Chapter 8

Applications

In this chapter, we provide two examples of applications of the various concepts and results we have developed in the previous chapters.

8.1 Discrete dynamical systems

8.1.1 Coyotes and roadrunners

Let us consider two species in a desert: coyotes and roadrunners. We denote by $c(n)$ the population of coyotes at year n, and by $r(n)$ the population of roadrunners at year n. The following equation models the transformation of the system, as a function of the year:

$$\begin{cases} c(n+1) = 0.86\,c(n) + 0.08\,r(n) \\ r(n+1) = -0.12\,c(n) + 1.14\,r(n) \end{cases} \tag{8.1.1}$$

Note that each of these coefficients has an interpretation: 0.86 and 1.14 correspond to the birth rate of the coyotes and or the roadrunners, respectively. The coefficient 0.08 can be interpreted as a favorable factor on the population of coyotes whenever roadrunners can be eaten, while -0.12 corresponds to the decrease in the population of roadrunners due to coyotes' appetite. Now, if one sets $X(n) = \begin{pmatrix} c(n) \\ r(n) \end{pmatrix}$ one can then rewrite this system as

$$X(n+1) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} X(n).$$

For later use, we set $\mathcal{A} := \left(\begin{smallmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{smallmatrix}\right)$.

Question: If at year $n = 0$ we observe the populations $X(0) = \begin{pmatrix} c(0) \\ r(0) \end{pmatrix}$, what about the populations $X(n)$ for large n, *i.e.* in the far future ?

For example, it $X(0) = \left(\begin{smallmatrix} 100 \\ 100 \end{smallmatrix}\right)$, then $X(1) = \mathcal{A}X(0) = \left(\begin{smallmatrix} 96 \\ 102 \end{smallmatrix}\right)$, $X(2) = \mathcal{A}^2X(0) = \mathcal{A}X(1)$, and $X(10) = \mathcal{A}^{10}X(0) \cong \left(\begin{smallmatrix} 80 \\ 170 \end{smallmatrix}\right)$. Here the computations are rather lengthy. On the other hand, if $X(0) = \binom{100}{300}$, then $X(1) = \binom{110}{330} = 1.1X(0)$, $X(2) = \mathcal{A}^2X(0) = \mathcal{A}X(1) = \mathcal{A}\big(1.1X(0)\big) = 1.1^2X(0)$, and thus $X(10) = 1.1^{10}X(0)$. In such a case, the computations are easier, but note that the populations are continuously increasing (there is no correlation between these two facts). Note also that a similar computation shows that if $X(0) = \binom{200}{100}$, then one has $X(n) \to \binom{0}{0}$ as n tends to infinity.

Now, a natural question is how can one organize these information more systematically ? In fact, this is possible with the help of the eigenvalues and the eigenvectors of the matrix $\mathcal{A}$. For that purpose, let us first observe that

$$P_{\mathcal{A}}(\lambda) = \lambda^2 - 2\lambda + 0.99 = (\lambda - 1.1)(\lambda - 0.9).$$

Thus, the eigenvalues of $\mathcal{A}$ are 1.1 and 0.9. Moreover, if X_1 and X_2 are eigenvectors of $\mathcal{A}$ associated with the eigenvalues 1.1 and 0.9 respectively, recall that we can define an invertible matrix $\mathcal{B}$ by $\mathcal{B} = \big(X_1\ X_2\big)$, and then that

$$\mathcal{B}^{-1}\mathcal{A}\mathcal{B} = \begin{pmatrix} 1.1 & 0 \\ 0 & 0.9 \end{pmatrix} \quad \text{or equivalently} \quad \mathcal{A} = \mathcal{B}\begin{pmatrix} 1.1 & 0 \\ 0 & 0.9 \end{pmatrix}\mathcal{B}^{-1}.$$

With this rather simple information at hands, some computations simplify a lot. For example, one directly obtains that

$$\mathcal{A}^n = \mathcal{B}\begin{pmatrix} 1.1^n & 0 \\ 0 & 0.9^n \end{pmatrix}\mathcal{B}^{-1}$$

which requires much less efforts than multiplying n times $\mathcal{A}$ by itself.

Alternatively, since $\{X_1, X_2\}$ generate a basis of $\mathbb{R}^2$, any initial condition $X(0)$ can be decomposed with respect to this basis and one has $X(0) = c_1X_1 + c_2X_2$, with $c_1, c_2 \in \mathbb{R}$. Then, one has

$$\begin{aligned}\mathcal{A}^nX(0) &= \mathcal{A}^n(c_1X_1 + c_2X_2) = c_1\mathcal{A}^nX_1 + c_2\mathcal{A}^nX_2 \\ &= c_1(1.1)^nX_1 + c_2(0.9)^nX_2 = 1.1^nc_1X_1 + 0.9^nc_2X_2.\end{aligned}$$

Thus, knowing the eigenvalues of $\mathcal{A}$, one can better understand why, depending on the initial populations, the populations can either increase as n tends to infinity (due to the eigenvalue 1.1), or vanish as n tends to infinity (due to the eigenvalue 0.9).

Question: Can one find $X(0)$ such that both populations remain constant as n goes to infinity ?

8.1.2 Discrete dynamical systems with real eigenvalues

More generally, suppose that we consider discrete evolution system given by the equation $X(n+1) = \mathcal{A}X(n)$ for some $\mathcal{A} \in M_m(\mathbb{R})$. Assume in addition that $\mathcal{A}$ is diagonalizable, with its m eigenvalues real, *i.e.* there exists an invertible matrix $\mathcal{B} \in M_m(\mathbb{R})$ such that $\mathcal{A} = \mathcal{B}\ \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\ \mathcal{B}^{-1}$, with $\lambda_j \in \mathbb{R}$ for any $j \in \{1, \ldots, m\}$. Then, the family of eigenvectors associated with the eigenvalues of $\mathcal{A}$ generates a basis of $\mathbb{R}^m$ and any initial state $X(0)$ can be decomposed with respect to this basis. Thus, if X_j denotes an eigenvector associated with the eigenvalue λ_j one has $X(0) = c_1X_1 + \cdots + c_mX_m$ for some $c_j \in \mathbb{R}$, and again

$$\mathcal{A}^nX(0) = \lambda_1^n c_1X_1 + \lambda_2^n c_2X_2 + \cdots + \lambda_m^n c_mX_m.$$

Then, depending on λ_j, the large n behavior of the system can be predicted. Note that in the following description, the index j is an arbitrary element of $\{1, \ldots, m\}$.

(i) If $\lambda_j > 1$, the corresponding part of the system grows infinitely,

(ii) If $\lambda_j = 1$, the corresponding part of the system remains the same forever,

(iii) If $0 < \lambda_j < 1$, the corresponding part of the system tends to vanish as n tends to infinity,

(iv) If $\lambda_j = 0$, the corresponding part of the system disappears already for $n = 1$ (this part of the system belongs to the kernel of $\mathcal{A}$),

(v) If $-1 < \lambda_j < 0$, the corresponding part of the system tends to vanish as n tends to infinity, but its sign is alternating for n even or n odd,

(vi) If $\lambda_j = -1$, the corresponding part of the system alternates between two states with a different sign,

(vii) If $\lambda_j < -1$, the corresponding grows infinitely in norm, but its sign is alternating for n even or n odd.

8.1.3 Discrete dynamical systems with complex eigenvalues

Let us now consider a discrete evolution system given by the equation $X(n+1) = \mathcal{A}X(n)$ with $\mathcal{A} \in M_2(\mathbb{R})$, but with $P_{\mathcal{A}}(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$ and $\lambda_1 = x + iy$, $\lambda_2 = x - iy$ and $y \neq 0$. In such a case, there again exists an invertible matrix $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}\begin{pmatrix} x+iy & 0 \\ 0 & x-iy \end{pmatrix}\mathcal{B}^{-1}$ and therefore

$$\mathcal{A}^n = \mathcal{B}\begin{pmatrix} (x+iy)^n & 0 \\ 0 & (x-iy)^n \end{pmatrix}\mathcal{B}^{-1}.$$

In order to compute $(x \pm iy)^n$, let us first observe that

$$x \pm iy = |x \pm iy|\,\frac{x \pm iy}{|x \pm iy|} = \sqrt{x^2+y^2}\Big(\frac{x}{\sqrt{x^2+y^2}} \pm i\frac{y}{\sqrt{x^2+y^2}}\Big).$$

As a consequence, one can write $x + iy = \sqrt{x^2+y^2}\big(\cos(2\pi\theta) + i\sin(2\pi\theta)\big)$ for a unique $\theta \in [0, 1)$. Then, by using de Moivre's formula, as shown in Exercise 9.6, one gets that

$$(x+iy)^n = (x^2+y^2)^{n/2}\big(\cos(2\pi n\theta) + i\sin(2\pi n\theta)\big).$$

Note that a similar formula holds for $(x - iy)$ and for $(x - iy)^n$. Then, depending if $\sqrt{x^2 + y^2}$ is bigger, equal or smaller than 0, and depending if θ is rational or irrational (*i.e.* if $\theta = p/q$ for some $p, q \in \mathbb{Z}$ or not), the asymptotic behavior of $(x + iy)^n$ changes drastically. For example, if $\sqrt{x^2 + y^2} > 1$, the $|(x + iy)^n|$ goes to infinity as n goes to infinity, while if $\sqrt{x^2 + y^2} < 1$, then $|(x + iy)^n|$ goes to 0 as n tends to infinity. If $\sqrt{x^2 + y^2} = 1$ and if $\theta = p/q$ for some $p, q \in \mathbb{Z}$, then $(x+iy)^n$ is periodic, with a period depending on p and q, while if $\theta \neq p/q$ for any $p, q \in \mathbb{Z}$, then $(x + iy)^n$ takes different values for any n. A notion of aperiodicity appears in fact in such a situation. Note that an enumeration of all possible behaviors as in the previous subsection could also be established in the present setting.

8.2 The $ 25’000’000’000 eigenvector

This section is inspired from the paper ”The $25’000’000’000 eigenvector: the linear algebra behind Google”[1], which gives an another opportunity to use the concepts introduced in the previous chapters.

Let us first list what a search engine for internet has to do:

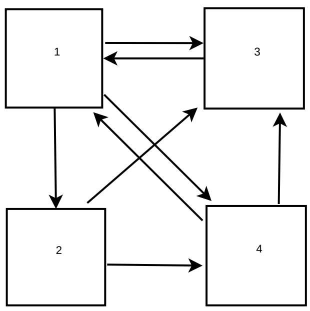
(i) Locate all web pages with public access,

(ii) Index these data with keywords,

(iii) Rate the importance of each page.

Question: How can one define and quantify the ”importance” of a web page ?

We shall call *the importance score* or simply *the score* such a quantitative rating. The main idea behind Google page ranking is derived from the links to that page (called backlinks). For example, let us look at the world wide web which contains only 4 pages as represented in the following figure. Each arrow $A \longrightarrow B$ represents a link from A to



B (a backlink for B) Let us also set $x_k \geq 0$ for the importance of the page k, with the convention that $x_j > x_k$ means that the page j is more important than the page k.

[1]Kurt Bryan, Tanya Leise, *The $25’000’000’000 eigenvector: the linear algebra behind Google*, SIAM REVIEW, Vol. 48, No. 3, pp. 569–581.

A first idea for the ranking could be to assign for x_k the number of backlinks. In this case, one would obtain $x_1 = 2$, $x_2 = 1$, $x_3 = 3$ and $x_4 = 2$, and the page 3 would then be the most important one. However, one also would like that a link to page j from an important page boosts the score of page j more than a link to page j from an unimportant page. For example, a link from BBC.com to your web page is certainly more important than a link from the web page of your neighbour to your web page. Thus, a more refined way to compute the score should be implemented.

A second idea for the ranking could be to score the page j with the sum of the score of the pages linking to page j. With this approach, an important page with a link to page j would boost the score of page j. Of course, the procedure becomes more complicated because it is self-referential. In addition, a single page should not gain any importance by containing too many links. For that purpose, we shall impose that each page has a total vote of 1, as in a democracy.

Taking these remarks into account, let us set

$$x_j := \sum_{\text{pages } k \text{ linking to } j} \frac{x_k}{n_k}$$

where n_k is the number of links emerging from page k. Note that if a page has a link to itself, this link is ignored. For the example shown above, we then obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

which is equivalent to $X = \mathcal{A}X$ with $X = {}^t(x_1, x_2, x_3, x_4)$ and $\mathcal{A}$ the matrix shown above. In other words, computing the scores x_j corresponds to finding an eigenvector X for the linear map $\mathrm{L}_{\mathcal{A}}$ associated with the eigenvalue 1, if such an eigenvalue exists. In this setting, the matrix $\mathcal{A}$ is called *the link matrix* for a given web.

Note that in the example shown above, any multiple of the vector ${}^t(12, 4, 9, 6)$ is an eigenvector of $\mathrm{L}_{\mathcal{A}}$ associated with the eigenvalue 1. In we impose in addition that $\sum_j x_j = 1$, then we get $x_1 = 12/31 \cong 0.387$, $x_2 = 4/31 \cong 0.129$, $x_3 = 9/31 \cong 0.290$ and $x_4 = 6/31 \cong 0.194$. Let us observe that page 3 is no more the most important one. Indeed, this important page has only one link to page 1, and this single link boosts the score of page 1 which then becomes the most important one.

Let us now try to think a little bit more generally. Assume that the web has no page with 0 outgoing link, which means that $n_k \neq 0$ for any k. With such an assumption, the entries of any column of a link matrix sum up to 1. Indeed, each page j gives $\frac{1}{n_j}$ of its vote to n_j different pages. With this observation, let's come back to mathematics.

Lemma 8.2.1. *If $n_k \neq 0$ for any k, then the linear map associated with any link matrix possesses the eigenvalue* 1.

Proof. Observe that 1 is an eigenvalue of $\mathrm{L}_{{}^t\mathcal{A}}$ with eigenvector ${}^t(1, 1, \dots 1)$. Indeed, the entries in each row of ${}^t\mathcal{A}$ sum to 1, and therefore

$$ {}^t\mathcal{A} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. $$

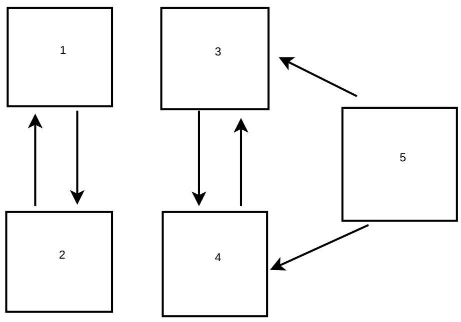
Since $\mathrm{L}_{{}^t\mathcal{A}}$ and $\mathrm{L}_{\mathcal{A}}$ share the same spectrum (see Lemma 7.2.9), it follows that 1 also belongs to the spectrum of $\mathrm{L}_{\mathcal{A}}$. □

Since the spectrum of the linear map associated with any link matrix contains the value 1, let us denote by V_1 the corresponding eigenspace. We can then wonder if this eigenspace is of dimension one (in which case the ranking is unique) or if it is of dimension higher than one (in which case there exist different rankings which can not be distinguished with our criteria) ? Another question is how to get rid of the assumption $n_k \neq 0$, since this assumption is not always satisfied (there exist interesting web pages without any outgoing link) ?

Let us now observe that the non-uniqueness of the ranking is possible if the web is disconnected. For example, consider the link matrix

$$ \mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} $$

associated with the following world wide web Such a disconnected web gives rise to a



matrix $\mathcal{A}$ which is block diagonal, see Exercises 4.28 and 4.29. In the above example, and more generally in any situation with a link matrix which is block diagonal, it is not difficult to see that the eigenspace associated with the eigenvalue 1 is of dimension 2 or higher.

One way to solve this problem is to replace the link matrix $\mathcal{A}$ by a slightly improved version of it. More precisely, consider the matrix

$$ \mathcal{A}_m = (1 - m)\mathcal{A} + m \begin{pmatrix} 1/n & \dots & 1/n \\ \vdots & \ddots & \vdots \\ 1/n & \dots & 1/n \end{pmatrix} $$

for some $m \in [0,1]$ and with n the number of web pages of the world wide web. Then, for any $m \in (0,1]$ the matrix $\mathcal{A}_m$ is no more block diagonal. In fact, this procedure corresponds to adding artificial links between each web pages on the web, which becomes no more disconnected. If $m = 1$, all pages are rated equally. At a certain time, Google used the value $m = 0.15$. In this setting the following statement can then be proved.

Proposition 8.2.2. *If $n_k \neq 0$ for any k, and if $m \in (0,1]$, then the dimension of the eigenspace associated with the eigenvalue 1 of $\mathrm{L}_{\mathcal{A}_m}$ is of dimension 1.*

The proof of this statement as well as much more information on how Google ranks the web pages of the world wide web are available in the paper of Kurt Bryan and Tanya Leise. Updated and more precise information are also available on internet. Just use Google to find them !