Chapter 6

The determinant

6.1 Multilinear maps

In this first section, we generalize the notions of linear maps and bilinear maps.

Definition 6.1.1. *Let V be a vector space over a field $\mathbb{F}$, and let $n \in \mathbb{N}$. A map*

$$\mathrm{T} : \underbrace{V \times V \times \cdots \times V}_{n \text{ terms}} \to \mathbb{F}$$

*is n-*linear *if it is linear in each argument,* i.e.

$$\begin{aligned}&\mathrm{T}(X_1, X_2, \dots, X_j + X_j', \dots, X_n)\\ &= \mathrm{T}(X_1, X_2, \dots, X_j, \dots, X_n) + \mathrm{T}(X_1, X_2, \dots, X_j', \dots, X_n)\end{aligned}$$

and

$$\mathrm{T}(X_1, X_2, \dots, \lambda X_j, \dots, X_n) = \lambda \mathrm{T}(X_1, X_2, \dots, X_j, \dots, X_n)$$

for any $X_1, \dots, X_j, X_j', \dots, X_n \in V$, $\lambda \in \mathbb{F}$ and $j \in \{1, \dots, n\}$. The set of all n-linear maps is denoted by $\mathrm{Mult}_n(V)$.

Note that if $n = 1$ one speaks about a linear map, while $n = 2$ corresponds to a bilinear map. Without difficulty one can show that the set of $\mathrm{Mult}_n(V)$ is a vector space.

Definition 6.1.2. *An element* $\mathrm{T} \in \mathrm{Mult}_n(V)$ *is* alternating *if*

$$\mathrm{T}(X_1, \dots, X_i, \dots, X_j, \dots, X_n) = 0$$

whenever $X_i = X_j$ for some $i, j \in \{1, \dots, n\}$ with $i \neq j$.

Example 6.1.3. *If $\mathcal{A} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, then the bilinear map $\mathrm{F}_\mathcal{A} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined in* (5.3.1) *is alternating. Indeed, if $X = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for any $x, y \in \mathbb{R}$, then one has*

$$\mathrm{F}_\mathcal{A}(X, X) = (x\ y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x\ y) \left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right) = xy - yx = 0.$$

On the other hand, observe also that if $X = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ *and* $Y = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$*, then*

$$\mathrm{F}_{\mathcal{A}}(X,Y) = (a\ b)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = ad - bc.$$

Lemma 6.1.4. *Let* V *be a vector space, and let* $\mathrm{T} \in \mathrm{Mult}_n(V)$ *be alternating. If* $X_1, \dots, X_n \in V$ *is a linearly dependent family, then* $\mathrm{T}(X_1, \dots, X_n) = 0$*.*

Proof. Since the vectors are linearly dependent, it means that one of them, let's say X_1, is a linear combination of the others: $X_1 = \sum_{i=2}^n \lambda_i X_i$ for some scalars λ_i. Then one has

$$\begin{aligned} \mathrm{T}(X_1, X_2, \dots, X_n) &= \mathrm{T}\Big(\sum_{i=2}^n \lambda_i X_i, X_2 \dots, X_n\Big) \\ &= \sum_{i=2}^n \lambda_i \mathrm{T}(X_i, X_2, \dots, X_n) = \sum_{i=2}^n \lambda_i 0 = 0. \end{aligned}$$

□

Note that a simple consequence of this lemma is that if $\dim(V) = m$ and if $\mathrm{T} \in \mathrm{Mult}_n(V)$ for some $n > m$ one must have $\mathrm{T}(X_1, \dots, X_n) = 0$ whenever T is alternating. Indeed, there does not exist a family of n linearly independent vectors in a vector space of dimension $m < n$.

Lemma 6.1.5. *Let* V *be a vector space, and let* $T \in \mathrm{Mult}_n(V)$ *be alternating. For any* $X_1, \dots, X_n \in V$ *one has*

$$\mathrm{T}(X_1, \dots, X_j, \dots, X_k, \dots, X_n) = -\mathrm{T}(X_1, \dots, X_k, \dots, X_j, \dots, X_n),$$

or in other words T *changes its sign when two arguments are exchanged.*

Proof. One has by linearity and since T is alternating:

$$\begin{aligned} 0 =& \mathrm{T}(X_1, \dots, X_j + X_k, \dots, X_j + X_k, \dots, X_n) \\ =& \mathrm{T}(X_1, \dots, X_j, \dots, X_j, \dots, X_n) + \mathrm{T}(X_1, \dots, X_k, \dots, X_k, \dots, X_n) \\ &+ \mathrm{T}(X_1, \dots, X_j, \dots, X_k, \dots, X_n) + \mathrm{T}(X_1, \dots, X_k, \dots, X_j, \dots, X_n) \\ =& 0 + 0 + \mathrm{T}(X_1, \dots, X_j, \dots, X_k, \dots, X_n) + \mathrm{T}(X_1, \dots, X_k, \dots, X_j, \dots, X_n) \end{aligned}$$

from which the statement follows directly. □

Let us finally state two useful properties of alternating maps which follow almost directly from the definition.

Lemma 6.1.6. *Let* V *be a vector space over a field* $\mathbb{F}$*, and let* $\mathrm{T} \in \mathrm{Mult}_n(V)$ *be alternating. Then for any* $X_1, \dots, X_n \in V$ *and any* $\lambda, \lambda_i \in \mathbb{F}$ *one has*

(i)

$$\mathrm{T}(X_1, \dots, \lambda X_j, \dots, X_n) = \lambda \mathrm{T}(X_1, \dots, X_j, \dots, X_n),$$

(ii)

$$\mathrm{T}\Big(X_1 + \sum_{i=2}^{n} \lambda_i X_i, X_2, \dots, X_n\Big) = \mathrm{T}(X_1, X_2, \dots, X_n),$$

and such linear combination can be performed at any entry.

Proof. The first statement is nothing but the linearity of T in its j^{th}-argument. The second statement is a consequence of the alternating property of T. □

6.2 The determinant

Let $\mathbb{F}$ be a field, and recall that a map

$$\mathrm{T} : \underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{m\ terms} \to \mathbb{F}$$

is multlinear alternating if T is linear in each of its m arguments and if

$$\mathrm{T}(X_1, \dots, X_i, \dots, X_j, \dots, X_m) = 0$$

whenever $X_i = X_j$ for some $i \neq j$. In the special case $m = n$, a very strong statement holds. For this, recall that the standard basis $\{E_j\}_{j=1}^n$ of $\mathbb{F}^n$ is given by $(E_j)_i = 1$ if $i = j$ and $(E_j)_i = 0$ if $i \neq j$.

Theorem 6.2.1. *For any field $\mathbb{F}$ there exists a unique $\mathrm{T} \in \mathrm{Mult}_n(\mathbb{F}^n)$ alternating such that*

$$\mathrm{T}(E_1, E_2, \dots, E_n) = 1.$$

In order to prove this statement, we need to introduce one more notation. For any indices $i_1, i_2, \dots, i_n$ with $i_j \in \{1, 2, \dots, n\}$ we define the number $\varepsilon_{i_1 i_2 \dots i_n}$ by

$$\begin{aligned}
\varepsilon_{i_1 i_2 \dots i_n} &= 0 && \text{if two of the indices are equal,} \\
\varepsilon_{i_1 i_2 \dots i_n} &= (-1)^m && \text{if } \{i_1, \dots, i_n\} = \{1, \dots, n\} \text{ and if } m \text{ is the number of transpo-} \\
& && \text{sitions (exchanges) needed to reorder } i_1 i_2 \dots i_n \text{ into } 1\,2 \dots n.
\end{aligned}$$

Note that the equality $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ has to be understood as an equality between sets, without any consideration about the order. For example, $\{1, 2\} = \{2, 1\}$ because both sets contain the same elements.

Example 6.2.2.

$$\varepsilon_{12} = 1, \varepsilon_{21} = -1, \varepsilon_{11} = 0 = \varepsilon_{22}$$
$$\varepsilon_{123} = 1 = \varepsilon_{231} = \varepsilon_{312}, \varepsilon_{132} = -1 = \varepsilon_{321} = \varepsilon_{213}, \varepsilon_{122} = 0 = \varepsilon_{111} = \cdots$$

Proof of Theorem 6.2.1. Let $X_1, \dots, X_n \in \mathbb{F}^n$, and let $\lambda_{ij} \in \mathbb{F}$ for $i, j \in \{1, \dots, n\}$ such that $X_j = \sum_{i=1}^n \lambda_{ji} E_i$. Thus, if T is any multilinear map one has

$$\begin{aligned} \mathrm{T}(X_1, \dots, X_n) &= \mathrm{T}\Big(\sum_{i_1=1}^n \lambda_{1i_1} E_{i_1}, \dots, \sum_{i_n=1}^n \lambda_{ni_n} E_{i_n}\Big) \\ &= \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \lambda_{1i_1} \dots \lambda_{ni_n} \mathrm{T}(E_{i_1}, \dots, E_{i_n}). \end{aligned} \tag{6.2.1}$$

In addition, if T is alternating as well, then $\mathrm{T}(E_{i_1}, \dots, E_{i_n}) = 0$ unless $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ and in this case one has

$$\mathrm{T}(E_{i_1}, \dots, E_{i_n}) = \varepsilon_{i_1 i_2 \dots i_n} \mathrm{T}(E_1, E_2, \dots, E_n).$$

Finally, by imposing $\mathrm{T}(E_1, E_2, \dots, E_n) = 1$ one gets from (6.2.1) that

$$\begin{aligned} \mathrm{T}(X_1, \dots, X_n) &= \sum_{\{i_1,\dots,i_n\}=\{1,\dots,n\}} \lambda_{1i_1} \dots \lambda_{ni_n} \varepsilon_{i_1 i_2 \dots i_n} \\ &= \sum_{\{i_1,\dots,i_n\}=\{1,\dots,n\}} \varepsilon_{i_1 i_2 \dots i_n} \lambda_{1i_1} \dots \lambda_{ni_n}. \end{aligned} \tag{6.2.2}$$

Note that the summation has to be performed on the set of all permutations of the n numbers $1, 2, \dots, n$. One concludes by observing that the r.h.s. of (6.2.2) does not depend on T, showing that there exists only one T satisfying the stated conditions. □

Corollary 6.2.3. *There exists a unique map* Det : $M_n(\mathbb{F}) \to \mathbb{F}$ *which is n-linear alternating as a function of the columns, and which is equal to* 1 *for the identity matrix* $\mathbf{1}_n$. *This map is called* the determinant.

Proof. It is sufficient to identify a matrix $\mathcal{A} \in M_n(\mathbb{F})$ with its n columns $\mathcal{A}^j$, each one belonging to $\mathbb{F}^n$, and to use the previous theorem. □

Note that the following two notations are used for the determinant of a matrix $\mathcal{A}$: either $\mathrm{Det}(\mathcal{A})$ or $|\mathcal{A}|$. In the next statement, we simply adapt the properties proved for n-linear maps to the determinant.

Lemma 6.2.4. *Let* $\mathcal{A} \in M_n(\mathbb{F})$ *with* $\mathcal{A} = (\mathcal{A}^1\ \mathcal{A}^2\ \dots \mathcal{A}^n)$. *Then*

(i) $\mathrm{Det}(\mathcal{A}) = 0$ *if the n columns of $\mathcal{A}$ are linearly dependent,*

(ii)
$$\mathrm{Det}(\mathcal{A}^1 \dots \mathcal{A}^j \dots \mathcal{A}^k \dots \mathcal{A}^n) = -\mathrm{Det}(\mathcal{A}^1 \dots \mathcal{A}^k \dots \mathcal{A}^j \dots \mathcal{A}^n),$$
or in other words the sign of the determinant changes whenever two columns of the matrix are exchanged,

(iii)
$$\mathrm{Det}(\mathcal{A}^1 \dots \lambda\mathcal{A}^j \dots \mathcal{A}^n) = \lambda \mathrm{Det}(\mathcal{A}^1 \dots \mathcal{A}^j \dots \mathcal{A}^n),$$

(iv) $\mathrm{Det}(\mathcal{A})$ *is not changed if one adds to a column a linear combination of the other columns.*

Let us also state two formulas for the computation of the determinant (see also Exercise 6.5). For this purpose, we introduce one more notation: For a matrix $\mathcal{A} \in M_n(\mathbb{F})$ and for $i, j \in \{1, \dots, n\}$ we denote by $\mathcal{A}(i,j) \in M_{n-1}(\mathbb{F})$ the matrix obtained by disregarding the row i and the column j of $\mathcal{A}$. Then the following formulas hold: for any $\mathcal{A} = (a_{ij})$ one has

$$\mathrm{Det}(\mathcal{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \,\mathrm{Det}\big(\mathcal{A}(i,j)\big) \qquad \text{for any fixed } i \in \{1, \dots, n\}, \tag{6.2.3}$$

or

$$\mathrm{Det}(\mathcal{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \,\mathrm{Det}\big(\mathcal{A}(i,j)\big) \qquad \text{for any fixed } j \in \{1, \dots, n\}. \tag{6.2.4}$$

Note that formula (6.2.3) corresponds to a development of the determinant with respect to the row i of $\mathcal{A}$, while (6.2.4) corresponds to the development of the determinant with respect to the column j of $\mathcal{A}$.

Examples 6.2.5. *(i) If* $\mathcal{A} \in M_1(\mathbb{F})$, i.e. $\mathcal{A} = (a) \in \mathbb{F}$, *then* $\mathrm{Det}(\mathcal{A}) = a$,

(ii) In $\mathcal{A} \in M_2(\mathbb{F})$ *with* $\mathcal{A} = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, *then*

$$\begin{aligned} \mathrm{Det}(\mathcal{A}) &= (-1)^2 a_{11} \,\mathrm{Det}\big(\mathcal{A}(1,1)\big) + (-1)^3 a_{12} \,\mathrm{Det}\big(\mathcal{A}(1,2)\big) \\ &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}a_{22} - a_{21}a_{12}, \end{aligned}$$

(iii) If $\mathcal{A} \in M_3(\mathbb{F})$ *with* $\mathcal{A} = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)$, *then*

$$\begin{aligned} &\mathrm{Det}(\mathcal{A}) \\ =&(-1)^2 a_{11} \,\mathrm{Det}\big(\mathcal{A}(1,1)\big) + (-1)^3 a_{12} \,\mathrm{Det}\big(\mathcal{A}(1,2)\big) + (-1)^4 a_{13} \,\mathrm{Det}\big(\mathcal{A}(1,3)\big) \\ =&a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ =&a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}. \end{aligned}$$

Remark that in the above examples, we have performed the development with respect to the first row, but the same result would have been obtained if the development was performed with respect to any other row or column.

In the sequel, we shall obtain various additional properties of the determinant.

Lemma 6.2.6. *Let* $\mathcal{A} \in M_n(\mathbb{F})$, *then* $\mathrm{Det}(\mathcal{A}) = \mathrm{Det}({}^t\mathcal{A})$.

Proof. The proof is performed by induction. Clearly, for $n = 1$ the statement is true since the matrix just corresponds to a single scalar. So we can assume that the statement is true for any matrix in $M_{n-1}(\mathbb{F})$ and prove it for any element of $M_n(\mathbb{F})$. Let $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{F})$ and let us set $\mathcal{B} = (b_{ij})$ with $\mathcal{B} = {}^t\mathcal{A}$. Then, by the formula (6.2.3) with $i = 1$ one gets

$$\operatorname{Det}(\mathcal{A}) = a_{11} \operatorname{Det}\big(\mathcal{A}(1,1)\big) - a_{12} \operatorname{Det}\big(\mathcal{A}(1,2)\big) + \cdots + (-1)^{n+1} a_{1n} \operatorname{Det}\big(\mathcal{A}(1,n)\big) \quad (6.2.5)$$

and by formula (6.2.4) with $j = 1$ one gets

$$\operatorname{Det}(\mathcal{B}) = b_{11} \operatorname{Det}\big(\mathcal{B}(1,1)\big) - b_{21} \operatorname{Det}\big(\mathcal{B}(2,1)\big) + \cdots + (-1)^{n+1} b_{n1} \operatorname{Det}\big(\mathcal{B}(n,1)\big). \quad (6.2.6)$$

Now, observe that $a_{1j} = b_{j1}$ because $\mathcal{B}$ is the transpose of $\mathcal{A}$, and similarly $\mathcal{B}(j,1) = {}^t\mathcal{A}(1,j) \in M_{n-1}(\mathbb{F})$. Since by assumption one has

$$\operatorname{Det}\big(\mathcal{A}(1,j)\big) = \operatorname{Det}\big({}^t\mathcal{A}(1,j)\big) = \operatorname{Det}\big(\mathcal{B}(j,1)\big)$$

one directly infers from (6.2.5) and (6.2.6) that $\operatorname{Det}(\mathcal{A}) = \operatorname{Det}(\mathcal{B})$, which corresponds to the statement. □

Corollary 6.2.7. *All the properties of* $\operatorname{Det}(\mathcal{A})$ *with respect to the columns of* $\mathcal{A}$ *also hold with respect to the rows of* $\mathcal{A}$.

In order to state an important result linking $\operatorname{Det}(\mathcal{A})$ and the invertibility of $\mathcal{A}$, let us recall some results of Chapter 2 but in the general framework of an arbitrary field $\mathbb{F}$.

1) Recall that the elementary matrices have been introduced in Definition 2.5.1 and that their definition holds for any field. One shows in Exercise 6.4 that

(i) $\operatorname{Det}(\mathbf{1}_n - I_{rr} + cI_{rr}) = c$, for $c \in \mathbb{F}$ with $c \neq 0$,

(ii) $\operatorname{Det}(\mathbf{1}_n + I_{rs} + I_{sr} - I_{rr} - I_{ss}) = -1$, for $r \neq s$,

(iii) $\operatorname{Det}(\mathbf{1}_n + cI_{rs}) = 1$, for $r \neq s$ and any $c \in \mathbb{F}$.

In addition, one also observes that for any $\mathcal{A} \in M_n(\mathbb{F})$ and any elementary matrix $\mathcal{B} \in M_n(\mathbb{F})$ one has

$$\operatorname{Det}(\mathcal{B}\mathcal{A}) = \operatorname{Det}(\mathcal{B}) \operatorname{Det}(\mathcal{A}). \quad (6.2.7)$$

Note that this property can be inferred from the general property of the determinant and from the action of an elementary matrix on $\mathcal{A}$, as seen in Exercise 2.14.

2) For any $\mathcal{A} \in M_n(\mathbb{F})$, let us recall that there exist a family of elementary matrices $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_p \in M_n(\mathbb{F})$ such that $\mathcal{A}' := \mathcal{B}_p \mathcal{B}_{p-1} \dots \mathcal{B}_1 \mathcal{A}$ is a matrix in the standard form, see Corollary 2.4.10 and the subsequent definition. In particular, it has been shown in Theorem 2.5.4 that $\mathcal{A}' = \mathbf{1}_n$ if and only if $\mathcal{A}$ is invertible. Equivalently, $\mathcal{A}$ is not invertible if and only if $\mathcal{A}'$ contains some 0 on its diagonal.

3) One easily observes that $\operatorname{Det}(\mathcal{A}') = 1$ if $\mathcal{A}' = \mathbf{1}_n$ and that $\operatorname{Det}(\mathcal{A}') = 0$ if $\mathcal{A}'$ contains some 0 on its diagonal.

Proposition 6.2.8. *For any field $\mathbb{F}$ and any $\mathcal{A} \in M_n(\mathbb{F})$, the following statements are equivalent:*

(i) $\mathrm{Det}(\mathcal{A}) \neq 0$*,*

(ii) $\mathcal{A}$ *is invertible,*

(iii) The columns $\mathcal{A}^1, \mathcal{A}^2, \dots, \mathcal{A}^n$ *of* $\mathcal{A}$ *are linearly independent,*

(iv) The rows $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ *of* $\mathcal{A}$ *are linearly independent.*

Proof. Since there exist elementary matrices $\mathcal{B}_j$ such that $\mathcal{A}' := \mathcal{B}_p\mathcal{B}_{p-1}\dots\mathcal{B}_1\mathcal{A}$ with $\mathcal{A}'$ in the standard form, one gets from (6.2.7) that

$$\begin{aligned}\mathrm{Det}(\mathcal{A}') &= \mathrm{Det}\big(\mathcal{B}_p\mathcal{B}_{p-1}\dots\mathcal{B}_1\mathcal{A}\big)\\ &= \mathrm{Det}(\mathcal{B}_p)\,\mathrm{Det}\big(\mathcal{B}_{p-1}\dots\mathcal{B}_1\mathcal{A}\big)\\ &= \dots\\ &= \underbrace{\mathrm{Det}(\mathcal{B}_p)\,\mathrm{Det}(\mathcal{B}_{p-1})\dots\mathrm{Det}(\mathcal{B}_1)}_{\neq 0}\,\mathrm{Det}(\mathcal{A}).\end{aligned}$$

Thus, one infers that $\mathrm{Det}(\mathcal{A}) \neq 0$ if and only if $\mathrm{Det}(\mathcal{A}') \neq 0$. Since by the above observations 2) and 3) one already knows that $\mathrm{Det}(\mathcal{A}') \neq 0$ if and only if $\mathcal{A}$ is invertible, one then concludes that $\mathrm{Det}(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ is invertible. This corresponds to the equivalence between (i) and (ii).

For the second equivalence, consider $\mathrm{L}_\mathcal{A} : \mathbb{F}^n \to \mathbb{F}^n$ be the linear map defined by $\mathrm{L}_\mathcal{A}X = \mathcal{A}X$ for any $X \in \mathbb{F}^n$. By definition of the rank, $\mathcal{A}^1, \dots, \mathcal{A}^n$ are linearly independent if and only if $\mathrm{rank}(\mathcal{A}) = n$. However, by Corollary 4.4.2, Theorem 4.3.5 and Lemma 4.7.8 one has

$$\mathrm{rank}(\mathcal{A}) = n \Leftrightarrow \dim\big(\mathrm{Ran}(\mathrm{L}_\mathcal{A})\big) = n \Leftrightarrow \dim\big(\mathrm{Ker}(\mathrm{L}_\mathcal{A})\big) = 0 \Leftrightarrow \mathrm{L}_\mathcal{A} \text{ is invertible.}$$

Finally, from Example 4.7.2 one also infers that $\mathrm{L}_\mathcal{A}$ is invertible if and only if $\mathcal{A}$ is invertible. Summing up these information, one has obtained that $\mathcal{A}^1, \dots, \mathcal{A}^n$ are linearly independent if and only if $\mathcal{A}$ is invertible, which corresponds to the equivalence between (ii) and (iii).

The equivalence between (iii) and (iv) corresponds to a reformulation of Corollary 6.2.7. □

Corollary 6.2.9. *Let* $\mathbb{F}$ *be any field and let* $X_1, \dots, X_n$ *be* n *elements of* $\mathbb{F}^n$*. Then* $X_1, \dots, X_n$ *are linearly independent if and only if* $\mathrm{Det}(X_1\,X_2\dots X_n) \neq 0$*.*

Let us now prove an extension of (6.2.7) valid for arbitrary matrices.

Proposition 6.2.10. *For any field* $\mathbb{F}$ *and any* $\mathcal{A}, \mathcal{C} \in M_n(\mathbb{F})$ *one has*

$$\mathrm{Det}(\mathcal{A}\mathcal{C}) = \mathrm{Det}(\mathcal{A})\,\mathrm{Det}(\mathcal{C}). \tag{6.2.8}$$

Proof. Since there exist elementary matrices $\mathcal{B}_j$ such that $\mathcal{A} := \mathcal{B}_1^{-1}\mathcal{B}_2^{-1}\dots\mathcal{B}_p^{-1}\mathcal{A}'$ with $\mathcal{A}'$ in the standard form, one gets from (6.2.7) that

$$\begin{aligned}\mathrm{Det}(\mathcal{A}\mathcal{C}) &= \mathrm{Det}\big(\mathcal{B}_1^{-1}\mathcal{B}_2^{-1}\dots\mathcal{B}_p^{-1}\mathcal{A}'\mathcal{C}\big)\\ &= \mathrm{Det}(\mathcal{B}_1^{-1})\,\mathrm{Det}\big(\mathcal{B}_2^{-1}\dots\mathcal{B}_p^{-1}\mathcal{A}'\mathcal{C}\big)\\ &= \dots\\ &= \underbrace{\mathrm{Det}(\mathcal{B}_1^{-1})\,\mathrm{Det}(\mathcal{B}_2^{-1})\dots\mathrm{Det}(\mathcal{B}_p^{-1})}_{\neq 0}\,\mathrm{Det}(\mathcal{A}'\mathcal{C}).\end{aligned}$$

Thus, if $\mathcal{A}' = \mathbf{1}_n$, one deduces that

$$\mathrm{Det}(\mathcal{A}\mathcal{C}) = \underbrace{\mathrm{Det}(\mathcal{B}_1^{-1})\,\mathrm{Det}(\mathcal{B}_2^{-1})\dots\mathrm{Det}(\mathcal{B}_p^{-1})}_{=\mathrm{Det}(\mathcal{A})}\,\mathrm{Det}(\mathcal{C}) = \mathrm{Det}(\mathcal{A})\,\mathrm{Det}(\mathcal{C}).$$

On the other hand, if $\mathcal{A}' \neq \mathbf{1}_n$, then the last row of $\mathcal{A}'$ is filled with 0 and one has $\mathrm{Det}(\mathcal{A}') = 0 = \mathrm{Det}(\mathcal{A})$, where we have used an argument from the previous proof for the last equality. However, one also deduces from the formula (2.2.4) on the product of two matrices that the last row of $\mathcal{A}'\mathcal{C}$ is also filled only with 0, and this implies that $\mathrm{Det}(\mathcal{A}'\mathcal{C}) = 0$ as well. As a consequence, one has both

$$\mathrm{Det}(\mathcal{A}\mathcal{C}) = \mathrm{Det}(\mathcal{B}_1^{-1})\,\mathrm{Det}(\mathcal{B}_2^{-1})\dots\mathrm{Det}(\mathcal{B}_p^{-1})\,\mathrm{Det}(\mathcal{A}'\mathcal{C}) = 0$$

and $\mathrm{Det}(\mathcal{A})\,\mathrm{Det}(\mathcal{C}) = 0\,\mathrm{Det}(\mathcal{C}) = 0$. Again, the equality (6.2.8) holds. □

6.3 Cramer's rule and the inverse of a matrix

The next proposition is usually referred as Cramer's rule.

Proposition 6.3.1. *Let $\mathcal{A} \in M_n(\mathbb{F})$ with $\mathrm{Det}(\mathcal{A}) \neq 0$, and consider the system of equations $\mathcal{A}X = B$ with $B \in \mathbb{F}^n$. Then its solution $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$ is given by*

$$x_j = \frac{1}{\mathrm{Det}(\mathcal{A})}\mathrm{Det}\big(\mathcal{A}^1\,\mathcal{A}^2\dots B\dots\mathcal{A}^n\big),$$

where B is replacing the column $\mathcal{A}^j$.

The proof of this statement is provided in Exercise 6.12.

Corollary 6.3.2. *If $\mathcal{A} \in M_n(\mathbb{F})$ is invertible, then its inverse is given by the following formula*

$$(\mathcal{A}^{-1})_{ij} = (-1)^{i+j}\frac{\mathrm{Det}\big(\mathcal{A}(j,i)\big)}{\mathrm{Det}(\mathcal{A})}.$$

Proof. For fixed $j \in \{1, \dots, n\}$, consider the equation $\mathcal{A}X = E_j$ with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$ and with $E_j \in \mathbb{F}^n$ the vector consisting in 1 at the position j and 0 everywhere else. Since $\mathcal{A}$ is invertible this equation is equivalent to $\mathcal{A}^{-1}E_j = X$, or more precisely $x_i = \sum_{k=1}^n (\mathcal{A}^{-1})_{ik}(E_j)_k$ for any $i \in \{1, \dots, n\}$. Since $(E_j)_k = 0$ whenever $j \neq k$ one gets $x_i = (\mathcal{A}^{-1})_{ij}$.

On the other hand, from the previous proposition with $B = E_j$ one also gets

$$x_i = \frac{1}{\mathrm{Det}(\mathcal{A})}\mathrm{Det}\big(\mathcal{A}^1\,\mathcal{A}^2 \dots E_j \dots \mathcal{A}^n\big) = \frac{1}{\mathrm{Det}(\mathcal{A})}(-1)^{i+j}\,\mathrm{Det}\big(\mathcal{A}(j,i)\big)$$

where formula (6.2.4) with respect to the column i has been used. By identifying the two expressions for x_i one gets the stated equality. $\square$

6.4 Exercises

Exercise 6.1. *Let us define the map* $\mathrm{F} : \underbrace{M_n(\mathbb{R}) \times \cdots \times M_n(\mathbb{R})}_{m \text{ arguments}} \to \mathbb{R}$ *by*

$$\mathrm{F}\big(\mathcal{A}_1, \ldots, \mathcal{A}_m\big) = \mathrm{Tr}\big(\mathcal{A}_1 \ldots \mathcal{A}_m\big).$$

Show that F *is a* m*-linear map.*

Exercise 6.2. *Show that the the cross product in* $\mathbb{R}^3$ *is a bilinear alternating map.*

Exercise 6.3. *Exhibit 3 different alternating bilinear maps on* $\mathbb{R}^3$*.*

Exercise 6.4. *For* $r \in \{1, \ldots, m\}$ *and* $s \in \{1, \ldots, m\}$*, let* $I_{rs} \in M_m(\mathbb{F})$ *be the matrix whose* rs*-component is* 1 *and all the other ones are equal to* 0*. For* $c \neq 0$*, consider the following 3 types of elementary matrices :*

1. $\mathbf{1}_m - I_{rr} + cI_{rr}$*, the matrix obtained from the identity matrix by multiplying the* r*-th diagonal component by* c*,*
2. *For* $r \neq s$*,* $(\mathbf{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})$*, the matrix obtained from the identity matrix by interchanging the* r*-th row with the* s*-th row,*
3. *For* $r \neq s$*,* $(\mathbf{1}_m + cI_{rs})$*, the matrix having the* rs*-th component equal to* c*, all other components* 0 *except the diagonal components which are equal to* 1*.*

Compute the determinant of these elementary matrices.

Exercise 6.5. *For an arbitrary field* $\mathbb{F}$ *let* $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{F})$ *and recall the formula:*

$$\begin{aligned} \mathrm{Det}(\mathcal{A}) &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \, \mathrm{Det}\big(\mathcal{A}(i,j)\big) && \text{for any fixed } j \in \{1, \ldots, n\} \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \, \mathrm{Det}\big(\mathcal{A}(i,j)\big) && \text{for any fixed } i \in \{1, \ldots, n\}. \end{aligned}$$

Show that $\mathrm{Det}(\mathbf{1}_n) = 1$ *for any* n*. For* $n = 2$*, show that*

(i) *the determinant is linear as a function of the columns of* $\mathcal{A}$*,*

(ii) *the determinant is alternating as a function of the columns of* $\mathcal{A}$*.*

Can you do it for $n = 3$*? For arbitrary* n *(a proof by induction over the dimension* n *is recommended).*

Exercise 6.6. *Compute the determinant of the following matrices:*

$$a) \begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 5 & 7 \end{pmatrix}, \quad b) \begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 8 \end{pmatrix} \quad c) \begin{pmatrix} 3 & 1 & 1 \\ 2 & 5 & 5 \\ 8 & 7 & 7 \end{pmatrix} \quad d) \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 5 \end{pmatrix}$$

Exercise 6.7. *Let* $\mathcal{A} = (a_{jk}) \in M_n(\mathbb{R})$ *be an upper triangular matrix. Compute* $\mathrm{Det}(\mathcal{A})$.

Exercise 6.8. *Show that two similar square matrices share the same determinant.*

Exercise 6.9. *Show that if* $\mathcal{A} \in M_n(\mathbb{F})$ *is invertible then the following equality holds:*

$$\mathrm{Det}\big(\mathcal{A}^{-1}\big) = \frac{1}{\mathrm{Det}(\mathcal{A})}.$$

Exercise 6.10. *Compute the determinant of the matrix* $\left(\begin{smallmatrix} x+1 & x-1 \\ x & 2x+5 \end{smallmatrix}\right)$.

Exercise 6.11. *Consider the matrix* $\mathcal{A} = \left(\begin{smallmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ 1 & 1 & \lambda \end{smallmatrix}\right)$ *with* $\lambda \in \mathbb{R}$.

(i) Compute the determinant of $\mathcal{A}$,

(ii) For which values of λ *is* $\mathcal{A}$ *invertible ?*

Exercise 6.12. *Prove Cramer's rule,* i.e. *show that if* $\mathcal{A} \in M_n(\mathbb{F})$ *is invertible and if* $X \in \mathbb{F}^n$ *satisfies* $\mathcal{A}X = B$ *for some* $B \in \mathbb{F}^n$, *then*

$$x_j = \frac{1}{\mathrm{Det}(\mathcal{A})}\mathrm{Det}\big(\mathcal{A}^1\,\mathcal{A}^2 \dots B \dots \mathcal{A}^n\big),$$

where B *is replacing the column* $\mathcal{A}^j$. *For that purpose, one should first recall that* $\mathcal{A}X = B$ *is equivalent to* $x_1\mathcal{A}^1 + x_2\mathcal{A}^2 + \dots + x_n\mathcal{A}^n = B$, *and insert this equality in the term* $\mathrm{Det}\big(\mathcal{A}^1\,\mathcal{A}^2 \dots B \dots \mathcal{A}^n\big)$.

Exercise 6.13. *By using determinants, find the inverse for the following matrices :*

$$a)\quad \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 7 \end{pmatrix} \qquad b)\quad \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{pmatrix}$$

Exercise 6.14. *By using determinants, solve the following systems of equations :*

$$a)\quad \begin{cases} x + 2y - z & = 1 \\ \quad y + z & = 1 \\ \quad 2y + 7z & = 1 \end{cases} \qquad b)\quad \begin{cases} 2x + y + 2z & = 0 \\ \quad 3y - z & = 1 \\ \quad 4x + y + z & = 2 \end{cases}$$

Exercise 6.15. *Let* X, Y *be two vectors in* $\mathbb{R}^2$. *Check that the area of the parallelogram spanned by* X *and* Y *is equal to the absolute value of the determinant of the matrix* $(X\ Y) \in M_2(\mathbb{R})$. *More generally, if* $X_1, \dots, X_n$ *are* n *vectors of* $\mathbb{R}^n$, *one writes* $\mathsf{Vol}(X_1, \dots, X_n)$ *for the volume of the* n*-dimensional box spanned by* $X_1, \dots, X_n$. *Why is it natural to have*

$$\mathsf{Vol}(X_1, \dots, X_n) = |\mathrm{Det}(X_1 \dots X_n)|\ ?$$

Exercise 6.16. *Let* $\{V_1, \dots, V_n\}$ *and* $\{V_1', \dots, V_n'\}$ *be two bases of* $\mathbb{R}^n$, *and let* $\mathcal{B} \in M_n(\mathbb{R})$ *be the matrix of change of bases,* i.e. $V_j' = \mathcal{B}V_j$ *for any* $j = 1, 2, \dots, n$. *What is the geometric interpretation of* $|\mathrm{Det}(\mathcal{B})|$ *in this setting ? For that purpose, one should first check that if* $(V_1\ V_2 \dots V_n)$ *denotes the matrix with columns* V_j *and* $(V_1'\ V_2' \dots V_n')$ *denotes the matrix with columns* V_j', *then one has*

$$(V_1'\ V_2' \dots V_n') = (\mathcal{B}V_1\ \mathcal{B}V_2 \dots \mathcal{B}V_n) = \mathcal{B}(V_1\ V_2 \dots V_n).$$