Chapter 2

C^*-algebras

This chapter is mainly based on the first chapters of the book [Mur90]. Material borrowed from other references will be specified.

2.1 Banach algebras

Definition 2.1.1. A Banach algebra $\mathscr{C}$ *is a complex vector space endowed with an associative multiplication and with a norm* $\|\cdot\|$ *which satisfy for any* $A, B, C \in \mathscr{C}$ *and* $\alpha \in \mathbb{C}$

(i) $(\alpha A)B = \alpha(AB) = A(\alpha B)$,

(ii) $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$,

(iii) $\|AB\| \leq \|A\|\,\|B\|$ *(submultiplicativity)*

(iv) $\mathscr{C}$ *is complete with the norm* $\|\cdot\|$.

One says that $\mathscr{C}$ is *abelian* or *commutative* if $AB = BA$ for all $A, B \in \mathscr{C}$. One also says that $\mathscr{C}$ is *unital* if $\mathbf{1} \in \mathscr{C}$, *i.e.* if there exists an element $\mathbf{1} \in \mathscr{C}$ with $\|\mathbf{1}\| = 1$ such that $\mathbf{1}B = B = B\mathbf{1}$ for all $B \in \mathscr{C}$. *A subalgebra* $\mathscr{J}$ *of* $\mathscr{C}$ is a vector subspace which is stable for the multiplication. If $\mathscr{J}$ is norm closed, it is a Banach algebra in itself.

Examples 2.1.2. *(i)* $\mathbb{C}$, $M_n(\mathbb{C})$, $\mathscr{B}(\mathcal{H})$, $\mathscr{K}(\mathcal{H})$ *are Banach algebras, where* $M_n(\mathbb{C})$ *denotes the set of* $n \times n$*-matrices over* $\mathbb{C}$. *All except* $\mathscr{K}(\mathcal{H})$ *are unital, and* $\mathscr{K}(\mathcal{H})$ *is unital if* $\mathcal{H}$ *is finite dimensional.*

(ii) If Ω *is a locally compact topological space,* $C_0(\Omega)$ *and* $C_b(\Omega)$ *are abelian Banach algebras, where* $C_b(\Omega)$ *denotes the set of all bounded and continuous complex functions from* Ω *to* $\mathbb{C}$, *and* $C_0(\Omega)$ *denotes the subset of* $C_b(\Omega)$ *of functions* f *which vanish at infinity,* i.e. *for any* $\varepsilon > 0$ *there exists a compact set* $K \subset \Omega$ *such that* $\sup_{x \in \Omega \setminus K} |f(x)| \leq \varepsilon$. *These algebras are endowed with the* L^∞*-norm, namely* $\|f\| = \sup_{x \in \Omega} |f(x)|$. *Note that* $C_b(\Omega)$ *is unital, while* $C_0(\Omega)$ *is not, except if* Ω *is compact. In this case, one has* $C_0(\Omega) = C(\Omega) = C_b(\Omega)$.

(iii) If (Ω, μ) is a measure space, then $L^\infty(\Omega)$, the (equivalent classes of) essentially bounded complex functions on Ω is a unital abelian Banach algebra with the essential supremum norm $\|\cdot\|_\infty$.

(iv) For any $n \in \mathbb{N}$, the set $BC_u(\mathbb{R}^d)$ of bounded and uniformly continuous complex functions on $\mathbb{R}^d$ is a unital abelian Banach algebra. Recall that $f : \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in \mathbb{R}^d$ with $|x - y| \leq \delta$ one has $|f(x) - f(y)| \leq \varepsilon$. Note that this property can be defined not only on $\mathbb{R}^d$ but on all uniform spaces.

If S is a subset of a Banach algebra $\mathscr{C}$, the smallest closed subalgebra of $\mathscr{C}$ which contains S is called *the closed algebra generated by S*.

Definition 2.1.3. An ideal *in a Banach algebra $\mathscr{C}$ is a (non-trivial) subalgebra $\mathscr{J}$ of $\mathscr{C}$ such that $AB \in \mathscr{J}$ and $BA \in \mathscr{J}$ whenever $A \in \mathscr{J}$ and $B \in \mathscr{C}$. An ideal $\mathscr{J}$ is* maximal *in $\mathscr{C}$ if $\mathscr{J}$ is proper ($\Leftrightarrow$ not equal to $\mathscr{C}$) and $\mathscr{J}$ is not contained in any other proper ideal of $\mathscr{C}$.*

In the examples presented above, $C_0(\Omega)$ is an ideal of $C_b(\Omega)$, while $\mathscr{K}(\mathcal{H})$ is an ideal of $\mathscr{B}(\mathcal{H})$.

Lemma 2.1.4. *If $\mathscr{C}$ is a Banach algebra and $\mathscr{J}$ is a closed ideal in $\mathscr{C}$, the quotient $\mathscr{C}/\mathscr{J}$ of $\mathscr{C}$ by $\mathscr{J}$, endowed with the multiplication $(A + \mathscr{J})(B + \mathscr{J}) = (AB + \mathscr{J})$ and with the quotient norm $\|A + \mathscr{J}\| := \inf_{B \in \mathscr{J}} \|A + B\|$, is a Banach algebra.*

Proof. The algebraic properties of the quotient are easily verified, and the submultiplicativity is shown below. The completeness of the quotient with respect to the norm is a standard result of normed vector spaces, see for example [Ped89, Prop. 2.1.5].

Let $\varepsilon > 0$ and let $A, B \in \mathscr{C}$. Then

$$\|A + A'\| < \|A + \mathscr{J}\| + \varepsilon \qquad \|B + B'\| < \|B + \mathscr{J}\| + \varepsilon$$

for some $A', B' \in \mathscr{J}$. Hence, by setting $C := A'B + AB' + A'B' \in \mathscr{J}$ one has

$$\|AB + C\| \leq \|A + A'\| \|B + B'\| \leq \big(\|A + \mathscr{J}\| + \varepsilon\big)\big(\|B + \mathscr{J}\| + \varepsilon\big).$$

Thus, $\|AB + \mathscr{J}\| \leq \big(\|A + \mathscr{J}\| + \varepsilon\big)\big(\|B + \mathscr{J}\| + \varepsilon\big)$. By letting then $\varepsilon \searrow 0$, we get $\|AB + \mathscr{J}\| \leq \|A + \mathscr{J}\| \|B + \mathscr{J}\|$, which corresponds to the submultiplicativity of the quotient norm. □

Definition 2.1.5. A homomorphism *φ between two Banach algebras $\mathscr{C}$ and $\mathscr{Q}$ is a linear map $\varphi : \mathscr{C} \to \mathscr{Q}$ which satisfies $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathscr{C}$. If $\mathscr{C}$ and $\mathscr{Q}$ are unital and if $\varphi(\mathbf{1}) = \mathbf{1}$, one says that φ is* unit preserving *or* a unital homomorphism.

It is easily seen that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a homomorphism, its kernel $\mathsf{Ker}(\varphi)$ is an ideal in $\mathscr{C}$ and its range $\varphi(\mathscr{C})$ is a subalgebra of $\mathscr{Q}$. Alternatively, if $\mathscr{J}$ is an ideal in a Banach algebra $\mathscr{C}$, then *the quotient map* $q : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ is a homomorphism.

Let us now consider an arbitrary unital Banach algebra $\mathscr{C}$, and let $A \in \mathscr{C}$. One says that A is *invertible* if there exists $B \in \mathscr{C}$ such that $AB = \mathbf{1} = BA$. In this case, the element B is denoted by A^{-1} and is called *the inverse of* A. The set of all invertible elements in a unital Banach algebra $\mathscr{C}$ is denoted by $\mathsf{Inv}(\mathscr{C})$.

Exercise 2.1.6. *By using the Neumann series, show that $\mathsf{Inv}(\mathscr{C})$ is an open set in a unital Banach algebra $\mathscr{C}$, and that the map $\mathsf{Inv}(\mathscr{C}) \ni A \mapsto A^{-1} \in \mathscr{C}$ is differentiable.*

On the other hand, let us show that maximal ideals in a unital Banach algebra $\mathscr{C}$ are closed. For this, observe first that for every ideal $\mathscr{J} \neq \mathscr{C}$ we have $\mathscr{J} \cap \mathsf{Inv}(\mathscr{C}) = \emptyset$. Indeed, if one has $A \in \mathscr{J} \cap \mathsf{Inv}(\mathscr{C})$, then for any $B \in \mathscr{C} \setminus \mathscr{J}$ one would have $B = A(A^{-1}B) \in \mathscr{J}$, which is absurd. As a consequence, it follows that $\|\mathbf{1} - A\| \geq 1$ since otherwise A would be invertible with the Neumann series. Consequently, $\mathscr{J}$ can not be dense in $\mathscr{C}$, and thus the closure $\overline{\mathscr{J}}$ of $\mathscr{J}$ is a proper and closed ideal in $\mathscr{C}$. One infers from this that any maximal ideal in $\mathscr{C}$ is closed.

2.2 Spectral theory

The main notions of spectral theory introduced before in the context of $\mathscr{B}(\mathcal{H})$ can be generalized to arbitrary unital Banach algebra.

For any A in a unital Banach algebra $\mathscr{C}$ we define *the spectrum $\sigma_{\mathscr{C}}(A)$ of A with respect to* $\mathscr{C}$ by

$$\sigma_{\mathscr{C}}(A) := \big\{ z \in \mathbb{C} \mid (A - z) \notin \mathsf{Inv}(\mathscr{C}) \big\}. \tag{2.2.1}$$

Note that the spectrum $\sigma_{\mathscr{C}}(A)$ of A is never empty, see for example [Mur90, Thm. 1.2.5]. This result is not completely trivial and its proof is based on Liouville's Theorem in complex analysis.

Based on this observation, we state two results which are often quite useful.

Theorem 2.2.1 (Gelfand-Mazur). *If $\mathscr{C}$ is a unital Banach algebra in which every non-zero element is invertible, then $\mathscr{C} = \mathbb{C}\mathbf{1}$.*

Proof. We know from the observation made above that for any $A \in \mathscr{C}$, there exists $z \in \mathbb{C}$ such that $A - z \equiv A - z\mathbf{1} \notin \mathsf{Inv}(\mathscr{C})$. By assumption, it follows that $A = z\mathbf{1}$. □

Lemma 2.2.2. *Let $\mathscr{J}$ be a maximal ideal of a unital abelian Banach algebra $\mathscr{C}$, then $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$.*

Proof. As seen in Lemma 2.1.4, $\mathscr{C}/\mathscr{J}$ is a Banach algebra with unit $\mathbf{1}+\mathscr{J}$; the quotient map $\mathscr{C} \to \mathscr{C}/\mathscr{J}$ is denoted by q. If $\mathscr{I}$ is an ideal in $\mathscr{C}/\mathscr{J}$, then $q^{-1}(\mathscr{I})$ is an ideal of $\mathscr{C}$ containing $\mathscr{J}$, which is therefore either equal to $\mathscr{C}$ or to $\mathscr{J}$, by the maximality of $\mathscr{J}$. Consequently, $\mathscr{I}$ is either equal to $\mathscr{C}/\mathscr{J}$ or to $\mathbf{0}$, and $\mathscr{C}/\mathscr{J}$ has no proper ideal.

Now, if $A \in \mathscr{C}/\mathscr{J}$ and $A \neq \mathbf{0}$, then $A \in \mathsf{Inv}\left(\mathscr{C}/\mathscr{J}\right)$, since otherwise $A(\mathscr{C}/\mathscr{J})$ would be a proper ideal of $\mathscr{C}/\mathscr{J}$. In other words, one has obtained that any non-zero element of $\mathscr{C}/\mathscr{J}$ is invertible, which implies that $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$, by Theorem 2.2.1. □

Lemma 2.2.3. *Let $\mathscr{C}$ be a unital Banach algebra and let $A \in \mathscr{C}$. Then $\sigma_{\mathscr{C}}(A)$ is a closed subset of the disc in the complex plane, centered at* 0 *and of radius $\|A\|$.*

Proof. If $|z| > \|A\|$, then $\|z^{-1}A\| < 1$, and therefore $(\mathbf{1} - z^{-1}A)$ is invertible (use the Neumann series). Equivalently, this means that $(z - A)$ is invertible, and therefore $z \notin \sigma_{\mathscr{C}}(A)$. Thus, one has obtained that if $z \in \sigma_{\mathscr{C}}(A)$, then $|z| \leq \|A\|$.

Since $\mathsf{Inv}(\mathscr{C})$ is an open set in $\mathscr{C}$, one easily infers that $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ is an open set in $\mathbb{C}$, which means that $\sigma_{\mathscr{C}}(A)$ is a closed set in $\mathbb{C}$. □

Another notion related to the spectrum of A is sometimes convenient. If A belongs to a unital Banach algebra $\mathscr{C}$, its *spectral radius* $r(A)$ is defined by

$$r(A) := \sup_{z \in \sigma_{\mathscr{C}}(A)} |z|.$$

Clearly, it follows from the previous lemma that $r(A) \leq \|A\|$. In addition, the following property holds:

Theorem 2.2.4 (Beurling)**.** *If A is an element of a unital Banach algebra, then*

$$r(A) = \inf_{n \geq 1} \|A^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n}.$$

Proof. See [Mur90, Thm. 1.2.7] or [Ped89, Thm. 4.1.13]. □

For the next statement, recall that if K is a non-empty compact set in $\mathbb{C}$, it complement $\mathbb{C} \setminus K$ admits exactly one unbounded component, and that the bounded components of $\mathbb{C} \setminus K$ are called the *holes* of K.

Proposition 2.2.5. *Let $\mathscr{C}$ be a closed subalgebra of a unital Banach algebra $\mathscr{A}$ which contains the unit of $\mathscr{A}$. Then,*

(i) The set $\mathsf{Inv}(\mathscr{C})$ is a clopen ($\Leftrightarrow$ open and closed) subset of $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$,

(ii) For each $A \in \mathscr{C}$,

$$\sigma_{\mathscr{A}}(A) \subseteq \sigma_{\mathscr{C}}(A) \qquad \text{and} \qquad \partial\sigma_{\mathscr{C}}(A) \subseteq \partial\sigma_{\mathscr{A}}(A),$$

(iii) If $A \in \mathscr{C}$ and $\sigma_{\mathscr{A}}(A)$ has no hole, then $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$.

Proof. Clearly $\mathsf{Inv}(\mathscr{C})$ is an open set in $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$. To see that it is also closed, let (A_n) be a sequence in $\mathsf{Inv}(\mathscr{C})$ converging to a point $A \in \mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$. Then, from the equality $A_n^{-1} - A^{-1} = A_n^{-1}(A - A_n)A^{-1}$, one infers that (A_n^{-1}) converges to A^{-1} in $\mathscr{A}$, so $A^{-1} \in \mathscr{C}$ (by the completeness of $\mathscr{C}$), which implies that $A \in \mathsf{Inv}(\mathscr{C})$. Hence, $\mathsf{Inv}(\mathscr{C})$ is clopen in $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$.

If $A \in \mathscr{C}$, the inclusion $\sigma_{\mathscr{A}}(A) \subseteq \sigma_{\mathscr{C}}(A)$ is immediate from the inclusion $\mathsf{Inv}(\mathscr{C}) \subseteq \mathsf{Inv}(\mathscr{A})$.

If $z \in \partial\sigma_{\mathscr{C}}(A)$, then there is a sequence (z_n) in $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ converging to z. Hence, $(A - z_n) \in \mathsf{Inv}(\mathscr{C})$, and $(A - z) \notin \mathsf{Inv}(\mathscr{C})$, so $(A - z) \notin \mathsf{Inv}(\mathscr{A})$, by the point (i). Also, $A - z_n \in \mathsf{Inv}(\mathscr{A})$, so $z_n \in \mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$. Therefore, $z \in \partial\sigma_{\mathscr{A}}(A)$. This proves the point (ii).

If $A \in \mathscr{C}$ and $\sigma_{\mathscr{A}}(A)$ has no hole, then $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$ is connected. Since $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ is a clopen subset of $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$ by the points (i) and (ii), it follows that $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A) = \mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$, and therefore $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$. ☐

Let us end this section with a construction which can be used if a Banach algebra $\mathscr{C}$ has no unit. Consider the set $\tilde{\mathscr{C}} := \mathscr{C} \oplus \mathbb{C}$ with the multiplication

$$(A, z)(B, y) = (AB + zB + yA, zy).$$

This algebra contains a unit $\mathbf{1} = (\mathbf{0}, 1)$ and is call *a unitization of* $\mathscr{C}$. Clearly, the map $\mathscr{C} \ni A \mapsto (A, 0) \in \tilde{\mathscr{C}}$ is an injective homomorphism, which can be used to identify $\mathscr{C}$ with an ideal of $\tilde{\mathscr{C}}$. It is quite common to write simply $A + z$ for the element (A, z) of $\tilde{\mathscr{C}}$. Endowed with the norm $\|A + z\| := \|A\| + |z|$, $\tilde{\mathscr{C}}$ is a unital Banach algebra, which is abelian if $\mathscr{C}$ is abelian.

If $\mathscr{C}$ is a non-unital Banach algebra and $A \in \mathscr{C}$, one sets $\sigma_{\mathscr{C}}(A) := \sigma_{\tilde{\mathscr{C}}}(A)$.

2.3 The Gelfand representation

In this section, we concentrate on abelian Banach algebras and state a fundamental result for these algebras. First of all, let us observe that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a unital homomorphism between the unital Banach algebras $\mathscr{C}$ and $\mathscr{Q}$, then $\varphi\big(\mathsf{Inv}(\mathscr{C})\big) \subset \mathsf{Inv}(\mathscr{Q})$, and therefore $\sigma_{\mathscr{Q}}(\varphi(A)) \subset \sigma_{\mathscr{C}}(A)$ whenever $A \in \mathscr{C}$.

Definition 2.3.1. A character τ *on an abelian algebra* $\mathscr{C}$ *is a non-zero homomorphism from* $\mathscr{C}$ *to* $\mathbb{C}$. *The set of all characters of* $\mathscr{C}$ *is denoted by* $\Omega(\mathscr{C})$.

Let us immediately observe that if $\tau \in \Omega(\mathscr{C})$ for a unital abelian Banach algebra $\mathscr{C}$, then $\|\tau\| = 1$. Indeed, if $A \in \mathscr{C}$, one has $\tau(A) \subset \sigma_{\mathscr{C}}(A)$, and therefore $|\tau(A)| \leq \|A\|$. Hence $\|\tau\| \leq 1$, but $\tau(\mathbf{1}) = 1$ since $\tau(\mathbf{1}) = \tau(\mathbf{1})^2$ and $\tau(\mathbf{1}) \neq 0$.

For the next statement, we introduce the notation $M(\mathscr{C})$ for the set of maximal ideals of a Banach algebra $\mathscr{C}$.

Proposition 2.3.2. *Let* $\mathscr{C}$ *be a unital abelian Banach algebra. There is a bijection* $\tau \leftrightarrow \mathsf{Ker}(\tau)$ *between the set* $\Omega(\mathscr{C})$ *of characters of* $\mathscr{C}$ *and the set* $M(\mathscr{C})$. *Additionally, for each* $A \in \mathscr{C}$ *one has*

$$\sigma_{\mathscr{C}}(A) = \big\{\tau(A) \mid \tau \in \Omega(\mathscr{C})\big\}.$$

Proof. Let us first take $\mathscr{J} \in M(\mathscr{C})$ and consider the quotient Banach algebra $\mathscr{C}/\mathscr{J}$. By Lemma 2.2.2, it follows that $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$, and therefore the quotient map $\tau : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ belongs $\Omega(\mathscr{C})$. Conversely, if $\tau \in \Omega(\mathscr{C})$, then $\mathsf{Ker}(\tau)$ is an ideal in $\mathscr{C}$. In addition, one has $\mathscr{C} = \mathsf{Ker}(\tau) + \mathbb{C}\mathbf{1}$, since $\big(A - \tau(A)\mathbf{1}\big) \in \mathsf{Ker}(\tau)$. Consequently, $\mathsf{Ker}(\tau)$ is of co-dimension 1, and therefore is maximal.

Now, we show that any $A \in \mathscr{C} \setminus \mathsf{Inv}(\mathscr{C})$ is contained in a maximal ideal. Indeed, one easily observes that $A \in \mathscr{C}A$, with $\mathscr{C}A$ an ideal of $\mathscr{C}$ which does not contain $\mathbf{1}$. Then, the set of ideals that contains A but not $\mathbf{1}$ is inductively ordered by induction (because a union of an increasing family of ideals is an ideal), and a maximal element of this ordering is a maximal ideal. From Zorn's Lemma, it follows that A is contained in a maximal ideal.

Finally, if $A \in \mathscr{C}$ and $z \in \sigma_{\mathscr{C}}(A)$, then $(A - z) \notin \mathsf{Inv}(\mathscr{C})$. Therefore, there exists a character $\tau \in \Omega(\mathscr{C})$ such that $(A - z) \equiv (A - z\mathbf{1})$ belongs to the corresponding maximal ideal $\mathsf{Ker}(\tau)$. Accordingly, $\tau(A - z\mathbf{1}) = 0 \iff \tau(A) = z$. Conversely, if $\tau(A) = z$ for some $\tau \in \Omega(\mathscr{C})$, then $z \in \sigma_{\mathbb{C}}\big(\tau(A)\big) \subset \sigma_{\mathscr{C}}(A)$, by the observation made at the beginning of the section. □

Remark 2.3.3. *In the previous statement, if $\mathscr{C}$ is not unital one has for any $A \in \mathscr{C}$*

$$\sigma_{\mathscr{C}}(A) = \big\{\tau(A) \mid \tau \in \Omega(\mathscr{C})\big\} \cup \{0\}. \tag{2.3.1}$$

Indeed, if $\tau_\infty : \tilde{\mathscr{C}} \to \mathbb{C}$ denotes the character defined by $\tau_\infty(A, z) = z$, then one has $\Omega(\tilde{\mathscr{C}}) = \big\{\tilde{\tau} \mid \tau \in \Omega(\mathscr{C})\big\} \cup \{\tau_\infty\}$ with $\tilde{\tau}(A, z) = \tau(A) + z$, and

$$\sigma_{\mathscr{C}}(A) = \sigma_{\tilde{\mathscr{C}}}(A) = \{\tau(A, 0) \mid \tau \in \Omega(\tilde{\mathscr{C}})\} = \{\tau(A) \mid \tau \in \Omega(\mathscr{C})\} \cup \{0\}. \tag{2.3.2}$$

Since for any abelian Banach algebra $\mathscr{C}$, any $A \in \mathscr{C}$ and any $\tau \in \Omega(\mathscr{C})$ one has $|\tau(A)| \leq \|A\|$, it follows that $\Omega(\mathscr{C})$ is contained in the closed unit ball of the dual space $\mathscr{C}^*$. Thus, we can endow $\Omega(\mathscr{C})$ with the relative weak* topology and call the topological space $\Omega(\mathscr{C})$ the *character space*, or *spectrum* of $\mathscr{C}$.

Proposition 2.3.4. *If $\mathscr{C}$ is an abelian Banach algebra, then $\Omega(\mathscr{C})$ is a locally compact Hausdorff*[1] *space. If $\mathscr{C}$ is unital, then $\Omega(\mathscr{C})$ is compact.*

Proof. If $\mathscr{C}$ is unital, then it can be checked that $\Omega(\mathscr{C})$ is weak* closed in the closed unital ball $\mathscr{B}$ of $\mathscr{C}^*$. Since $\mathscr{B}$ is weak* compact (Banach-Alaoglu Theorem), it follows that $\Omega(\mathscr{C})$ is weak* compact.

If $\mathscr{C}$ is not unital, then $\Omega(\mathscr{C}) \cong \Omega(\tilde{\mathscr{C}}) \setminus \{\tau_\infty\}$, and therefore one obtains that $\Omega(\tilde{\mathscr{C}})$ is only locally compact. □

For any A in an abelian algebra $\mathscr{C}$ one defines the function $\hat{A}$ by

$$\hat{A} : \Omega(\mathscr{C}) \ni \tau \mapsto \hat{A}(\tau) \in \mathbb{C}$$

with $\hat{A}(\tau) := \tau(A)$. The topology of $\Omega(\mathscr{C})$ makes this function continuous. In addition, since for any $\varepsilon > 0$ the set $\{\tau \in \Omega(\mathscr{C}) \mid |\tau(A)| \geq \varepsilon\}$ is weak* closed in the closed unit ball of $\mathscr{C}^*$, and weak* compact by the Banach-Alaoglu Theorem, it follows that $\hat{A} \in C_0\big(\Omega(\mathscr{C})\big)$. Note that the map $A \mapsto \hat{A}$ is called *the Gelfand transform*.

[1] A Hausdorff space is a topological space in which distinct points have disjoint neighbourhoods. The weak* topology is Hausdorff.

Theorem 2.3.5. *Let $\mathscr{C}$ be an abelian Banach algebra. Then the map*

$$\mathscr{C} \ni A \mapsto \hat{A} \in C_0\big(\Omega(\mathscr{C})\big)$$

is a norm decreasing homomorphism, and $\|\hat{A}\|_\infty = r(A)$. If $\mathscr{C}$ is unital, then $\sigma_{\mathscr{C}}(A) = \hat{A}\big(\Omega(\mathscr{C})\big)$, while if $\mathscr{C}$ is not unital, $\sigma_{\mathscr{C}}(A) = \hat{A}\big(\Omega(\mathscr{C})\big) \cup \{0\}$, for any $A \in \mathscr{C}$.

Proof. It is easily checked that the mentioned map is a homomorphism. The spectral properties are direct consequences of (2.3.1) and (2.3.2), while the property on the norm follows from the observation that $\|\hat{A}\|_\infty = r(A) \leq \|A\|$. □

Note that the interpretation of the character space as a sort of generalized spectrum is motivated by the following result.

Lemma 2.3.6. *Let $\mathscr{C}$ be a unital Banach algebra, and let $\mathscr{A}$ be the unital subalgebra generated by $\mathbf{1}$ and an element $A \in \mathscr{C}$. Then $\mathscr{A}$ is abelian and the map*

$$\phi_A : \Omega(\mathscr{A}) \to \sigma_{\mathscr{A}}(A), \quad \phi_A(\tau) := \tau(A) \tag{2.3.3}$$

is a homeomorphism.

Proof. It is clear that the algebra $\mathscr{A}$ is abelian, and that ϕ_A is a continuous bijection. Since $\Omega(\mathscr{A})$ and $\sigma_{\mathscr{A}}(A)$ are compact Hausdorff spaces, the map ϕ_A is a homeomorphism (open mapping theorem). □

2.4 Basics on C^*-algebras

Definition 2.4.1. *A* Banach $*$-algebra *or* B^*-algebra *is a Banach algebra $\mathscr{C}$ together with an involution * satisfying for any $A, B \in \mathscr{C}$ and $\alpha \in \mathbb{C}$*

(i) $(A^*)^* = A$,

(ii) $(A+B)^* = A^* + B^*$,

(iii) $(\alpha A)^* = \overline{\alpha} A^*$,

(iv) $(AB)^* = B^* A^*$.

Clearly, if $\mathscr{C}$ is a unital B^*-algebra, then $\mathbf{1}^* = \mathbf{1}$.

Exercise 2.4.2. *Show that $\|A^*\| = \|A\|$ whenever A belongs to a B^*-algebra.*

Definition 2.4.3. A C^*-algebra *is a B^*-algebra $\mathscr{C}$ for which the following additional property is satisfied:*

$$\|A^*A\| = \|A\|^2 \qquad \forall\, A \in \mathscr{C}. \tag{2.4.1}$$

Examples 2.4.4. *All examples mentioned in Examples 2.1.2 are in fact C^*-algebras, once complex conjugation is considered as the involution for complex functions. In addition, let us observe that for a family $\{\mathscr{C}_i\}_{i\in I}$ of C^*-algebras, the direct sum $\oplus_{i\in I}\mathscr{C}_i$, with the pointwise involution and the supremum norm, is also a C^*-algebra.*

Note that a C^*-*subalgebra* of a C^*-algebra $\mathscr{C}$ is a norm closed subalgebra of $\mathscr{C}$ which is stable for the involution. It is clearly a C^*-algebra in itself. Note also that if $\mathscr{C}$ and $\mathscr{Q}$ are C^*-algebras, then $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism if φ is a homomorphism and if in addition $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathscr{C}$. An ideal $\mathscr{J}$ in a C^*-algebra is *self-adjoint* if it is stable for the involution.

Definition 2.4.5. *Let $\mathscr{C}$ be a C^*-algebra. An element $A \in \mathscr{C}$ satisfying $A = A^*$ is called* self-adjoint *or* hermitian, *an element $P \in \mathscr{C}$ satisfying $P = P^2 = P^*$ is called* an orthogonal projection, *and an element $A \in \mathscr{C}$ satisfying $AA^* = A^*A$ is called* a normal element of $\mathscr{C}$. *In addition, if $\mathscr{C}$ is unital, an element $U \in \mathscr{C}$ satisfying $UU^* = \mathbf{1} = U^*U$ is called* a unitary,

Note that it then follows from relation (2.4.1) that $\|U\| = 1$ for any unitary in $\mathscr{C}$, and that $\|P\| = 1$ for any (non-trivial) orthogonal projection in $\mathscr{C}$.

For the next statement, let us set

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Lemma 2.4.6. *Any self-adjoint element A in a unital C^*-algebra $\mathscr{C}$ satisfies $\sigma_{\mathscr{C}}(A) \subset \mathbb{R}$. If U is a unitary element of $\mathscr{C}$, then $\sigma_{\mathscr{C}}(U) \subset \mathbb{T}$.*

Proof. First of all, from the equality $\big((C - z)^{-1}\big)^* = (C^* - \overline{z})^{-1}$, one infers that if $z \in \sigma_{\mathscr{C}}(C)$, then $\overline{z} \in \sigma_{\mathscr{C}}(C^*)$, for any $C \in \mathscr{C}$. Furthermore, from the equality

$$z^{-1}(z - C)C^{-1} = -(z^{-1} - C^{-1}),$$

one also deduces that if $z \in \sigma_{\mathscr{C}}(C)$ for some $C \in \mathsf{Inv}(\mathscr{C})$, then $z^{-1} \in \sigma_{\mathscr{C}}(C^{-1})$.

Now, for a unitary $U \in \mathscr{C}$, one deduces from the above computations that if $z \in \sigma_{\mathscr{C}}(U)$, then $\overline{z}^{-1} \in \sigma_{\mathscr{C}}\big((U^*)^{-1}\big) = \sigma_{\mathscr{C}}(U)$. Since $\|U\| = 1$ one then infers from Lemma 2.2.3 that $|z| \leq 1$ and $|z^{-1}| \leq 1$, which means $z \in \mathbb{T}$.

If $A = A^* \in \mathscr{C}$, one sets $e^{iA} := \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}$ and observes that

$$(e^{iA})^* = e^{-iA} = (e^{iA})^{-1}.$$

Therefore, e^{iA} is a unitary element of $\mathscr{C}$ and it follows that $\sigma_{\mathscr{C}}\big(e^{iA}\big) \subset \mathbb{T}$. Now, let us assume that $z \in \sigma_{\mathscr{C}}(A)$, set $B := \sum_{n=1}^{\infty} \frac{i^n(A-z)^{n-1}}{n!}$, and observe that B commutes with A. Then one has

$$e^{iA} - e^{iz} = (e^{i(A-z)} - 1)e^{iz} = (A - z)Be^{iz}.$$

It follows from this equality that $e^{iz} \in \sigma_{\mathscr{C}}(e^{iA})$. Indeed, if $\big(e^{iA} - e^{iz}\big) \in \mathsf{Inv}(\mathscr{C})$, then $Be^{iz}\big(e^{iA} - e^{iz}\big)^{-1}$ would be an inverse for $(A - z)$, which can not be since $z \in \sigma_{\mathscr{C}}(A)$. From the preliminary computation, one deduces that $|e^{iz}| = 1$, which holds if and only if $z \in \mathbb{R}$. One has thus obtains that $\sigma_{\mathscr{C}}(A) \subset \mathbb{R}$. □

The following statement is an important result for the spectral theory in the framework of C^*-algebras. It shows that the computation of the spectrum does not depend on the surrounding algebra.

Theorem 2.4.7. *Let $\mathscr{C}$ be a C^*-subalgebra of a unital C^*-algebra $\mathscr{A}$ which contains the unit of $\mathscr{A}$. Then for any $A \in \mathscr{C}$,*

$$\sigma_{\mathscr{C}}(A) = \sigma_{\mathscr{A}}(A).$$

Proof. First of all, suppose that A is a self-adjoint element of $\mathscr{C}$. Then, since $\sigma_{\mathscr{A}}(A) \subset \mathbb{R}$, it follows from Proposition 2.2.5.(iii) that $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$. Alternatively, this means that A is invertible in $\mathscr{C}$ if and only if A is invertible in $\mathscr{A}$.

Now suppose that A is an arbitrary element of $\mathscr{C}$ which is invertible in $\mathscr{A}$, *i.e.* there exists $B \in \mathscr{A}$ such that $AB = BA = \mathbf{1}$. Then $A^*B^* = B^*A^* = \mathbf{1}$, so that $AA^*B^*B = \mathbf{1} = B^*BAA^*$, and this means that AA^* is invertible in $\mathscr{A}$, and therefore also in $\mathscr{C}$. Hence, there exists $C \in \mathscr{C}$ such that $AA^*C = \mathbf{1} = CAA^*$. One infers then that $A^*C = B$, which implies that $B \in \mathscr{C}$ and thus that A is invertible in $\mathscr{C}$. As a consequence, for any $A \in \mathscr{C}$ its invertibility in $\mathscr{A}$ is equivalent to its invertibility in $\mathscr{C}$, which directly implies the statement of the theorem. □

Because of the previous result, it is common to denote by $\sigma(A)$ the spectrum of an element A of a C^*-algebra, without specifying in which algebra the spectrum is computed. Let us also mention an additional result concerning the spectral radius:

Exercise 2.4.8. *If A is a self-adjoint element of a C^*-algebra $\mathscr{C}$, show that $r(A) = \|A\|$.*

Let us observe that this simple result has an important corollary:

Corollary 2.4.9. *There is at most one norm on a $*$-algebra making it a C^*-algebra.*

Proof. If $\|\cdot\|_1$, $\|\cdot\|_2$ are norms on a $*$-algebra $\mathscr{C}$ making it a C^*-algebra, then for any $A \in \mathscr{C}$ one has $\|A\|_j^2 = \|A^*A\|_j = r(A^*A)$, and therefore $\|A\|_1 = \|A\|_2$. □

We have already seen at the end of Section 2.2 how we can construct a unital Banach algebra $\tilde{\mathscr{C}}$ from a non-unital Banach algebra $\mathscr{C}$. However, if $\mathscr{C}$ is a C^*-algebra, the resulting algebra $\tilde{\mathscr{C}}$ is not a C^*-algebra in general. We shall now see how the construction can be adapted.

A double centralizer for a C^*-algebra $\mathscr{C}$ is a pair (L, R) of bounded linear maps on $\mathscr{C}$ such that for all $A, B \in \mathscr{C}$ one has

$$L(AB) = L(A)\,B, \quad R(AB) = A\,R(B), \quad \text{and} \quad R(A)\,B = A\,L(B).$$

For example, if $C \in \mathscr{C}$, then one can define a double centralizer (L_C, R_C) by $L_C(A) := CA$ and $R_C(A) := AC$. One then easily checks that

$$\|C\| = \sup_{\|A\|\leq 1} \|CA\| = \sup_{\|A\|\leq 1} \|AC\|,$$

and therefore $\|L_C\| = \|R_C\| = \|C\|$.

More generally one has:

Exercise 2.4.10. *If (L, R) is a double centralizer for a C^*-algebra, show that $\|L\| = \|R\|$.*

Thus, for any C^*-algebra $\mathscr{C}$, one denotes by $\mathscr{M}(\mathscr{C})$ the set of double centralizers of $\mathscr{C}$ and endows it with the norm $\|(L, R)\| := \|R\| = \|L\|$. $\mathscr{M}(\mathscr{C})$ becomes then a closed vector subspace of $\mathscr{B}(\mathscr{C}) \oplus \mathscr{B}(\mathscr{C})$. If in addition, one endows this set with the multiplication

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$$

and with the involution $(L, R)^* = (R^*, L^*)$ with $L^*(A) = \big(L(A^*)\big)^*$ and $R^*(A) = \big(R(A^*)\big)^*$, then one ends up with:

Proposition 2.4.11. *If $\mathscr{C}$ is a C^*-algebra, then $\mathscr{M}(\mathscr{C})$ is also a C^*-algebra.*

Proof. We only prove the property that $\|(L, R)^*(L, R)\| = \|(L, R)\|^2$, the other conditions being quite straightforward. For that purpose, let $A \in \mathscr{C}$ with $\|A\| \leq 1$. Then one has

$$\begin{aligned} \|L(A)\|^2 = \big\|\big(L(A)\big)^* L(A)\big\| = \|L^*(A^*)L(A)\| &= \big\|A\, R^*\big(L(A)\big)\big\| \\ &\leq \|R^* L\| = \big\|(L, R)^*(L, R)\big\|, \end{aligned}$$

which implies that

$$\|(L, R)\|^2 = \sup_{\|A\| \leq 1} \|L(A)\|^2 \leq \|(L, R)^*(L, R)\| \leq \|(L, R)\|^2.$$

One thus infers that $\|(L, R)^*(L, R)\| = \|(L, R)\|^2$. □

The C^*-algebra $\mathscr{M}(\mathscr{C})$ is called *the multiplier algebra* of $\mathscr{C}$, and the map $\mathscr{C} \ni A \mapsto (L_A, R_A) \in \mathscr{M}(\mathscr{C})$ is an isometric $*$-homomorphism of $\mathscr{C}$ into $\mathscr{M}(\mathscr{C})$. We can therefore identify $\mathscr{C}$ with a C^*-subalgebra of $\mathscr{M}(\mathscr{C})$. In fact, $\mathscr{C}$ is an ideal in $\mathscr{M}(\mathscr{C})$, and since $\mathbf{1} \in \mathscr{B}(\mathscr{C})$ the algebra $\mathscr{M}(\mathscr{C})$ is a unital C^*-algebra with unit $(\mathbf{1}, \mathbf{1})$. Note that $\mathscr{C} = \mathscr{M}(\mathscr{C})$ if and only if $\mathscr{C}$ is unital, and that $\mathscr{M}(\mathscr{C})$ is in fact the largest *unitization* of $\mathscr{C}$ in the following sense:

Theorem 2.4.12. *If $\mathscr{J}$ be a closed self-adjoint ideal in a C^*-algebra $\mathscr{C}$, then there exists a unique $*$-homomorphism $\varphi : \mathscr{C} \to \mathscr{M}(\mathscr{J})$ such that φ is the identity map on $\mathscr{J}$. Moreover, φ is injective if and only $\mathscr{J}$ is essential[2] in $\mathscr{C}$.*

Proof. See Proposition 2.2.14 of [W-O93] or Theorem 3.1.8 of [Mur90]. □

Let us recall that a $*$-isomorphism is a bijective $*$-homomorphism. In the next lemma, we deduce a consequence of the previous theorem.

Lemma 2.4.13. *If $\mathscr{C}$ is a C^*-algebra, then there exists a unique norm on its unitization $\tilde{\mathscr{C}}$ making it a C^*-algebra.*

[2]One says that a closed ideal $\mathscr{J}$ in a C^*-algebra $\mathscr{C}$ is *essential* if $AB = \mathbf{0}$ for all $B \in \mathscr{J}$ implies $A = \mathbf{0}$.

Proof. Uniqueness of the norm is given by Corollary 2.4.9. The proof of the existence falls into two cases, depending on whether $\mathscr{C}$ is unital or not.

Let us consider first the case of a unital C^*-algebra $\mathscr{C}$. Then, the map $\varphi : \tilde{\mathscr{C}} \to \mathscr{C} \oplus \mathbb{C}$ defined by $\varphi(A, z) = (A + z\mathbf{1}, z)$ is a $*$-isomorphism. Hence, one gets a C^*-norm on $\tilde{\mathscr{C}}$ by setting $\|(A, z)\| := \|\varphi(A, z)\|$.

Suppose now that $\mathscr{C}$ has no unit. If $\mathbf{1}$ denotes the unit of $\mathscr{M}(\mathscr{C})$, then $\mathscr{C} \cap \mathbb{C}\mathbf{1} = 0$. The map φ from $\tilde{\mathscr{C}}$ to the subalgebra $\mathscr{C} \oplus \mathbb{C}\mathbf{1}$ of $\mathscr{M}(\mathscr{C})$ defined by $\varphi(A, z) = A + z\mathbf{1}$ is a $*$-isomorphism, so we get a C^*-norm on $\tilde{\mathscr{C}}$ by setting $\|(A, z)\| := \|\varphi(A, z)\|$. □

From now on, we shall always consider the unitization $\tilde{\mathscr{C}}$ of a C^*-algebra endowed with its C^*-norm. Note in addition, that $\mathscr{M}(\mathscr{C})$ is usually much bigger than $\tilde{\mathscr{C}}$. For example, if $\mathscr{C} = C_0(\Omega)$ for a locally compact space Ω, then $\mathscr{M}(\mathscr{C}) = C_b(\Omega)$.

It is easily observed that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism between $*$-algebras, then φ extends uniquely to a unital $*$-homomorphism $\tilde{\varphi} : \tilde{\mathscr{C}} \to \tilde{\mathscr{Q}}$.

Lemma 2.4.14. *A $*$-homomorphism $\varphi : \mathscr{C} \to \mathscr{Q}$ from a B^*-algebra $\mathscr{C}$ to a C^*-algebra $\mathscr{Q}$ is necessarily norm decreasing.*

Proof. Without lost of generality, one can consider $\mathscr{C}$ and $\mathscr{Q}$ unital (by going to $\tilde{\mathscr{C}}$ and $\tilde{\mathscr{Q}}$ if necessary). For $A \in \mathscr{C}$ one has $\sigma_{\mathscr{Q}}\big(\varphi(A)\big) \subset \sigma_{\mathscr{C}}(A)$, and therefore

$$\|\varphi(A)\|^2 = \big\|\varphi(A)^*\varphi(A)\big\| = \|\varphi(A^*A)\| = r\big(\varphi(A^*A)\big) \leq r(A^*A) \leq \|A^*A\| \leq \|A\|^2.$$

It thus follows that $\|\varphi(A)\| \leq \|A\|$. □

Let us observe that an important corollary can be deduced from the previous lemma, namely any $*$-isomorphism between C^*-algebras is necessarily isometric.

Our next aim is to show that the Gelfand representation contained in Theorem 2.3.5 can be improved in the context of abelian C^*-algebras. For that purpose, observe first that any character on a C^*-algebra preserves adjoints. Indeed, let $\mathscr{C}$ be a C^*-algebra and let τ be a character on $\mathscr{C}$. Then, for any $A \in \mathscr{C}$, let us set $A = \Re(A) + i\Im(A)$ (with $\Re(A) := \frac{A+A^*}{2}$ and $\Im(A) := \frac{A-A^*}{2i}$ self-adjoint) and observe that

$$\tau(A^*) = \tau\big(\Re(A) - i\Im(A)\big) = \tau\big(\Re(A)\big) - i\tau\big(\Im(A)\big) = \overline{\tau\big(\Re(A) + i\Im(A)\big)} = \overline{\tau(A)}.$$

Theorem 2.4.15 (Gelfand representation)**.** *For any non-zero abelian C^*-algebra $\mathscr{C}$, the Gelfand representation*

$$\mathscr{C} \ni A \mapsto \hat{A} \in C_0\big(\Omega(\mathscr{C})\big) \tag{2.4.2}$$

is an isometric $*$-isomorphism.

Proof. Let us denote by φ the homomorphism defined in (2.4.2). It follows from Theorem 2.3.5 that φ is a norm decreasing homomorphism, with $\|\hat{A}\| = r(A)$, for any $A \in \mathscr{C}$. Now, if $\tau \in \Omega(\mathscr{C})$ one has $[\varphi(A^*)](\tau) = \tau(A^*) = \overline{\tau(A)} = \overline{[\varphi(A)](\tau)} = [\varphi(A)^*](\tau)$, which means that φ is a $*$-homomorphism. Moreover, φ is an isometry since

$$\|\varphi(A)\|^2 = \|\varphi(A)^*\varphi(A)\| = \|\varphi(A^*A)\| = r(A^*A) = \|A^*A\| = \|A\|^2.$$

Then, $\varphi(\mathscr{C})$ is a closed $*$-subalgebra of $C_0\big(\Omega(\mathscr{C})\big)$ separating the points of $\Omega(\mathscr{C})$, and having the property that for any $\tau \in \Omega(\mathscr{C})$ there is an element $A \in \mathscr{C}$ such that $[\varphi(A)](\tau) = \tau(A) \neq 0$. The Stone-Weierstrass Theorem implies therefore that $\varphi(\mathscr{C}) = C_0\big(\Omega(\mathscr{C})\big)$. □

The following exercise shows the coherence of the theory:

Exercise 2.4.16. *Let $\boldsymbol{\Omega}$ be a compact Hausdorff space, and for each $x \in \boldsymbol{\Omega}$ let τ_x be the character on $C(\boldsymbol{\Omega})$ defined by $\tau_x(f) = f(x)$ for any $f \in C(\boldsymbol{\Omega})$. Show that the map*

$$\boldsymbol{\Omega} \ni x \mapsto \tau_x \in \Omega\big(C(\boldsymbol{\Omega})\big)$$

is a homeomorphism.

The Gelfand representation has various useful applications. One is contained in the proof of the following statement. For this proof, we also need the following observation: If $\phi : \Omega \to \Omega'$ is a continuous map between compact Hausdorff spaces Ω and Ω', then the transpose map:

$$\phi^t : C(\Omega') \to C(\Omega), \qquad \phi^t(f) := f \circ \phi$$

is a unital $*$-homomorphism. Moreover, if ϕ is a homeomorphism, then ϕ^t is a $*$-isomorphism.

Proposition 2.4.17. *Let A be a normal element of a unital C^*-algebra $\mathscr{C}$, and let z be the inclusion map of $\sigma(A)$ in $\mathbb{C}$. Then there exists a unique unital $*$-homomorphism $\varphi : C\big(\sigma(A)\big) \to \mathscr{C}$ such that $\varphi(z) = A$. Moreover, φ is isometric and the image of φ is the C^*-subalgebra of $\mathscr{C}$ generated by A and $\mathbf{1}$.*

Proof. Let $\mathscr{A}$ be the unital C^*-subalgebra of $\mathscr{C}$ generated by A and $\mathbf{1}$, and let $\psi : \mathscr{A} \to C\big(\Omega(\mathscr{A})\big)$ be the Gelfand representation. By Theorem 2.4.15 ψ is a $*$-isomorphism. In addition, we know from Lemma 2.3.6 that the map ϕ_A defined in (2.3.3) is a homeomorphism, and therefore the map $\phi_A^t : C\big(\sigma(A)\big) \to C\big(\Omega(\mathscr{A})\big)$ is also a $*$-isomorphism. It then follows that the composed map $\varphi := \psi^{-1} \circ \phi_A^t : C\big(\sigma(A)\big) \to \mathscr{A}$ is a unital $*$-homomorphism, with $\varphi(z) = A$ since $\varphi(z) = \psi^{-1}\big(\phi_A^t(z)\big) = \psi^{-1}(\hat{A}) = A$. From the Stone-Weierstrass Theorem, we know that $C\big(\sigma(A)\big)$ is generated by 1 and z; φ is therefore the unique unital $*$-homomorphism from $C\big(\sigma(A)\big)$ to $\mathscr{C}$ such that $\varphi(z) = A$.

The remaining part of the proof is rather clear. □

Based on the idea developed in the previous proof, it is natural to set the following definitions: If S is any subset of a C^*-algebra, we denote by $C^*(S)$ the smallest C^*-algebra generated by S. Clearly, $C^*(S) \subset \mathscr{C}$, and $C^*(A) := C^*(\{A\})$ is an abelian algebra if A is normal. If A is self-adjoint, $C^*(A)$ is the closure of the set of polynomials in A with zero constant term. On the other hand, $C^*(\{A, \mathbf{1}\})$ is the closure of the set of polynomials in A with constant terms.

Let us finally mention that a bounded functional calculus similar to the one developed in Section 1.7.3 can also be defined in the C^*-algebraic framework. We mention below a useful result, but refer to [Mur90, Thm. 2.1.14] for its proof.

Theorem 2.4.18 (Spectral mapping)**.** *Let A be a normal element in a unital C^*-algebra $\mathscr{C}$, and let $\varphi \in C\big(\sigma(A)\big)$. Then the following equality holds:*

$$\sigma\big(\varphi(A)\big) = \varphi\big(\sigma(A)\big).$$

Moreover, if $\psi \in C\big(\sigma(\varphi(A))\big)$, then $[\psi \circ \varphi](A) = \psi\big(\varphi(A)\big)$.

2.5 Additional material on C^*-algebras

In this section we add some standard material on C^*-algebras. More information can be found in Chapters 2 and 3 of [Mur90].

Let us first observe that if $\mathscr{C} = C_0(\Omega)$ for a locally compact space Ω, then a natural notion of positivity on $\mathscr{C}$ exists. Indeed, if $\mathscr{C}_{sa}$ denote the subset of $\mathscr{C}$ made of real functions on Ω, then for $f \in \mathscr{C}_{sa}$ one writes $f \geq 0$ if and only if $f(x) \geq 0$ for any $x \in \Omega$. In addition, any $f \geq 0$ has a unique positive square root in $\mathscr{C}$, namely the function $x \mapsto \sqrt{f(x)}$. This notion of positivity endowed $\mathscr{C}_{sa}$ with a partial order: if $f, g \in \mathscr{C}_{sa}$ one sets $f \geq g$ if and only if $f - g \geq 0$. We shall now define a similar partial order on an arbitrary C^*-algebra.

Let $\mathscr{C}$ be a C^*-algebra, and $A \in \mathscr{C}$. One says that A is *positive* if A is self-adjoint, and $\sigma(A) \subset [0, \infty)$. We also write $A \geq 0$ to mean that A is positive, and denote by $\mathscr{C}^+$ the set of positive elements in $\mathscr{C}$. If $\mathscr{J}$ is a subalgebra of $\mathscr{C}$, one clearly has $\mathscr{J}^+ = \mathscr{J} \cap \mathscr{C}^+$.

Theorem 2.5.1. *Let $\mathscr{C}$ be a C^*-algebra and let $A \in \mathscr{C}^+$. Then there exists a unique $B \in \mathscr{C}^+$ such that $B^2 = A$.*

Proof. That there exists $B \in C^*(A)$ such that $B \geq 0$ and $B^2 = A$ follows from the Gelfand representation, since we may use it and identify $C^*(A)$ with $C_0(\Omega)$, where $\Omega := \Omega\big(C^*(A)\big)$, and then apply the above observation, see also Proposition 2.4.17.

Now, suppose that there exists another element $C \in \mathscr{C}^+$ such that $C^2 = A$. Since C commute with A, C also commute with the elements generated by A, and therefore C commute with B. So, let us set $\mathscr{Q} := C^*(\{B, C\})$ which is an abelian C^*-subalgebra of $\mathscr{C}$, and let $\varphi : \mathscr{Q} \to C_0\big(\Omega(\mathscr{Q})\big)$ be its Gelfand representation. Then, $\varphi(C)$ and $\varphi(B)$ are positive square root of $\varphi(A)$, which means that $\varphi(C) = \varphi(B)$. Since φ is is an isometric $*$-isomorphism, it follows that $C = B$. □

If A is a positive element of a C^*-algebra $\mathscr{C}$, we usually write $A^{1/2}$ for its unique positive square root in $\mathscr{C}$. For $A, B \in \mathscr{C}_{sa}$ we also set $A \geq B$ if $A - B \geq 0$. Let us add some elementary information about $\mathscr{C}^+$

Proposition 2.5.2. *Let $\mathscr{C}$ be a C^*-algebra. Then,*

(i) The sum of two positive elements of $\mathscr{C}$ is a positive element of $\mathscr{C}$,

(ii) The set $\mathscr{C}^+$ is equal to $\{A^*A \mid A \in \mathscr{C}\}$,

(iii) *If $A, B \in \mathscr{C}_{as}$ and $C \in \mathscr{C}$, then $A \geq B \Rightarrow C^*AC \geq C^*BC$,*

(iv) *If $A \geq B \geq 0$, then $A^{1/2} \geq B^{1/2}$,*

(v) *If $A \geq B \geq 0$, then $\|A\| \geq \|B\|$,*

(vi) *If $\mathscr{C}$ is unital and A, B are positive and invertible elements of $\mathscr{C}$, then $A \geq B \Rightarrow B^{-1} \geq A^{-1} \geq 0$,*

(vii) *For any $A \in \mathscr{C}$ there exist $A_1, A_2, A_3, A_4 \in \mathscr{C}^+$ such that*

$$A = A_1 - A_2 + iA_3 - iA_4.$$

Proof. See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur90]. □

Let us stress that the implication $A \geq B \geq 0 \Rightarrow A^2 \geq B^2$ is NOT true in general.

Definition 2.5.3. *For a C^*-algebra $\mathscr{C}$,* an approximate unit *is an upwards-directed set $\{I_j\}_{j\in J} \subset \mathscr{C}^+$ with $\|I_j\| \leq 1$ and such that $A = \lim_j I_jA$ for any $A \in \mathscr{C}$.*

In order to show that each C^*-algebra $\mathscr{C}$ possesses such an approximate unit, let us first observe that the set of elements of $\mathscr{C}^+$ with norm strictly less than 1 is a partially ordered set which is upwards-directed ($\Leftrightarrow$ if $A, B \in \mathscr{C}^+$ then there exists $C \in \mathscr{C}^+$ such that $C \geq A$ and $C \geq B$). For that purpose, let us set $\mathscr{C}_1^+ := \{A \in \mathscr{C}^+ \mid \|A\| < 1\}$. Observe first that if $A \in \mathscr{C}^+$, then $\mathbf{1} + A$ is invertible in $\tilde{\mathscr{C}}$, and $A(\mathbf{1} + A)^{-1} = \mathbf{1} - (\mathbf{1} + A)^{-1} \in \mathscr{C}$. We next show that if $A, B \in \mathscr{C}^+$ with $B \geq A$, then $B(\mathbf{1} + B)^{-1} \geq A(\mathbf{1} + A)^{-1}$. Indeed, if $B \geq A \geq 0$, then $\mathbf{1} + B \geq \mathbf{1} + A$ in $\tilde{\mathscr{C}}$, and by Proposition 2.5.2.(vi) it follows that $(\mathbf{1} + A)^{-1} \geq (\mathbf{1} + B)^{-1}$. As a consequence, $\mathbf{1}-(\mathbf{1}+B)^{-1} \geq \mathbf{1}-(\mathbf{1}+A)^{-1}$, that is $B(\mathbf{1}+B)^{-1} \geq A(\mathbf{1}+A)^{-1}$ in $\mathscr{C}$. Observe now that if $A \in \mathscr{C}^+$, then $A(\mathbf{1}+A)^{-1} \in \mathscr{C}_1^+$ (use the Gelfand representation applied to $C^*(\{A, \mathbf{1}\})$). Suppose finally that $A, B \in \mathscr{C}_1^+$, and set $A' := A(\mathbf{1} - A)^{-1}$, $B' := B(\mathbf{1} - B)^{-1}$ and $C := (A' + B')(\mathbf{1} + A' + B')^{-1}$. Then, $C \in \mathscr{C}_1^+$, and since $A' + B' \geq A'$ we have $C \geq A'(\mathbf{1} + A')^{-1} = A$. Similarly, $C \geq B$, and therefore $\mathscr{C}_1^+$ is upwards-directed, as claimed.

Theorem 2.5.4. *Every C^*-algebra $\mathscr{C}$ admits an approximate unit.*

The idea of the proof is to show that the upwards-directed set $\mathscr{C}_1^+$ provide such an approximate unit. More precisely, for any $\Lambda \in \mathscr{C}_1^+$, we set $I_\Lambda := \Lambda$ and show that the family $\{I_\Lambda\}_{\Lambda\in\mathscr{C}_1^+}$ is an approximate unit. This approximate unit is called *the canonical approximate unit*. We refer to [Mur90, Thm. 3.1.1] for the details. Note that in the applications, more natural approximate units appear quite often.

If $\{I_j\}_{j\in J}$ is an approximate unit for a C^*-algebra, then, one has by definition $\lim_j \|(\mathbf{1} - I_j)A\| = 0$ for all $A \in \mathscr{C}$. Let us also observe that $\lim_j \|A(\mathbf{1} - I_j)\| = 0$. Indeed, from the relations

$$\|A(\mathbf{1} - I_j)\|^2 = \|(\mathbf{1} - I_j)A^*A(\mathbf{1} - I_j)\| \leq \|(\mathbf{1} - I_j)A^*A\|$$

one directly infers the statement.

Theorem 2.5.5. *Let $\mathscr{J}$ be a closed self-adjoint ideal in a C^*-algebra $\mathscr{C}$. Since $\mathscr{J}$ is itself a C^*-algebra, there exists an approximate unit $\{I_j\}_{j\in J}$ for $\mathscr{J}$, and then for each $A \in \mathscr{C}$ one has*

$$\|A + \mathscr{J}\| = \lim_j \|A - I_j A\| = \lim_j \|A - AI_j\|$$

Proof. Let $A \in \mathscr{C}$ and let $\varepsilon > 0$. From the definition of the norm of $A + \mathscr{J}$ there exists $B \in \mathscr{J}$ such that $\|A + B\| < \|A + \mathscr{J}\| + \varepsilon/2$. Since $B = \lim_j I_j B$ there exists j_0 such that $\|(\mathbf{1} - I_i)B\| < \varepsilon/2$ for all $j \geq j_0$, and therefore

$$\begin{aligned}\|A - I_j A\| &\leq \|(\mathbf{1} - I_j)(A + B)\| + \|(\mathbf{1} - I_j)B\| \leq \|A + B\| + \|(\mathbf{1} - I_j)B\| \\ &< \|A + \mathscr{J}\| + \varepsilon.\end{aligned}$$

It follows that $\|A + \mathscr{J}\| = \lim_j \|A - I_j A\|$. The second equality can be shown similarly. □

Let us now state three useful corollaries which can be deduced from this statement, and refer to [Mur90, Sec. 3.1] for their proofs. These statements correspond to extensions to the framework of C^*-algebras of results which have already been discussed for Banach algebras.

Corollary 2.5.6. *If $\mathscr{J}$ is a closed self-adjoint ideal in a C^*-algebra, then the quotient algebra $\mathscr{C}/\mathscr{J}$ is a C^*-algebra.*

Corollary 2.5.7. *If $\varphi : \mathscr{C} \to \mathscr{Q}$ is an injective $*$-homomorphism between C^*-algebras, then φ is necessarily isometric.*

Corollary 2.5.8. *If $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism between C^*-algebras, then $\varphi(\mathscr{C})$ is a C^*-subalgebra of $\mathscr{Q}$.*

Extension 2.5.9. *With the use of an approximate unit, give the proof the three corollaries.*

We now state an important result for the theory of C^*-algebra, the GNS construction. It will then allow us to consider any C^*-algebra as a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

Definition 2.5.10. A representation *of a C^*-algebra $\mathscr{C}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathscr{C} \to \mathscr{B}(\mathcal{H})$ is a $*$-homomorphism. This representation is* faithful *if π is injective.*

Theorem 2.5.11 (Gelfand-Naimark-Segal (GNS) representation)**.** *For any C^*-algebra $\mathscr{C}$ there exists a faithful representation.*

Extension 2.5.12. *The proof of this theorem is based on the notion of* states *(positive linear functionals) on a C^*-algebra, and on the existence of sufficiently many such states. The construction is rather explicit and can be studied.*

With the GNS construction at hand, we can end this chapter by considering again the multiplier algebra $\mathscr{M}(\mathscr{C})$ for a C^*-algebra $\mathscr{C}$, and add some information concerning this algebra. More precisely, let us assume that the C^*-algebra $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ acts non-degenerately on $\mathcal{H}$, *i.e.* for any $f \in \mathcal{H} \setminus \{0\}$ there exists $A \in \mathscr{C}$ such that $Af \neq 0$. Note that this is not really any constraint since one can always "eliminate" any superfluous part of the Hilbert space. Then it is natural to set

$$\mathscr{M}_{\mathcal{H}}(\mathscr{C}) := \big\{B \in \mathscr{B}(\mathcal{H}) \mid BA \in \mathscr{C} \text{ and } AB \in \mathscr{C} \text{ for all } A \in \mathscr{C}\big\}.$$

Theorem 2.5.13. *Let $\mathscr{C}$ be a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ acting non-degenerately on $\mathcal{H}$. Then, the correspondence*

$$\mathscr{M}_{\mathcal{H}}(\mathscr{C}) \ni C \mapsto (L_C, R_C) \in \mathscr{M}(\mathscr{C})$$

is an isometric $*$-isomorphism.

We refer to [W-O93, Prop. 2.2.11] for the proof of this statement. Note that the non-trivial part of the proof consists in constructing the inverse map $\mathscr{M}(\mathscr{C}) \to \mathscr{M}_{\mathcal{H}}(\mathscr{C})$. Because of the previous results, we shall simply write $\mathscr{M}(\mathscr{C})$ for $\mathscr{M}_H(\mathscr{C})$ and also call it *the multiplier algebra*. This should not lead to any confusion.

Definition 2.5.14. *Let $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra acting non-degenerately on $\mathcal{H}$.* The strict topology *on $\mathscr{M}(\mathscr{C})$ is the weakest topology making the maps $B \mapsto BA$ and $B \mapsto AB$ norm continuous, for any $B \in \mathscr{M}(\mathscr{C})$ and $A \in \mathscr{C}$. In other words, the strict topology is the topology generated by the family of seminorms $B \mapsto \|BA\|$ and $B \mapsto \|AB\|$.*

It can be shown that $\mathscr{M}(\mathscr{C})$ is strictly complete, or equivalently that every strict Cauchy net in $\mathscr{M}(\mathscr{C})$ is strictly convergent in $\mathscr{M}(\mathscr{C})$. In fact, $\mathscr{M}(\mathscr{C})$ is the strict completion of $\mathscr{C}$. We refer to Section 2.3 of [W-O93] for a friendly approach to the strict topology.