WSkS (Weakly Second-order monadic logic with k Successors)

- Logic with variables on strings of k alphabet symbols and variables on sets on the strings
- Ex.: $\forall x.(x \in X \Rightarrow x \in Y)$
- Satisfiability is decidable

- **Syntax of WSkS**
  - **term**: **1st order variables** $x, y, z \ldots$ **and strings of alphabet** $\{1, \ldots, k\}$ **(variable can occur on left side)**

    Ex.: $x1123$, 211, ε
  - **Atomic formula**: $s = t$, $s \leq t$, $s \geq t$, $t \in X$ (s,t **for terms,** X,Y **for 2nd order variables**)
  - **Formula**: **constructed atomic formulae by** $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftarrow$, $\Leftrightarrow$, $\exists x$, $\forall x$, $\exists X$, $\forall X$

- **Semantics of WSkS**
  - **Terms are interpreted as strings**
  - **1st order variable x represents a string, 2nd order variable X represents a set of strings**
  - **$=$ for equality on strings, $\in$ for membership, $\leq$ for prefix relation** **Ex.:** $13 \leq 1322$、$11 \not\leq 121$
  - **Let $t_i \in \{1, \ldots, k\}^*$, and $S_i \subseteq \{1, \ldots, k\}^*$.**
    Let ϕ be formula with free variables $x_1, \ldots, x_n$ and $X_1, \ldots, X_m$.
    $t_1, \ldots, t_n, S_1, \ldots, S_m \models \phi$ denotes that ϕ holds for assignment t_i to x_i and S_i to X_i

- **Examples of formula, and their abbreviation**
  - **Subset** $X \subseteq Y$

    $\forall x.(x \in X \Rightarrow x \in Y)$
  - **Set equality** $Y = X$

    $Y \subseteq X \wedge X \subseteq Y$
  - **Set emptiness** $X = \emptyset$

    $\forall Y.(Y \subseteq X \Rightarrow Y = X)$
  - **Intersection emptiness** $X \cap Y = \emptyset$

    $\forall x.((x \in X \Rightarrow x \notin Y) \wedge (x \in Y \Rightarrow x \notin X))$
  - **Singleton set** $Sing(X)$

    $X \neq \emptyset \wedge \forall Y.(Y \subseteq X \Rightarrow (Y = X \vee Y = \emptyset))$

- **Examples of formula, and abbrev. (cont.)**
  - **Union of sets** $X = \bigcup_{i=1}^{n} X_i$

    $$\bigwedge_{i=1}^{n} X_i \subseteq X \wedge \forall x.(x \in X \Rightarrow \bigvee_{i=1}^{n} x \in X_i)$$
  - **Partition** $Partition(X, X_1, \ldots, X_n)$

    $$X = \bigcup_{i=1}^{n} X_i \wedge (\bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} X_i \cap X_j = \emptyset)$$
  - **Prefix** $x \leq y$**: (i.e.** $\leq$ **is not essential)**

    $$\forall X.(y \in X \wedge (\forall z.(\bigvee_{i=1}^{k} zi \in X) \Rightarrow z \in X) \\ \Rightarrow x \in X)$$
  - **Prefix closed** $PrefixClosed(X)$**:**

    $$\forall x.\forall y.((x \in X \wedge y \leq x) \Rightarrow y \in X)$$

- **Q8: Show WSkS formulae**

  (a) Set X contains at least one string that begins from 1

  (b) Set X consists of exactly two elements

- Restriction of the syntax: no 1st order variables with following atomic formula in preserving the power:
  - $X \subseteq Y$, $Sing(X)$, $X = Yi$, $X = \varepsilon$
  - $X = Yi$ is interpreted as both X and Y are singleton sets $\{t\}$ and $\{s\}$ satisfying $t = si$
  - For a restricted WSkS ϕ and sets S_i's,

    $$S_1, \ldots, S_n \models \phi$$

    denotes that ϕ holds for the assignment S_i to X_i

- **Prop. 1: There exists a transformation T from WSkS formula to an equivalent restricted formula, i.e.,**

  $s_1, \dots, s_n, S_1, \dots, S_m \models \phi$

  iff

  $\{s_1\}, \dots, \{s_n\}, S_1, \dots, S_m \models T(\phi)$

  An inverse transformation T' exists.

- **Proof: Construction of T. T' is omitted**

  $$T(ti \in X) \;=\; T(\exists y.(y = ti \wedge y \in X))$$
  $$T(y \in X) \;=\; X_y \subseteq X \wedge Sing(X_y)$$

- **Proof (cont.):**

$$
\begin{aligned}
T(t = s) &= T(\exists z. z = t \wedge z = s) \\
&\quad \textbf{(if } t \textbf{ and } s \textbf{ are non-variables)} \\
T(x = ti) &= T(\exists z. z = t \wedge x = zi) \\
&\quad \textbf{(if } t \textbf{ is not a variable)} \\
T(x = yi) &= X_x = X_y i \\
T(x = \varepsilon) &= X_x = \varepsilon \\
T(x = y) &= X_x \supseteq X_y \wedge X_x \subseteq X_y \\
&\quad \wedge Sing(X_x) \wedge Sing(X_y)
\end{aligned}
$$

$$T(\phi \vee \psi) = T(\phi) \vee T(\psi)$$

$$T(\neg \phi) = \neg T(\phi)$$

$$T(\exists X. \phi) = \exists X. T(\phi)$$

$$T(\exists x. \phi) = \exists X_y. (Sing(X_y) \wedge T(\phi))$$

Relation definable by WSkS is recognizable by an NFTA

- **A relation R on sets of strings R is definalble by WSkS:**

  For $S_1, \ldots, S_n \subseteq \{1, \ldots, k\}^*$, there exists a WSkS formula ϕ such that

  $(S_1, \ldots, S_n) \in R$ **iff** $S_1, \ldots, S_n \models \phi$

- **tree representation** $t = (S_1, \ldots, S_n)^{\sim}$ **of** $(S_1, \ldots, S_n)$

  $\mathsf{Pos}(t) =$

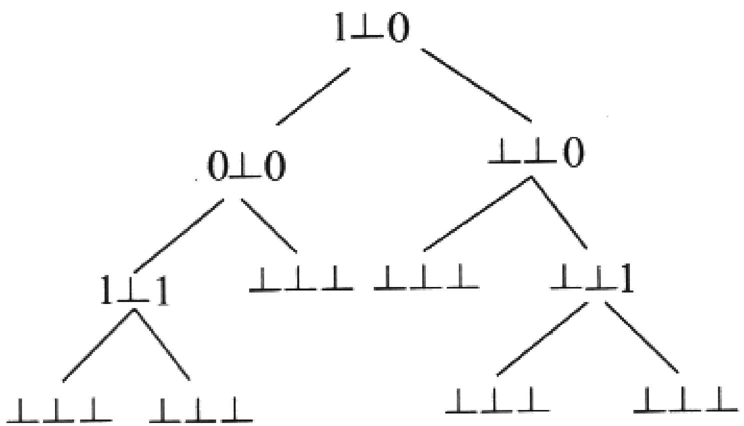
  $\{\varepsilon\} \cup \{pi \mid \exists p' \in \cup_{j=1}^{n} S_j, p \leq p', i \in \{1, \ldots, k\}\}$

  $t(p) = \alpha_1 \cdots \alpha_n$,

  where each α_i is

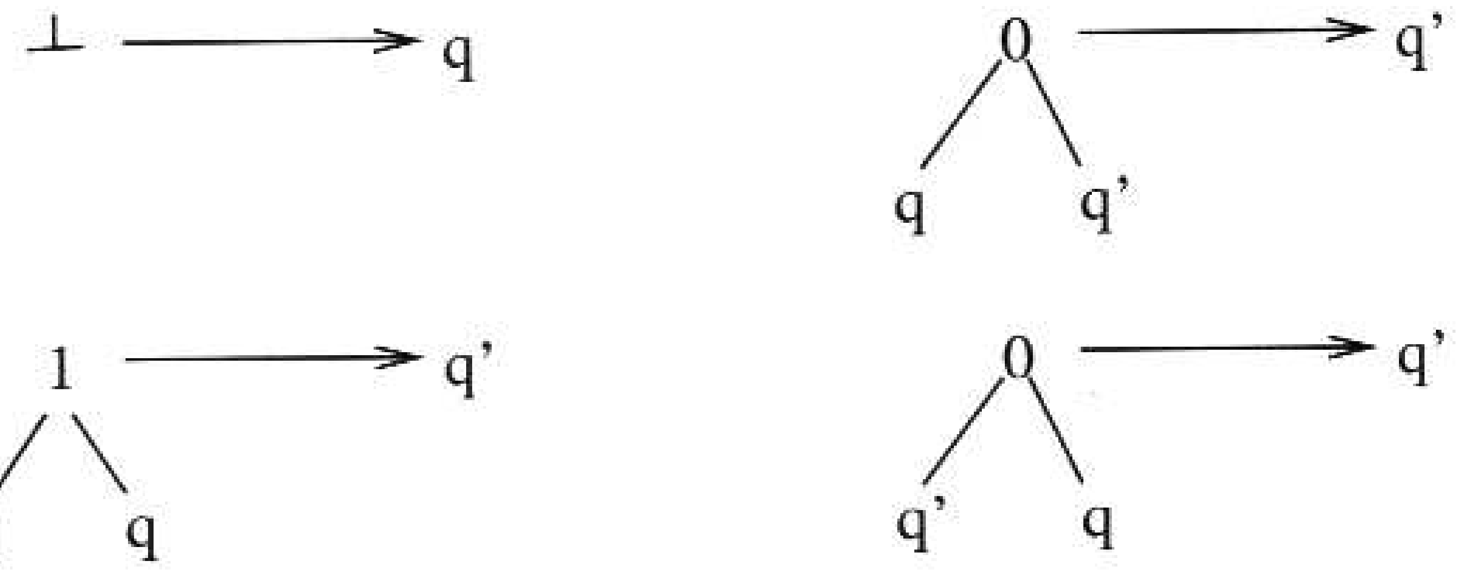
  $$\alpha_i = \begin{cases} 1 & \text{if } p \in S_i \\ 0 & \text{if } p \notin S-i, \exists p' \in S_i\, p < p' \\ \perp & \text{otherwise} \end{cases}$$

- **Ex.: For** $k = 2$, $A = \{\varepsilon, 11\}$, $B = \emptyset$, **and** $C = \{11, 22\}$, $(A, B, C)^{\sim} =$

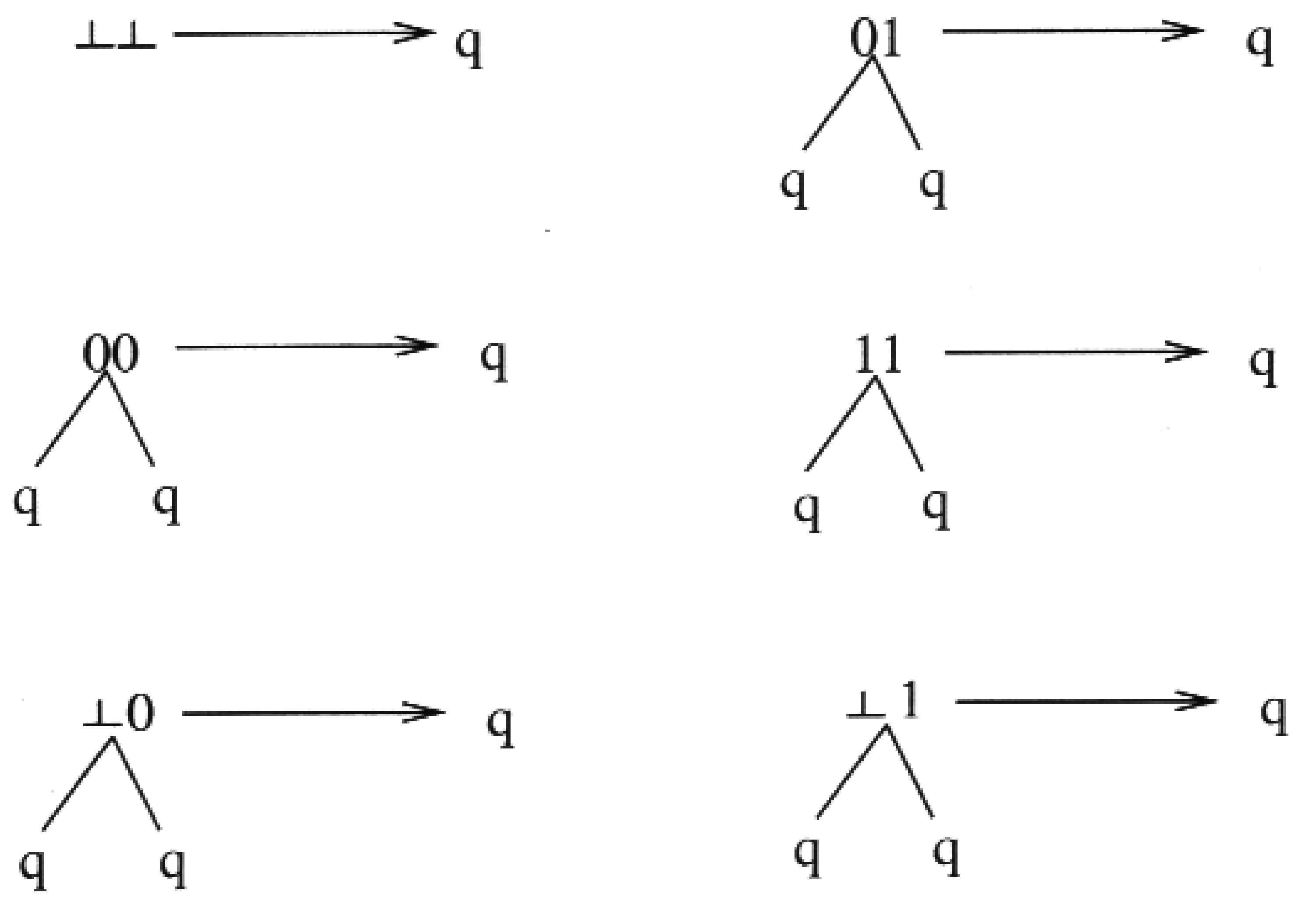


- **Th. 2: If a relation R on sets of strings is definable by WSkS, then there exists an NFTA that recognizes the following language**
  $\tilde{R} = \{(S_1, \ldots, S_n)^{\sim} \mid (S_1, \ldots, S_n) \in R\}$
- **Proof: From Prop. 1, we can assume ϕ are restricted WSkS formula. We prove by structural induction on ϕ. We show the proof fixed with $k = 2$**

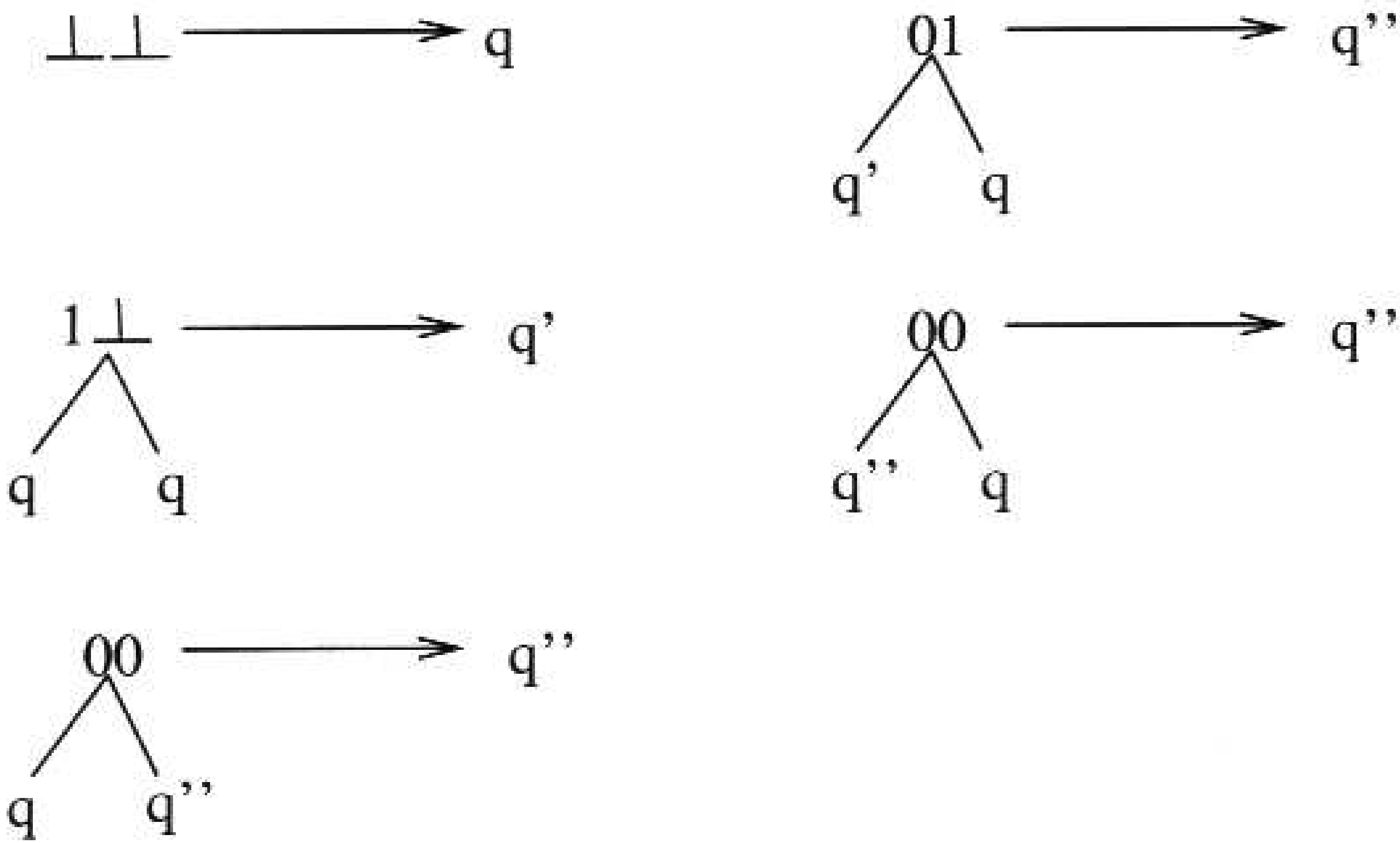
- **Proof (cont.)**
  - **If ϕ is $Sing(X)$, for a final state q'**



- **Proof (cont.)**
  - **If ϕ is $X \subseteq Y$, for a final state q**



- **Proof (cont.)**
  - **If ϕ is $X = Y1$, for a final state q''**



- **Proof (cont.)**
  - **If ϕ is $X = \varepsilon$, for a final state q'**

  - **If ϕ is $\neg\phi'$, there exists TA A' that recognizes R' defined by ϕ' by IH. We construct TA that recognizes the complement of R'**
  - **If ϕ is $\exists X_i.\phi'$, relation by ϕ is i'th projection of relation by ϕ'. Thus we convert each rule in TA A' for ϕ'**

    $\cdots\alpha_i\cdots(\cdots) \to q$ **into** $\cdots \quad \cdots(\cdots) \to q$

- **Proof (cont.)**
  - **Consider that ϕ is $\phi' \vee \phi''$. For simplicity, let**

    Free variables in ϕ' are $X_1 \ldots X_i \ldots X_n$

    Free variables in ϕ'' are $X_1 \ldots \quad \ldots X_n$

    Replace each rule of A'' by ϕ''

    $\cdots \quad \cdots(\cdots) \to q$ **into rules**

    $\cdots 0 \cdots (\cdots) \to q$

    $\cdots 1 \cdots (\cdots) \to q$

    $\cdots \perp \cdots (\cdots) \to q$

    Take union this and A' by ϕ'

- **Col.: Satifiability of WSkS formula is decidable**
- **Proof: Let R by the relation defined by WSkS formula ϕ. By Th. 2, there exists a TA A that recognizes $\tilde{R}$. It is enoough to decide the emptiness of A, because "$R = \emptyset$ iff $\tilde{R} = \emptyset$"**

Relation recognizable by NFTA is definable by WSkS

- **Coding $\bar{t}$ of trees t**
  - Ex.: $f(g(a), a)$ is representable by 4 sets (S, S_f, S_g, S_a), where $S = \{\varepsilon, 1, 11, 2\}$, $S_f = \{\varepsilon\}$, $S_g = \{1\}$, $S_a = \{11, 2\}$
- **Validity of coded tree :** $Term(X, X_{f_1}, \ldots, X_{f_n})$:

$$X \neq \emptyset \ \wedge Partition(X, X_{f_1}, \ldots, X_{f_n}) \wedge PrefixClosed(X)$$

$$\wedge \bigwedge_{i=0}^{k} \bigwedge_{\text{arity}(f_j)=i} \forall x \left(x \in X_{f_j} \Rightarrow \left(\bigwedge_{\ell=1}^{i} x\ell \in X \ \wedge \bigwedge_{\ell=i+1}^{k} x\ell \notin X \right) \right)$$

- **Th. 4: Let L be regular language on $\mathcal{F}$. There exists a WSkS formula ϕ such that**
  $t \in L$ **iff** $\bar{t} \models \phi$
- **Proof: Let $A = (Q, \mathcal{F}, Q^f, \Delta)$ be an NFTA such that $L = L(A)$, where**
  $\mathcal{F} = \{f_1, \ldots, f_n\}$, $Q = \{q_1, \ldots, q_m\}$, **and**
  $\bar{t} = (S, S_{f_1}, \ldots, S_{f_n})$

- **Proof (cont.)**

  ϕ **is the following formula with free variables** $X, X_{f_1}, \ldots, X_{f_n}$**:**

$$
\begin{aligned}
&\exists Y_{q_1}, \ldots, \exists Y_{q_m}. Term(X, X_{f_1}, \ldots, X_{f_n}) \\
&\wedge Partition(X, Y_{q_1}, \ldots, Y_{q_m}) \\
&\wedge \bigvee_{p \in Q_f} \varepsilon \in Y_p \\
&\wedge \forall x. \bigwedge_{f \in \mathcal{F}} \bigwedge_{p \in Q} ((x \in X_f \wedge x \in Y_p) \\
&\qquad\qquad \Rightarrow \bigvee_{f(p_1, \ldots, p_\ell) \to p \in \Delta} \bigwedge_{i=1}^{\ell} xi \in Y_{p_i})
\end{aligned}
$$

- **Col. 5:** Rec **is definable by WSkS**
- **Proof: For a relation** R **in** Rec**, by Th. 4 there exists a WSkS formula** ϕ **such that**

$$(t_1, \ldots, t_n) \in R \textbf{ iff } \overline{[t_1, \ldots, t_n]} \models \phi$$