Chapter 10

Application: Levinson's theorem

In this chapter, we briefly describe how the formalism introduced in the previous chapters leads to some index theorems in the context of scattering theory. Obviously, we shall only scratch the surface, and most the previous material is not really necessary for the example presented hereafter. However, for more involved examples this material turns out to be essential. We refer to [Ric15] for more information on the subject.

10.1 The ■-anisotropic algebra

In this section we briefly construct a C^*-algebra which will play a major role later on. This algebra has been introduced in [GI03, Sec. 3.5] for a different purpose, and we refer to this paper for the details of the construction.

In the Hilbert space $L^2(\mathbb{R})$ we consider the two canonical self-adjoint operators X of multiplication by the variable, and $D = -i\frac{\mathrm{d}}{\mathrm{d}x}$ of differentiation. These operators satisfy the canonical commutation relation written formally $[iD, X] = 1$, or more precisely $e^{-isX}e^{-itD} = e^{-ist}e^{-itD}e^{-isX}$. We recall that the spectrum of both operators is $\mathbb{R}$. Then, for any functions $\varphi, \eta \in L^\infty(\mathbb{R})$, one can consider by bounded functional calculus the operators $\varphi(X)$ and $\eta(D)$ in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. And by mixing some operators $\varphi_i(X)$ and $\eta_i(D)$ for suitable functions φ_i and η_i, we are going to produce an algebra $\mathcal{C}$ which will be useful in many applications.

Let us consider the closure in $\mathcal{B}\big(L^2(\mathbb{R})\big)$ of the C^*-algebra generated by elements of the form $\varphi_i(D)\eta_i(X)$, where φ_i, η_i are continuous functions on $\mathbb{R}$ which have limits at $\pm\infty$. Stated differently, φ_i, η_i belong to $C([-\infty, +\infty])$. Note that this algebra is clearly unital. In the sequel, we shall use the following notation:

$$\mathcal{C}_{(D,X)} := C^*\Big(\varphi_i(D)\eta_i(X) \mid \varphi_i, \eta_i \in C([-\infty, +\infty])\Big).$$

Let us also consider the C^*-algebra generated by $\varphi_i(D)\eta_i(X)$ with $\varphi_i, \eta_i \in C_0(\mathbb{R})$, which means that these functions are continuous and vanish at $\pm\infty$. As easily observed, this algebra is a closed ideal in $\mathcal{C}_{(D,X)}$ and is equal to the C^*-algebra $\mathcal{K}\big(L^2(\mathbb{R})\big)$ of compact operators in $L^2(\mathbb{R})$, see for example [GI03, Corol. 2.18].

Let us now study the quotient C^*-algebra $\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)$. For that purpose, we consider the square $\blacksquare := [-\infty,+\infty]\times[-\infty,+\infty]$ whose boundary $\square$ is the union of four parts: $\square = C_1\cup C_2\cup C_3\cup C_4$, with $C_1=\{-\infty\}\times[-\infty,+\infty]$, $C_2=[-\infty,+\infty]\times\{+\infty\}$, $C_3=\{+\infty\}\times[-\infty,+\infty]$ and $C_4=[-\infty,+\infty]\times\{-\infty\}$. We can also view $C(\square)$ as the subalgebra of

$$C([-\infty,+\infty])\oplus C([-\infty,+\infty])\oplus C([-\infty,+\infty])\oplus C([-\infty,+\infty]) \tag{10.1}$$

given by elements $\Gamma := (\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$ which coincide at the corresponding end points, that is, $\Gamma_1(+\infty)=\Gamma_2(-\infty)$, $\Gamma_2(+\infty)=\Gamma_3(+\infty)$, $\Gamma_3(-\infty)=\Gamma_4(+\infty)$, and $\Gamma_4(-\infty)=\Gamma_1(-\infty)$. Then $\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)$ is isomorphic to $C(\square)$, and if we denote the quotient map by

$$q:\mathcal{C}_{(D,X)}\to\mathcal{C}_{(D,X)}/\mathcal{K}\big(L^2(\mathbb{R})\big)\cong C(\square)$$

then the image $q\big(\varphi(D)\eta(X)\big)$ in (10.1) is given by $\Gamma_1=\varphi(-\infty)\eta(\cdot)$, $\Gamma_2=\varphi(\cdot)\eta(+\infty)$, $\Gamma_3=\varphi(+\infty)\eta(\cdot)$ and $\Gamma_4=\varphi(\cdot)\eta(-\infty)$. Note that this isomorphism is proved in [GI03, Thm. 3.22]. In summary, we have obtained the short exact sequence

$$0\to\mathcal{K}\big(L^2(\mathbb{R})\big)\hookrightarrow\mathcal{C}_{(D,X)}\xrightarrow{q}C(\square)\to 0 \tag{10.2}$$

with $\mathcal{K}\big(L^2(\mathbb{R})\big)$ and $\mathcal{C}_{(D,X)}$ represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$, but with $C(\square)$ which is not naturally represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. Note however that each of the four functions summing up in an element of $C(\square)$ can individually be represented in $\mathcal{B}\big(L^2(\mathbb{R})\big)$, either as a multiplication operator or as a convolution operator.

We shall now construct several isomorphic versions of these algebras. First of all, let us consider the Hilbert space $L^2(\mathbb{R}_+)$ and the action of the dilation group. More precisely, we consider the unitary group $\{U_t\}_{t\in\mathbb{R}}$ acting on any $f\in L^2(\mathbb{R}_+)$ as

$$[U_tf](x)=e^{t/2}f\big(e^tx\big),\qquad \forall x\in\mathbb{R}_+ \tag{10.3}$$

which is usually called *the unitary group of dilations*, and denote its self-adjoint generator by A and call it *the generator of dilations*.

Let also B be the operator of multiplication in $L^2(\mathbb{R}_+)$ by the function $-\ln$, *i.e.* $[Bf](\lambda)=-\ln(\lambda)f(\lambda)$ for any $f\in C_c(\mathbb{R}_+)$ and $\lambda\in\mathbb{R}_+$. Note that if one sets L for the self-adjoint operator of multiplication by the variable in $L^2(\mathbb{R}_+)$, *i.e.*

$$[Lf](\lambda):=\lambda f(\lambda)\qquad f\in C_c(\mathbb{R}_+)\text{ and }\lambda\in\mathbb{R}_+, \tag{10.4}$$

then one has $B=-\ln(L)$. Now, the equality $[iB,A]=1$ holds (once suitably defined), and the relation between the pair of operators (D,X) in $L^2(\mathbb{R})$ and the pair (B,A) in $L^2(\mathbb{R}_+)$ is well-known and corresponds to the Mellin transform. Indeed, let $\mathscr{V}:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ be defined by $(\mathscr{V}f)(x):=e^{x/2}f(e^x)$ for $x\in\mathbb{R}$, and remark that $\mathscr{V}$ is a unitary map with adjoint $\mathscr{V}^*$ given by $(\mathscr{V}^*g)(\lambda)=\lambda^{-1/2}g(\ln\lambda)$ for $\lambda\in\mathbb{R}_+$. Then, the Mellin transform $\mathscr{M}:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ is defined by $\mathscr{M}:=\mathcal{F}\mathscr{V}$ with $\mathcal{F}$ the usual unitary Fourier transform[1] in $L^2(\mathbb{R})$. The main property of $\mathscr{M}$ is that it diagonalizes the generator of dilations, namely, $\mathscr{M}A\mathscr{M}^* = X$. Note that one also has $\mathscr{M}B\mathscr{M}^* = D$.

Before introducing a first isomorphic algebra, observe that if $\eta \in C([-\infty, +\infty])$, then

$$\mathscr{M}^*\eta(D)\mathscr{M} = \eta(\mathscr{M}^*D\mathscr{M}) = \eta(B) = \eta\big(-\ln(L)\big) \equiv \psi(L)$$

for some $\psi \in C([0, +\infty])$. Thus, by taking these equalities into account, it is natural to define in $\mathcal{B}\big(L^2(\mathbb{R}_+)\big)$ the C^*-algebra

$$\mathcal{C}_{(L,A)} := C^*\Big(\psi_i(L)\eta_i(A) \mid \psi_i \in C([0, +\infty]) \text{ and } \eta_i \in C([-\infty, +\infty])\Big),$$

and clearly this algebra is isomorphic to the C^*-algebra $\mathcal{C}_{(D,X)}$ in $\mathcal{B}\big(L^2(\mathbb{R})\big)$. Thus, through this isomorphism one gets again a short exact sequence

$$0 \to \mathcal{K}\big(L^2(\mathbb{R}_+)\big) \hookrightarrow \mathcal{C}_{(L,A)} \xrightarrow{q} C(\square) \to 0$$

with the square $\square$ made of the four parts $\square = B_1 \cup B_2 \cup B_3 \cup B_4$ with $B_1 = \{0\} \times [-\infty, +\infty]$, $B_2 = [0, +\infty] \times \{+\infty\}$, $B_3 = \{+\infty\} \times [-\infty, +\infty]$, and $B_4 = [0, +\infty] \times \{-\infty\}$. In addition, the algebra $C(\square)$ of continuous functions on $\square$ can be viewed as a subalgebra of

$$C\big([-\infty, +\infty]\big) \oplus C\big([0, +\infty]\big) \oplus C\big([-\infty, +\infty]\big) \oplus C\big([0, +\infty]\big) \tag{10.5}$$

given by elements $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ which coincide at the corresponding end points, that is, $\Gamma_1(+\infty) = \Gamma_2(0)$, $\Gamma_2(+\infty) = \Gamma_3(+\infty)$, $\Gamma_3(-\infty) = \Gamma_4(+\infty)$, and $\Gamma_4(0) = \Gamma_1(-\infty)$.

Finally, if one sets $\mathcal{F}_\mathrm{s}$ for the unitary Fourier sine transformation in $L^2(\mathbb{R}_+)$, defined for $x, k \in \mathbb{R}_+$ and any $f \in C_\mathrm{c}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ by

$$[\mathcal{F}_\mathrm{s}f](k) := (2/\pi)^{1/2} \int_0^\infty \sin(kx) f(x)\,\mathrm{d}x \tag{10.6}$$

then the equalities $-A = \mathcal{F}_\mathrm{s}^* A \mathcal{F}_\mathrm{s}$ and $\sqrt{H_\mathrm{D}} = \mathcal{F}_\mathrm{s}^* L \mathcal{F}_\mathrm{s}$ hold, where H_D corresponds to the Dirichlet Laplacian on $\mathbb{R}_+$ (see the next section for its definition). As a consequence, note that the formal equality $[i\frac{1}{2}\ln(H_\mathrm{D}), A] = 1$ can also be fully justified. Moreover, by using this new unitary transformation one gets that the C^*-subalgebra of $\mathcal{B}\big(L^2(\mathbb{R}_+)\big)$ defined by

$$\mathcal{C}_{(H_\mathrm{D},A)} := C^*\Big(\psi_i(H_\mathrm{D})\eta_i(A) \mid \psi_i \in C([0, +\infty]) \text{ and } \varphi_i \in C([-\infty, +\infty])\Big), \tag{10.7}$$

is again isomorphic to $\mathcal{C}_{(D,X)}$. In addition, the following short exact sequence takes place

$$0 \to \mathcal{K}\big(L^2(\mathbb{R}_+)\big) \hookrightarrow \mathcal{C}_{(H_\mathrm{D},A)} \xrightarrow{q} C(\square) \to 0, \tag{10.8}$$

and $C(\square)$ can naturally be viewed as a subalgebra of the algebra introduced in (10.5) with suitable compatibility conditions at end points.

[1]For $f \in C_\mathrm{c}(\mathbb{R})$ and $x \in \mathbb{R}$ we set $[\mathcal{F}f](x) = (2\pi)^{-1/2} \int_\mathbb{R} e^{-ixy} f(y)\,\mathrm{d}y$.

10.2 Elementary scattering system

In this section we introduce an example of a scattering system for which everything can be computed explicitly. It will allow us to describe more precisely the kind of results we are looking for, without having to introduce too much information on scattering theory. In fact, we shall keep the content of this section as simple as possible.

Let us start by considering the Hilbert space $L^2(\mathbb{R}_+)$ and the Dirichlet Laplacian H_{D} on $\mathbb{R}_+ := (0,\infty)$. More precisely, we set $H_{\mathrm{D}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the domain $\mathsf{Dom}(H_{\mathrm{D}}) = \{f \in \mathcal{H}^2(\mathbb{R}_+) \mid f(0) = 0\}$. Here $\mathcal{H}^2(\mathbb{R}_+)$ means the usual Sobolev space on $\mathbb{R}_+$ of order 2. For any $\alpha \in \mathbb{R}$, let us also consider the operator H^α defined by $H^\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $\mathsf{Dom}(H^\alpha) = \{f \in \mathcal{H}^2(\mathbb{R}_+) \mid f'(0) = \alpha f(0)\}$. It is well-known that if $\alpha < 0$ the operator H^α possesses only one eigenvalue, namely $-\alpha^2$, and the corresponding eigenspace is generated by the function $x \mapsto e^{\alpha x}$. On the other hand, for $\alpha \geq 0$ the operators H^α have no eigenvalue, and so does H_{D}.

A common object of scattering theory is defined by the following formula:

$$W_\pm^\alpha := s - \lim_{t\to\pm\infty} e^{itH^\alpha} e^{-itH_{\mathrm{D}}},$$

and these limits in the strong sense are known to exist for this model. These operators are called *the wave operators*, they are isometries, and their existence allows one to study the operator H^α with respect to H_{D}. Moreover, we shall provide below a very explicit formula for these operators. Let us still stress that scattering theory is a comparison theory, one always study pairs of operators.

Our first result for this model then reads, see [Ric15, Cor. 9.3] for its proof:

Lemma 10.2.1. *The following equalities hold:*

$$W_-^\alpha = 1 + \tfrac{1}{2}\big(1 + \tanh(\pi A) - i\cosh(\pi A)^{-1}\big)\Big[\frac{\alpha + i\sqrt{H_{\mathrm{D}}}}{\alpha - i\sqrt{H_{\mathrm{D}}}} - 1\Big],$$
$$W_+^\alpha = 1 + \tfrac{1}{2}\big(1 - \tanh(\pi A) + i\cosh(\pi A)^{-1}\big)\Big[\frac{\alpha - i\sqrt{H_{\mathrm{D}}}}{\alpha + i\sqrt{H_{\mathrm{D}}}} - 1\Big].$$

It clearly follows from these explicit formulas that the operators $W_\pm^\alpha$ belong to the algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}$ introduced in (10.7). Since these operators are also isometries with a finite dimensional co-kernel, they can be considered as lifts for their image in the quotient algebra $\mathcal{C}_{(H_{\mathrm{D}},A)}/\mathcal{K}\big(L^2(\mathbb{R}_+)\big)$. We shall come back to this approach involving algebras in the next session, and work very explicitly for the time being.

Motivated by the above formula for W_-^α, let us now introduce the complex function

$$\Gamma_{\blacksquare}^\alpha : [0,+\infty] \times [-\infty,+\infty] \ni (x,y) \mapsto 1 + \tfrac{1}{2}\big(1 + \tanh(\pi y) - i\cosh(\pi y)^{-1}\big)\Big[\frac{\alpha + i\sqrt{x}}{\alpha - i\sqrt{x}} - 1\Big].$$

Since this function is continuous on the square $\blacksquare := [0,+\infty]\times[-\infty,+\infty]$, its restriction on the boundary $\square$ of the square is also well defined and continuous. Note that this boundary is made of four parts: $\square = B_1 \cup B_2 \cup B_3 \cup B_4$ with $B_1 = \{0\} \times [-\infty,+\infty]$,

$B_2 = [0, +\infty] \times \{+\infty\}$, $B_3 = \{+\infty\} \times [-\infty, +\infty]$, and $B_4 = [0, +\infty] \times \{-\infty\}$, and that the algebra $C(\square)$ of continuous functions on $\square$ can be viewed as a subalgebra of (10.5) with the necessary compatibility conditions at the end points. With these notations, the restriction function $\Gamma^\alpha_\square := \Gamma^\alpha_\blacksquare\big|_\square$ is given for $\alpha \neq 0$ by

$$\Gamma^\alpha_\square = \Big(1, \frac{\alpha + i\sqrt{\cdot}}{\alpha - i\sqrt{\cdot}}, -\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, 1\Big) \tag{10.9}$$

and for $\alpha = 0$ by

$$\Gamma^0_\square := \Big(-\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, -1, -\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1}, 1\Big). \tag{10.10}$$

For simplicity, we have directly written this function in the representation provided by (10.5).

Let us now observe that the boundary $\square$ of $\blacksquare$ is homeomorphic to the circle $\mathbb{S}$. Observe in addition that the function $\Gamma^\alpha_\square$ takes its values in the unit circle $\mathbb{T}$ of $\mathbb{C}$. Then, since $\Gamma^\alpha_\square$ is a continuous function on the closed curve $\square$ and takes values in $\mathbb{T}$, its winding number $\mathrm{Wind}(\Gamma^\alpha_\square)$ is well defined and can easily be computed. So, let us compute separately the contribution $w_j(\Gamma^\alpha_\square)$ to this winding number on each component B_j of $\square$. By convention, we shall turn around $\square$ clockwise, starting from the left-down corner, and the increase in the winding number is also counted clockwise. Let us stress that the contribution on B_3 has to be computed from $+\infty$ to $-\infty$, and the contribution on B_4 from $+\infty$ to 0. Without difficulty one gets:

	$w_1(\Gamma^\alpha_\square)$	$w_2(\Gamma^\alpha_\square)$	$w_3(\Gamma^\alpha_\square)$	$w_4(\Gamma^\alpha_\square)$	$\mathrm{Wind}(\Gamma^\alpha_\square)$
$\alpha < 0$	0	1/2	1/2	0	1
$\alpha = 0$	−1/2	0	1/2	0	0
$\alpha > 0$	0	−1/2	1/2	0	0

By comparing the last column of this table with the information on the eigenvalues of H^α mentioned at the beginning of the section one gets:

Proposition 10.2.2. *For any $\alpha \in \mathbb{R}$ the following equality holds:*

$$\mathrm{Wind}(\Gamma^\alpha_\square) = \textit{ number of eigenvalues of } H^\alpha. \tag{10.11}$$

The content of this proposition is an example of Levinson's theorem. Indeed, it relates the number of bound states of the operator H^α to a quantity computed on the scattering part of the system. Let us already mention that the contribution $w_2(\Gamma^\alpha_\square)$ is the only one usually considered in the literature. However, we can immediately observe that if $w_1(\Gamma^\alpha_\square)$ and $w_3(\Gamma^\alpha_\square)$ are disregarded, then no meaningful statement can be obtained.

Obviously, the above result should now be recast in a more general framework, and the algebraic background should be taken into account. Indeed, except for very specific models, it is usually not possible to compute precisely both sides of (10.11), but such an equality still holds in a much more general setting. The next section shows how K-theory can provide an insight on Levinson's theorem.

10.3 The abstract topological Levinson's theorem

Before stating the main result of this chapter, let us reformulate the content of Proposition 6.2.4.(*ii*). The key point in the next statement is that the central role is played by the partial isometry in $\mathcal{C}$ instead of the unitary element in $\mathcal{Q}$. In fact, the following statement is at the root of our topological approach of Levinson's theorem.

Proposition 10.3.1. *Consider the short exact sequence*

$$0 \to \mathcal{J} \hookrightarrow \mathcal{C} \xrightarrow{q} \mathcal{Q} \to 0$$

with $\mathcal{C}$ unital. Let W be a partial isometry in $M_n(\mathcal{C})$ and assume that $\Gamma := q(W)$ is a unitary element of $M_n(\mathcal{Q})$. Then $\mathbf{1}_n - W^*W$ and $\mathbf{1}_n - WW^*$ are projections in $M_n(\mathcal{J})$, and

$$\mathrm{ind}([q(W)]_1) := \delta_1([q(W)]_1) = [\mathbf{1}_n - W^*W]_0 - [\mathbf{1}_n - WW^*]_0 \ .$$

In order to go one more step in our construction, let us add some information about some special K-groups, as already mentioned in Example 4.3.6 and in Example 7.4.3.

Example 10.3.2. *(i) Let $C(\mathbb{S})$ denote the C^*-algebra of continuous functions on the unit circle $\mathbb{S}$, with the L^∞-norm, and let us identify this algebra with $\big\{\zeta \in C([0,2\pi]) \mid \zeta(0) = \zeta(2\pi)\big\}$, also endowed with the L^∞-norm. Some unitary elements of $C(\mathbb{S})$ are provided for any $m \in \mathbb{Z}$ by the functions*

$$\zeta_m : [0, 2\pi] \ni \theta \mapsto e^{-im\theta} \in \mathbb{T}.$$

Clearly, for two different values of m the functions ζ_m are not homotopic, and thus define different classes in $K_1\big(C(\mathbb{S})\big)$. With some more efforts one can show that these elements define in fact all elements of $K_1\big(C(\mathbb{S})\big)$, and indeed one has

$$K_1\big(C(\mathbb{S})\big) \cong \mathbb{Z}.$$

Note that this isomorphism is implemented by the winding number $\mathrm{Wind}(\cdot)$*, which is roughly defined for any continuous function on $\mathbb{S}$ with values in $\mathbb{T}$ as the number of times this function turns around 0 along the path from 0 to 2π. Clearly, for any $m \in \mathbb{Z}$ one has $\mathrm{Wind}(\zeta_m) = m$. More generally, if* $\det$ *denotes the determinant on $M_n(\mathbb{C})$ then the mentioned isomorphism is given by* $\mathrm{Wind} \circ \det$ *on $\mathcal{U}_n\big(C(\mathbb{S})\big)$.*

(ii) Let $\mathcal{K}(\mathcal{H})$ denote the C^*-algebra of all compact operators on a infinite dimensional and separable Hilbert space $\mathcal{H}$. For any n one can consider the orthogonal projections on subspaces of dimension n of $\mathcal{H}$, and these finite dimensional projections belong to $\mathcal{K}(\mathcal{H})$. It is then not too difficult to show that two projections of the same dimension are Murray-von Neumann equivalent, while projections corresponding to two different values of n are not. With some more efforts, one shows that the dimension of these projections plays the crucial role for the definition of $K_0\big(\mathcal{K}(\mathcal{H})\big)$, and one has again

$$K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}.$$

In this case, the isomorphism is provided by the usual trace Tr *on finite dimensional projections, and by the tensor product of this trace with the trace* tr *on* $M_n(\mathbb{C})$. *More precisely, on any element of* $\mathcal{P}_n\big(\mathcal{K}(\mathcal{H})\big)$ *the mentioned isomorphism is provided by* $\mathrm{Tr}\circ\mathrm{tr}$.

Let us now add the different pieces of information we have presented so far, and get an abstract version of our Levinson's theorem. For that purpose, we consider an arbitrary separable Hilbert space $\mathcal{H}$ and a unital C^*-subalgebra $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$ which contains the ideal of $\mathcal{K}(\mathcal{H})$ of compact operators. We can thus look at the short exact sequence of C^*-algebras

$$0 \to \mathcal{K}(\mathcal{H}) \hookrightarrow \mathcal{C} \xrightarrow{q} \mathcal{C}/\mathcal{K}(\mathcal{H}) \to 0.$$

Let us assume in addition that $\mathcal{C}/\mathcal{K}(\mathcal{H})$ is isomorphic to $C(\mathbb{S})$. Then, if we take the results presented in the previous example into account, one infers that

$$\mathbb{Z} \cong K_1\big(C(\mathbb{S})\big) \xrightarrow{\text{ind}} K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$$

with the first isomorphism realized by the winding number and the second isomorphism realized by the trace. As a consequence, one infers from this together with Proposition 10.3.1 that there exists $\mathfrak{n} \in \mathbb{Z}$ such that for any partial isometry $W \in \mathcal{C}$ with unitary $\Gamma := q(W) \in C(\mathbb{S})$ the following equality holds:

$$\mathrm{Wind}(\Gamma) = \mathfrak{n}\mathrm{Tr}\big([\mathbf{1} - W^*W] - [\mathbf{1} - WW^*]\big). \tag{10.12}$$

We emphasize once again that the interest in this equality is that the left hand side is independent of the choice of any special representative in $[\Gamma]_1$. On the other hand, in the context of scattering theory the r.h.s. of (10.12) is well understood, see the next statement. Let us also mention that the number $\mathfrak{n}$ depends on the choice of the extension of $\mathcal{K}(\mathcal{H})$ by $C(\mathbb{S})$, see [W-O93, Chap. 3.2], but also on the convention chosen for the computation of the winding number.

If we summarize all this in a single statement, one gets:

Theorem 10.3.3 (Abstract topological Levinson's theorem)**.** *Let* $\mathcal{H}$ *be a separable Hilbert space, and let* $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ *be a unital* C^**-algebra such that* $\mathcal{K}(\mathcal{H}) \subset \mathcal{C}$ *and* $\mathcal{C}/\mathcal{K}(\mathcal{H}) \cong C(\mathbb{S})$ *(with quotient morphism denoted by* q*). Then there exists* $\mathfrak{n} \in \mathbb{Z}$ *such that for any partial isometry* $W \in \mathcal{C}$ *with unitary* $\Gamma := q(W) \in C(\mathbb{S})$ *the following equality holds:*

$$\mathrm{Wind}(\Gamma) = \mathfrak{n}\mathrm{Tr}\big([\mathbf{1} - W^*W] - [\mathbf{1} - WW^*]\big). \tag{10.13}$$

In particular if $W = W_- = s - \lim_{t\to-\infty} e^{itH}e^{-itH_0}$ *for some suitable scattering pair* (H, H_0)*, then the previous equality reads*

$$\mathrm{Wind}\big(q(W_-)\big) = -\mathfrak{n}\big(\textit{number of eigenvalues of } H - \textit{number of eigenvalues of } H_0\big).$$

Note that in applications, the factor $\mathfrak{n}$ is determined by computing both sides of the equality on an explicit example.

Let us finally show that the example presented in Section 10.2 can be recast in the previous framework. We consider the Hilbert space $L^2(\mathbb{R}_+)$ and the unital C^*-algebra $\mathcal{C}_{(H_{\rm D},A)}$ introduced in (10.7). As already mentioned, the wave operator W_-^α is an isometry which clearly belongs to the C^*-algebra $\mathcal{C}_{(H_{\rm D},A)} \subset \mathcal{B}\big(L^2(\mathbb{R}_+)\big)$. In addition, the image of W_-^α in the quotient algebra $\mathcal{C}_{(H_{\rm D},A)}/\mathcal{K}\big(L^2(\mathbb{R}_+)\big) \cong C(\square)$ is precisely the function $\Gamma_\square^\alpha$, defined in (10.9) for $\alpha \neq 0$ and in (10.10) for $\alpha = 0$, which are unitary elements of $C(\square)$. Finally, since $C(\square)$ and $C(\mathbb{S})$ are clearly isomorphic, the winding number $\mathrm{Wind}(\Gamma_\square^\alpha)$ of $\Gamma_\square^\alpha$ can be computed, and in fact this has been performed and recorded in the table of Section 10.2.

If one sets $E_p(H^\alpha)$ for the orthogonal projection on the subspace generated by the bound states of the operator H^α, then one has

$$\mathrm{Tr}\big([\mathbf{1} - (W_-^\alpha)^* W_-^\alpha] - [\mathbf{1} - W_-^\alpha (W_-^\alpha)^*]\big) = -\mathrm{Tr}\big(E_p(H^\alpha)\big) = \begin{cases} -1 & \text{if } \alpha < 0\,, \\ 0 & \text{if } \alpha \geq 0\,. \end{cases} \tag{10.14}$$

Thus, this example fits in the framework of Theorem 10.3.3, and in addition both sides of (10.13) have been computed explicitly. By comparing (10.14) with the results obtained for $\mathrm{Wind}(\Gamma_\square^\alpha)$, one gets that the factor $\mathfrak{n}$ mentioned in (10.13) is equal to -1 for these algebras. Finally, since $E_p(H^\alpha)$ is related to the point spectrum of H^α, the content of Proposition 10.2.2 can be rewritten as

$$\mathrm{Wind}(\Gamma_\square^\alpha) = \sharp\sigma_{\rm p}(H^\alpha).$$

This equality corresponds to a topological version of Levinson's theorem for the elementary model. Obviously, this result was already obtained in Section 10.2 and all the above framework was not necessary for its derivation. However, we have now in our hands a very robust framework which can be applied to several other situations, see [Ric15] and the references therein.

Remark 10.3.4. *As a concluding remark, let us mention how the algebraic framework could still be extended. For that purpose, consider a short exact sequence*

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{C} \longrightarrow \mathcal{Q} \longrightarrow 0$$

and the corresponding index map $\mathrm{ind} : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$*. Assume that* η *is an even* n*-trace on* $\mathcal{J}$ *which can be paired with* $K_0(\mathcal{J})$*, see Theorem 9.5.8. Then one can wonder if there exists a map on higher traces which is dual to the index map, i.e. a map* $\#$ *which assigns to an even trace* η *an odd trace* $\#\eta$ *such that the equality*

$$\big\langle [\mathrm{ind}(\Gamma)]_0, [\eta] \big\rangle = \big\langle [\Gamma]_1, [\#\eta] \big\rangle \tag{10.15}$$

holds, for any $\Gamma \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ *? Except for some special cases (like in Theorem 10.3.3 for a* 0*-trace and a* 1*-trace), the answer to this question is apparently not known.*