Properties of Tree Automata

- Closure

  The following operations preserve regularity:

  - Union: $L_1 \cup L_2$
  - Complementation: $\overline{L_1}$

    via complete DFTA
  - Intersection: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$
  - Linear tree homomorphisms
  - Inversion of linear tree homomorphisms

- **Proof for union:**
  Construct NFTA $A = (Q, \mathcal{F}, Q^f, \Delta)$ **recognizing** $L_1 \cup L_2$ **from NFTA** $A_i = (Q_i, \mathcal{F}, Q_i^f, \Delta_i)$ **recognizing** L_i **for each** i
  - $Q = Q_1 \cup Q_2$
  - $\Delta = \Delta_1 \cup \Delta_2$
  - $Q^f = Q_1^f \cup Q_2^f$

- **Tree homomorphism:**

  - $h : \mathsf{T}(\mathcal{F}) \to \mathsf{T}(\mathcal{F}')$ **determined by** $h_{\mathcal{F}} : \mathcal{F} \to \mathsf{T}(\mathcal{F}', \mathcal{X})$,

    where $h_{\mathcal{F}}(f) \in \mathsf{T}(\mathcal{F}', \mathcal{X}_n)$ **for** $arity(f) = n$

    $h(a) = h_{\mathcal{F}}(a)$, **and**

    $h(f(t_1, \ldots, t_n)) =$

    $(h_{\mathcal{F}}(f))\{x_1 \leftarrow h(t_1), \ldots, x_n \leftarrow h(t_n)\}$

- **For** $h_{\mathcal{F}}(g) = f(x_1, f(x_2, x_1))$, $h_{\mathcal{F}}(a) = a$, $h_{\mathcal{F}}(b) = b$, $h(g(a, g(b, b))) = f(a, f(f(b, f(b, b)), a))$.

- h **is linear:** $h_{\mathcal{F}}(f)$ **is linear for any** $f \in \mathcal{F}$

- **image $h(L)$ and inverse image $h^{-1}(L)$ by tree homomorphism h of L:**
  $h(L) = \{h(t) \mid t \in L\}$, **and**
  $h^{-1}(L) = \{t \mid h(t) \in L\}$
- **Ex. that non-linearity prevents preservation of regularity:**
  for $\mathcal{F} = \{f(), g(), a\}$, $\mathcal{F}' = \{f'(,), g(), a\}$,
  $h_{\mathcal{F}}(f) = f'(x_1, x_1)$, $h_{\mathcal{F}}(g) = g(x_1)$, $h_{\mathcal{F}}(a) = a$,
  and $L = \{f(g^m(a)) \mid m \geq 0\}$,
  $h(L) = \{f'(g^m(a), g^m(a)) \mid m \geq 0\}$

- **Proof for linear homomoriphisms**
  For an NFTA $A = (Q, \mathcal{F}, Q^f, \Delta)$ recognizing L, we construct an NFTA $A' = (Q', \mathcal{F}, Q^f, \Delta')$ recognizing $h(L)$.
  - **For each $r = f(q_1, \ldots, q_n) \to q \in \Delta$,**
    $Q_r = \{q^r_p \mid p \in \mathsf{Pos}(h(f))\}$, **and**
    Δ_r **is the set of following rules (**$p \in \mathsf{Pos}(h(f))$**):**
    - $g(q^r_{p1}, \ldots, q^r_{pk}) \to q^r_p$ **for** $(h(f))(p) = g$ **and** $\mathsf{arity}(g) = k$,
    - $q_i \to q^r_p$ **for** $(h(f))(p) = x_i$, **and**
    - $q^r_\varepsilon \to q$
  - $Q' = Q \cup \bigcup_{r \in \Delta} Q_r$ **and** $\Delta' = \bigcup_{r \in \Delta} \Delta_r$

- **Proof for inversion of linear homomorphisms**
  For a DFTA $A = (Q, \mathcal{F}, Q^f, \Delta)$ **recognizing** L**, we construct an NFTA** $A' = (Q \cup \{s\}, \mathcal{F}, Q^f, \Delta')$ **recognizing** $h^{-1}(L)$**.**

  Δ' **is the set of rules given as follows:**

  - $a \to q$ **for** q **such that** $a \in \mathcal{F}$ **and** $h(a) \to_A^* q$**,**

  - $f(q_1, \ldots, q_n) \to q$ **for** $f \in \mathcal{F}$**,** $p_1, \ldots, p_n \in Q$**,** $(h(f))\{x_1 \leftarrow p_1, \ldots, x_n \leftarrow p_n\} \to_A^* q$ **(**$n > 0$**), and** $q_i = p_i$ **for** i **such that** x_i **occurs in** $h(f)$**, and** $q_i = s$ **for other** i**'s**

  - $a \to s$ **for** $a \in \mathcal{F}$

  - $f(s, \ldots, s) \to s$ **for** $f \in \mathcal{F}$ **(**$\text{arity}(f) = n > 0$**)**