}.$$

Consequently, $w \in \mathcal{U}(\mathcal{C})$.

ii) The continuity of the map $a \mapsto a^{-1}$ in $\mathcal{GL}(\mathcal{C})$ can easily be obtained by the Neumann series, as recalled in Exercise 1.2.1. Thus, to show that the map $a \mapsto w(a)$ is continuous, it is sufficient to show that the map $a \mapsto (a^*a)^{1/2}$ is continuous. Clearly, the map $a \mapsto a^*a$ is continuous, because involution and multiplication are continuous. It remains to show the continuity of the map $b \mapsto b^{1/2}$ on any bounded subset F of $\mathcal{C}^+$. However, this directly follows from Lemma 1.2.13 since any bounded subset F of $\mathcal{C}^+$ is contained in some F_K (in the notation of the mentioned lemma) with $K = [0, R]$ and $R := \sup\{\|a\| \mid a \in F\}$.

If u is unitary, one has $u^*u = \mathbf{1}$ and thus $|u| = \mathbf{1}$, which implies that $w(u) = u$. On the other hand, for $a \in \mathcal{GL}(\mathcal{C})$, let us set $v(t) = w(a)\big(t|a| + (1-t)\mathbf{1}\big)$ with $t \in [0,1]$. Clearly, $v(0) = w(a)$ and $v(1) = a$, and let us show $v(t) \in \mathcal{GL}(\mathcal{C})$ for any t. Indeed, since $|a|$ is positive and invertible, it follows that $\lambda := \inf \sigma(|a|) > 0$, from which one infers that $t|a| + (1-t)\mathbf{1} \geq \min\{\lambda, 1\}\mathbf{1} > 0$. As a consequence of Proposition 1.2.7.(vi), it follows that $t|a| + (1-t)\mathbf{1}$ is invertible, and therefore $v(t)$ is invertible as well. Since the map $t \mapsto v(t)$ is continuous, one concludes that $w(a) \sim_h a$ in $\mathcal{GL}(\mathcal{C})$.

iii) If $t \mapsto v(t)$ is a continuous path in $\mathcal{GL}(\mathcal{C})$ between v_0 and v_1, then $t \mapsto w\big(v(t)\big)$ is a continuous path in $\mathcal{U}(\mathcal{C})$ between v_0 and v_1. □

The above proposition says that $\mathcal{U}(\mathcal{C})$ is *a retract*[1] *of* $\mathcal{GL}(\mathcal{C})$. Note also that the above decomposition $a = w(a)|a|$ for any invertible element a of $\mathcal{C}$ is called *the polar decomposition of* a. This decomposition is often written $a = u|a|$ with $u := w(a)$.

Finally, let us state a useful result:

Lemma 2.1.9. *Let $\mathcal{C}$ be a unital C^*-algebra, and let $a \in \mathcal{C}$ be invertible. Assume that $b \in \mathcal{C}$ satisfies $\|b - a\| < \|a^{-1}\|^{-1}$. Then b is invertible, with*

$$\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\|,$$

and $a \sim_h b$ in $\mathcal{GL}(\mathcal{C})$.

Exercise 2.1.10. *Provide a proof of the previous lemma, with the possible help of [RLL00, Prop. 2.1.11].*

2.2 Equivalence of projections

We start with the definition of a (self-adjoint) projection in the setting of a C^*-algebra.

Definition 2.2.1. *An element p in a C^*-algebra $\mathcal{C}$ is called* a projection *if $p = p^2 = p^*$. The set of all projections in $\mathcal{C}$ is denoted by $\mathcal{P}(\mathcal{C})$.*

Exercise 2.2.2. *Let $\mathcal{C}$ be a unital C^*-algebra, and let $p \in \mathcal{P}(\mathcal{C})$. Show that $\sigma(p) \subset \{0, 1\}$.*

Clearly, the equivalence by homotopy $\sim_h$ can be considered on $\mathcal{P}(\mathcal{C})$, but let us consider two additional equivalence relations. Namely, for any $p, q \in \mathcal{P}(\mathcal{C})$, one writes $p \sim q$ if there exists $v \in \mathcal{C}$ such that $p = v^*v$ and $q = vv^*$ and calls it *the Murray-von Neumann equivalence*. Alternatively, one writes $p \sim_u q$ if there exists an element $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$ and calls it *the unitary equivalence*. Note that an element v of $\mathcal{C}$ satisfying $v^*v, vv^* \in \mathcal{P}(\mathcal{C})$ is called *a partial isometry*. The projection $p := v^*v$ is called *the support projection of* v, and the projection $q := vv^*$ is called *the range projection of* v. We can then observe that in this setting one has

$$v = qv = vp = qvp. \tag{2.2}$$

Exercise 2.2.3. *Show that for any v in a C^*-algebra such that v^*v is a projection, then automatically vv^* is also a projection. By using the equalities provided in* (2.2)*, show that the Murray-von Neumann relation is transitive.*

Lemma 2.2.4. *Let $\mathcal{C}$ be a unital C^*-algebra, and let $p, q \in \mathcal{P}(\mathcal{C})$. Then the following statements are equivalent:*

(i) $p \sim_u q$,

[1]A *retract* of a topological space Ω consists in a subspace Ω_0 such that there exists a continuous map $\tau : \Omega \to \Omega_0$ satisfying $x \sim_h \tau(x)$ in Ω, for any $x \in \Omega$, and such that $\tau(x) = x$ for all $x \in \Omega_0$.

(ii) $q = upu^*$ *for some* $u \in \mathcal{U}(\mathcal{C})$*,*

(iii) $p \sim q$ *and* $\mathbf{1} - p \sim \mathbf{1} - q$*.*

Proof. Let us denote by $\tilde{\mathbf{1}}$ the unit of $\widetilde{\mathcal{C}}$ and keep the notation $\mathbf{1}$ for the unit of $\mathcal{C}$. We set $1 := \tilde{\mathbf{1}} - \mathbf{1}$, and one can observe that 1 is a projection in $\widetilde{\mathcal{C}}$. In addition, one has

$$\widetilde{\mathcal{C}} = \big\{a + \alpha 1 \mid a \in \mathcal{C}, \alpha \in \mathbb{C}\big\}$$

and $a1 = 1a = 0$ for any $a \in \mathcal{C}$.

(i) $\Rightarrow$ *(ii)*: Assume that $q = vpv^*$ for some $v \in \mathcal{U}(\widetilde{\mathcal{C}})$. By the previous observation, one has $v = u + \alpha 1$ for some $u \in \mathcal{C}$ and $\alpha \in \mathbb{C}$. By computing v^*v and vv^*, one readily infers that $u \in \mathcal{U}(\mathcal{C})$ and then that $q = upu^*$.

(ii) $\Rightarrow$ *(iii)*: Suppose that $q = upu^*$ for some $u \in \mathcal{U}(\mathcal{C})$. By setting $v := up$ and $w := u(\mathbf{1} - p)$ one gets

$$v^*v = p, \quad vv^* = q, \quad w^*w = \mathbf{1} - p, \quad ww^* = \mathbf{1} - q. \tag{2.3}$$

(iii) $\Rightarrow$ *(i)*: Suppose that there are partial isometries v and w in $\mathcal{C}$ satisfying (2.3). By setting $u := v + w + 1$ and by taking (2.3) and the definition of 1 into account one gets

$$uu^* = vv^* + ww^* + wv^* + vw^* + (\tilde{\mathbf{1}} - \mathbf{1}) = wv^* + vw^* + \tilde{\mathbf{1}}$$

and

$$u^*u = v^*v + w^*w + w^*v + v^*w + (\tilde{\mathbf{1}} - \mathbf{1}) = w^*v + v^*w + \tilde{\mathbf{1}}.$$

Then, by inserting the support and the range projections one readily obtains $wv^* = w(1-p)pv^* = 0$, and similarly $vw^* = 0$, $w^*v = 0$ and $v^*w = 0$, which imply that $u \in \mathcal{U}(\widetilde{\mathcal{C}})$. We finally find that $upu^* = vpv^* = vv^* = q$, as expected. □

Let us now state a short technical result, which proof can be found in [RLL00, Lem. 2.2.3].

Lemma 2.2.5. *Let* $\mathcal{C}$ *be a* C^**-algebra and let* $p \in \mathcal{P}(\mathcal{C})$ *and* $a \in \mathcal{C}$ *be self-adjoint. By setting* $\delta := \|p - a\|$*, one has*

$$\sigma(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

Based on the previous lemma, one can now show the following statement:

Proposition 2.2.6. *Let* $\mathcal{C}$ *be a* C^**-algebra, and let* $p, q \in \mathcal{P}(\mathcal{C})$ *with* $\|p - q\| < 1$*. Then* $p \sim_h q$ *in* $\mathcal{P}(\mathcal{C})$*.*

Proof. For any $t \in [0,1]$, let us set $a(t) := (1-t)p + tq$. Clearly, $a(t)$ is self-adjoint, it satisfies

$$\min\big\{\|a(t) - p\|, \|a(t) - q\|\big\} \le \|p - q\|/2 < 1/2,$$

and the map $t \mapsto a(t)$ is continuous. Moreover, by Lemma 2.2.5 and with the notation of Lemma 1.2.13, each $a(t)$ belongs to F_K with $K := [-\delta, \delta] \cup [1-\delta, 1+\delta]$ and $\delta = \|p-q\|/2$. Note that since $\|p-q\| < 1$, these two intervals are disjoint.

Now, let f be the continuous function on K given by $f(x) = 0$ if $x \in [-\delta, \delta]$ and $f(x) = 1$ if $x \in [1-\delta, 1+\delta]$. Then, since $f = f^2 = \overline{f}$, it follows that $f\big(a(t)\big)$ is a projection for each $t \in [0,1]$. In addition, the map $t \mapsto f\big(a(t)\big) \in \mathcal{P}(\mathcal{C})$ is continuous by Lemma 1.2.13, and hence one has in $\mathcal{P}(\mathcal{C})$

$$p = f(p) = f\big(a(0)\big) \sim_h f\big(a(1)\big) = f(q) = q.$$

□

One usually says that two elements a, b in a unital C^*-algebra are *similar* if there exists $c \in \mathcal{GL}(\mathcal{C})$ such that $b = cac^{-1}$. In the next statement, we show that if two self-adjoint elements are similar, then they are unitarily equivalent.

Proposition 2.2.7. *Let a, b be self-adjoint elements in a unital C^*-algebra $\mathcal{C}$, and suppose that there exists $c \in \mathcal{GL}(\mathcal{C})$ such that $b = cac^{-1}$. Let $c = u|c|$ be the polar decomposition of c, with $u \in \mathcal{U}(\mathcal{C})$. Then $b = uau^*$.*

Proof. Since a and b are self-adjoint, the equation $b = cac^{-1}$ implies that $bc = ca$ and that $ac^* = c^*b$. As a consequence, one infers that

$$|c|^2 a = c^*ca = c^*bc = ac^*c = a|c|^2,$$

which means that a and $|c|^2$ commute. One then deduces that a commutes with all elements of $C^*\big(\{|c|^2, \mathbf{1}\}\big)$ and in particular a commutes with $|c|^{-1}$ (which exists since c is invertible). It thus follows that

$$uau^* = c|c|^{-1}au^* = ca|c|^{-1}u^* = bc|c|^{-1}u^* = buu^* = b.$$

□

Let us add one more information on the relation between $\sim_h$ and the unitary equivalence.

Proposition 2.2.8. *Let $\mathcal{C}$ be a C^*-algebra, and let $p, q \in \mathcal{P}(\mathcal{C})$. Then $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$ if and only if there exists a unitary element $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$ such that $q = upu^*$.*

Proof. Let $\mathbf{1}$ denote the unit of $\widetilde{\mathcal{C}}$, and assume that there exists $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$ which verifies $q = upu^*$. Let $t \mapsto u(t)$ be a continuous path in $\mathcal{U}(\widetilde{\mathcal{C}})$ with $u(0) = \mathbf{1}$ and $u(1) = u$. Because $\mathcal{C}$ is an ideal in $\widetilde{\mathcal{C}}$ it follows that $u(t)pu(t)^*$ is a projection in $\mathcal{C}$ for any t, and thus the map $t \mapsto u(t)pu(t)^*$ is a continuous path in $\mathcal{P}(\mathcal{C})$ from p to q.

Conversely, if $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$, then there are projections $p_0, p_1, \dots p_n$ in $\mathcal{C}$ with $p_0 = p$ and $p_n = q$ such that $\|p_{j+1} - p_j\| < 1/2$ for any $j = 0, 1, \dots n-1$. By concatenation, it is sufficient to show the statement for $\|p - q\| < 1/2$. Thus, let us set

$$b := pq + (\mathbf{1} - p)(\mathbf{1} - q) \in \widetilde{\mathcal{C}}$$

and observe that

$$pb = pq = bq, \tag{2.4}$$

and that

$$\|b - \mathbf{1}\| = \big\|p(q-p) + (\mathbf{1}-p)(p-q)\big\| \leq 2\|p-q\| < 1.$$

By Lemma 2.1.9 it follows that b is invertible and that $b \sim_h \mathbf{1}$ in $\mathcal{GL}(\widetilde{\mathcal{C}})$. In addition, by considering the polar decomposition $b = u|b|$ with $u \in \mathcal{U}(\widetilde{\mathcal{C}})$, one obtains from (2.4) and from Proposition 2.2.7 that $p = uqu^*$. Finally, from Proposition 2.1.8.(ii) one deduces that $u \sim_h b \sim_h \mathbf{1}$ in $\mathcal{GL}(\widetilde{\mathcal{C}})$, from which one gets that $u \in \mathcal{U}_0(\widetilde{\mathcal{C}})$, again from Proposition 2.1.8.(iii). □

Up to now, we have considered three equivalence relation, the homotopy relation $\sim_h$, the Murray-von Neumann relation $\sim$ and the unitary relation $\sim_u$. It can be shown on examples that these three relations are different from each other, see for example [RLL00, Ex. 2.2.9]. In fact, in the next lemma we shall show that homotopy equivalence is stronger than unitary equivalence, which is itself stronger that Murray-von Neumann equivalence. However, we shall see subsequently that these relations are equal modulo passing to matrix algebras.

Lemma 2.2.9. *Let p, q be projections in a C^*-algebra $\mathcal{C}$. Then:*

(i) If $p \sim_h q$ in $\mathcal{P}(\mathcal{C})$, then $p \sim_u q$,

(ii) If $p \sim_u q$, then $p \sim q$.

Proof. Clearly, the first statement is a consequence of Proposition 2.2.8. For the second one, let $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$. Then $v := up$ belongs to $\mathcal{C}$ and satisfies $v^*v = p$ and $vv^* = upu^* = q$. □

Proposition 2.2.10. *Let p, q be projections in a C^*-algebra $\mathcal{C}$. Then:*

(i) If $p \sim q$, then $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_u \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_2(\mathcal{C})$,

(ii) If $p \sim_u q$, then $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathcal{P}\big(M_2(\mathcal{C})\big)$.

Let us mention that both algebras $M_2(\widetilde{\mathcal{C}})$ and $\widetilde{M_2(\mathcal{C})}$ (the smallest unitization of $M_2(\mathcal{C})$) will be used during the proof of this proposition. It is easily observed that these two algebras are not equal, as illustrated in the following proof.

Proof. i) Let $v \in \mathcal{C}$ such that $p = v^*v$ and $q = vv^*$. By taking (2.2) into account and by denoting by $\mathbf{1}$ the unit of $\widetilde{\mathcal{C}}$, one readily infers that

$$u := \begin{pmatrix} v & \mathbf{1}-q \\ \mathbf{1}-p & v^* \end{pmatrix}, \qquad w := \begin{pmatrix} q & \mathbf{1}-q \\ \mathbf{1}-q & q \end{pmatrix}$$

are unitary elements of $M_2(\widetilde{\mathcal{C}})$, with $u^* = \begin{pmatrix} v^* & \mathbf{1}-p \\ \mathbf{1}-q & v \end{pmatrix}$ and $w^* = w$. Then, one observes that

$$wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* w^* = w \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, one has $wu \in M_2(\widetilde{\mathcal{C}})$, but by an explicit computation one observes that

$$wu = \begin{pmatrix} v + (\mathbf{1}-q)(\mathbf{1}-p) & (\mathbf{1}-q)v^* \\ q(\mathbf{1}-p) & (\mathbf{1}-q) + qv^* \end{pmatrix}$$

belongs to $\widetilde{M_2(\mathcal{C})} \subset M_2(\widetilde{\mathcal{C}})$, the claim (i) is thus proved.

ii) Let $u \in \mathcal{U}(\widetilde{\mathcal{C}})$ such that $q = upu^*$. By (2.1), there exists a continuous path $v : [0,1] \to \mathcal{U}\big(M_2(\widetilde{\mathcal{C}})\big)$ such that $v(0) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ and $v(1) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. Then by setting $w(t) := v(t) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} v(t)^*$, one gets that $w(t) \in \mathcal{P}\big(M_2(\mathcal{C})\big)$ for any t, that the map $t \mapsto w(t)$ is continuous, and that $w(0) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $w(1) = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. □

2.3 Liftings

Let us now consider two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, and let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism. Given an element $b \in \mathcal{Q}$, an element $a \in \mathcal{C}$ satisfying $\varphi(a) = b$ is called *a lift for b*. The set of all lifts for b is then given by $a + \mathsf{Ker}(\varphi)$. Now, if b has some additional properties, like being a projection or a unitary element, we shall be interested in looking at lifts for b which share similar properties (if possible). In the following statement, we collect several results in this direction.

Proposition 2.3.1. *Let $\varphi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism between C^*-algebras. Then:*

(i) Every $b \in \mathcal{Q}$ has a lift $a \in \mathcal{C}$ satisfying $\|a\| = \|b\|$,

(ii) Every self-adjoint $b \in \mathcal{Q}$ has a self-adjoint lift $a \in \mathcal{C}$. Moreover, this self-adjoint lift can be chosen such that $\|a\| = \|b\|$,

(iii) Every positive $b \in \mathcal{Q}$ has a positive lift $a \in \mathcal{C}$, and this lift can be chosen such that $\|a\| = \|b\|$,

(iv) A normal element $b \in \mathcal{Q}$ does not in general lift to a normal element in $\mathcal{C}$,

(v) A projection in $\mathcal{Q}$ does not in general lift to a projection in $\mathcal{Q}$,

(vi) When $\mathcal{C}$ and $\mathcal{Q}$ are unital, a unitary element $b \in \mathcal{Q}$ does not in general lift to a unitary element in $\mathcal{C}$.

Proof. ii) Consider a self-adjoint element $b \in \mathcal{Q}$, and let $x \in \mathcal{C}$ be any lift for b. Then $a_0 := (x + x^*)/2$ defines a self-adjoint lift for b. In order to impose the equality of the norms, let us consider $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$f(t) = \begin{cases} -\|b\| & \text{if } t \leq -\|b\| \\ t & \text{if } -\|b\| \leq t \leq \|b\| \\ \|b\| & \text{if } t \geq \|b\| \end{cases}$$

and set $a := f(a_0)$. Then a is self-adjoint, being obtained by functional calculus of a self-adjoint element, and one has $\sigma(a) = \{f(t) \mid t \in \sigma(a_0)\} \subset [-\|b\|, \|b\|]$. One infers from this inequality that $\|a\| \leq \|b\|$, since $r(a) = \|a\|$ for any self-adjoint element. On the other hand, one has

$$\varphi(a) = \varphi\big(f(a_0)\big) = f\big(\varphi(a_0)\big) = f(b) = b,$$

because of the definition of f. Since φ is a $*$-homomorphism, one infers that $\|\varphi\| \leq 1$, from which one concludes that $\|b\| \leq \|a\|$. By collecting these inequalities one obtains that $\|a\| = \|b\|$.

i) Let b be an arbitrary element of $\mathcal{Q}$, and set $y = \left(\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\right)$. Then y is a self-adjoint element in $M_2(\mathcal{Q})$, and

$$\|y\|^2 = \|y^*y\| = \|\left(\begin{smallmatrix} bb^* & 0 \\ 0 & b^*b \end{smallmatrix}\right)\| = \max\{\|bb^*\|, \|b^*b\|\} = \|b\|^2.$$

It follows then by (ii) that there exists a self-adjoint lift $x = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \in M_2(\mathcal{C})$ for y with $\|x\| = \|y\| = \|b\|$. Clearly, $a := x_{12}$ is then a lift for b, and from (1.4) one infers that $\|a\| \leq \|x\| = \|b\|$. As in the proof of (ii), one also has $\|b\| \leq \|a\|$, from which one deduces that $\|a\| = \|b\|$.

iii) Let b be a positive element in $\mathcal{Q}$, and let $x \in \mathcal{C}$ be any lift for b. Set $a_0 := (x^*x)^{1/2}$, which is positive, and observe that

$$\varphi(a_0) = \big(\varphi(x)^*\varphi(x)\big)^{1/2} = (b^*b)^{1/2} = b.$$

We can then set $a := f(a_0)$ with the function f introduced in the proof of (ii), and one gets that a is self-adjoint with $\sigma\big(f(a)\big) \subset [0, \|b\|]$. Thus, a is positive and satisfies $\varphi(a) = b$ together with $\|a\| = \|b\|$.

The remaining three assertions are based on counterexamples. For (iv), a counterexample is provided in [RLL00, Ex. 9.4.(iii)] and is based on the unilateral shift. For (v), consider the algebras $\mathcal{C} := C\big([0, 1]\big)$ and $\mathcal{Q} := \mathbb{C} \oplus \mathbb{C}$, with $\varphi : \mathcal{C} \to \mathcal{Q}$ defined by $\varphi(f) = \big(f(0), f(1)\big)$ for any $f \in \mathcal{C}$. Clearly, $(0, 1)$ is a projection in $\mathcal{Q}$, but there is no lift f in $\mathcal{C}$ which is a projection and which satisfies $\big(f(0), f(1)\big) = (0, 1)$. For (vi), a counterexample is provided in [RLL00, Ex. 2.12.(ii)] for the algebras $\mathcal{C} := C(\mathbb{D})$ and $\mathcal{Q} := C(\mathbb{T})$, with $\mathbb{D} := \{z \in \mathbb{C} \mid \|z\| \leq 1\}$. □