3 Representation

3.1 Eigenvectors and eigenvalues in linear algebra

Look at the action on monomials x^m:

$$\begin{aligned} Hx^m &= (-2m+\lambda)x^m, \\ Xx^m &= -mx^{m-1}, \\ Yx^m &= (m-\lambda)x^{m+1}, \end{aligned}$$

where

$$\begin{aligned} H &= -2x\partial + \lambda, \\ X &= -\partial, \\ Y &= x^2\partial - \lambda x. \end{aligned}$$

Exercise 3.1. *Show the above.*

Key word: Euler operator=degree-counting operator.
Let $\theta = x\partial = x\frac{d}{dx}$. We have

$$\begin{aligned} \theta x^m &= x\partial(x^m) = mx^m, \\ Hx^m &= (-2\theta+\lambda)x^m = (-2m+\lambda)x^m, \\ Yx^m &= x(x\partial-\lambda)x^m = x(\theta-\lambda)x^m. \end{aligned}$$

$\theta x^m = mx^m$	$A\mathbf{v} = \mathbf{v}$
θ: linear operator	A: matrix
x^m: function	$\mathbf{v}$: vector
m: scalar	λ: scalar

Thus x^m is an eigenfunction (eigenvector) of θ with an eigenvalue (spectrum) m. In this terminology, x^m is an eigenfunction of H with an eigenvalue $-2m+\lambda$.

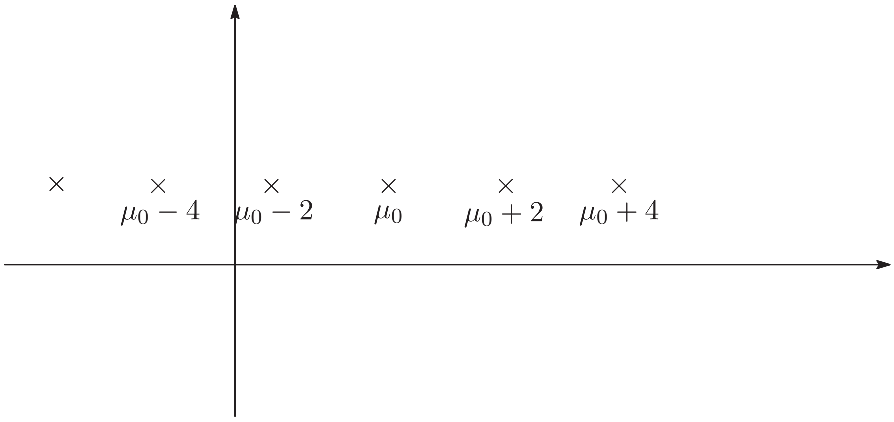
We will change the variable m into μ, $\mu = -2m+\lambda$, and x^m is denoted by v_μ. This notation v_μ indicated that the eigenvalue of v_μ is μ.

Remark 3.2. *An eigenvalue of H is also called a* weight *in representation theory.*

We obtain

$$\begin{aligned} Hv_\mu &= \mu v_\mu, \\ Xv_\mu &= \frac{\mu-\lambda}{2}v_{\mu+2}, \\ Yv_\mu &= \frac{-\mu-\lambda}{2}v_{\mu-2}. \end{aligned}$$

Notation: Fix $\mu_0 \in \mathbb{C}$, then we define $\mu_0 + 2\mathbb{Z} := \{\mu_0 + 2n \mid n \in \mathbb{Z}\}$.



This $\mu_0 + 2\mathbb{Z}$ is a (infinite and countable) subset of $\mathbb{C}$. We often consider a subset $I \subset \mu_0 + 2\mathbb{Z}$.

Definition 3.3. Let $I \subset \mu_0 + 2\mathbb{Z}$. We define a vector space

$$V(\lambda, I) := \bigoplus_{\mu \in I} \mathbb{C} v_\mu .$$

Note that $\{v_\mu\}_{\mu \in I}$ is a basis of $V(\lambda, I)$.

In particular, for $I = \mu_0 + 2\mathbb{Z}$, we often write as

$$V := V(\lambda, \mu_0 + 2\mathbb{Z}) = \bigoplus_{\mu \in \mu_0 + 2\mathbb{Z}} \mathbb{C} v_\mu = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_{\mu_0 + 2n} .$$

Note that λ and μ_0 are hidden in the notation V.

We also consider the linear maps

$$\begin{aligned} H &: V \longrightarrow V, \\ X &: V \longrightarrow V, \\ Y &: V \longrightarrow V \end{aligned}$$

given by

$$\begin{aligned} X\Big(\sum_\mu c_\mu v_\mu\Big) &= \sum_\mu c_\mu X(v_\mu), \\ Y\Big(\sum_\mu c_\mu v_\mu\Big) &= \sum_\mu c_\mu Y(v_\mu), \\ H\Big(\sum_\mu c_\mu v_\mu\Big) &= \sum_\mu c_\mu H(v_\mu). \end{aligned}$$

The vector space V together with H, X, Y is called a *representation* of $\mathfrak{sl}(2, \mathbb{C})$.

What happens for a general I?

When a subspace $W \subset V$ is a subrepresentation?

Definition 3.4. If a subspace $W \subset V$ is stable under the linear maps H, X and Y, then W is called a *subrepresentation* of V. Recall that W is H-*stable* (*stable* under H) if $HW \subset W$, that is, $w \in W \implies Hw \in W$.

Question 3.5 (above)**.** *For which I, the subspace $V(\lambda, I)$ is subrepresentation?*

Question 3.6. *Determine all I's such that*

$$\forall w \in V(\lambda, I) \implies Hw, Hw, Yw \in V(\lambda, I).$$

Lemma 3.7. *For a subspace $W \subset V$, the following conditions are equivalent:*

(1) $HW \subset W$.

(2) There exist $I \subset \mu_0 + 2\mathbb{Z}$ such that $W = V(\lambda, I)$.

Proof. (2)⇒(1) (Obvious part). Let $\forall w = \sum_{\mu \in I} c_\mu v_\mu \in W = V(\lambda, I)$. Then

$$Hw = \sum_{\mu \in I} c_\mu \mu v_\mu \in W = V(\lambda, I).$$

(1)⇒(2) (Nontrivial part). Suppose $W \subset V$ such that $HW \subset W$. Define

$$I := \{\mu \in \mu_0 + 2\mathbb{Z} \mid \exists w \in W \text{ such that } w = \sum_a c_a v_a \text{ with } c_\mu \neq 0\}.$$

Note that, at this moment, we don't know $v_\mu \in W$.

By definition, $W \subset V(\lambda, I)$, because for $\forall w = \sum c_a v_a \in W$, $c_a \neq 0 \Rightarrow a \in I$ and $\sum c_a v_a \in V(\lambda, I)$.

Conversely, suppose $\mu \in I$. Then $\exists w = \sum_{a \in A} c_a v_a \in W$ such that $c_\mu \neq 0$. By definition linear combination is a finite sum. So, A is a finite subset.

$$f(H) := \prod_{\substack{a \in A \\ a \neq \mu}} (H - a) = (H - a_1)(H - a_2) \cdots (H - a_{m-1}),$$

where $A = \{a_1, a_2, \ldots, a_{m-1}, \mu\}$. By assumption $HW \subset W$, for any $w \in W$,

$$f(H)w \in W.$$

On the other hand,

$$f(H)w = \sum_{a \in A} c_a f(H) v_a = \sum_{a \in A} c_a f(a) v_a = c_\mu f(\mu) v_\mu,$$

where we have used the fact that

$$f(a) = 0 \text{ if } a \neq \mu.$$

Since $c_\mu f(\mu) \neq 0$, $v_\mu \in W$. Thus $h \in I \Rightarrow v_\mu \in W$. So $V(\lambda, I) \subset W$.

Thus $W = V(\lambda, I)$ which is the condition (2). $\square$

As a corollary of Lemma 3.7, the previous Question 3.5 is rephrased as

Question 3.8. *Classify all the subrepresentations $W \subset V$.*

3.2 Raising/ lowering operators

Now we examine the condition $XW \subset W$.

Lemma 3.9. *Suppose that $W = V(\lambda, I)$ satisfies $XW \subset W$. Then*

$$\mu \in I \Rightarrow \mu + 2 \in I \text{ or } \mu = \lambda.$$

Lemma 3.10. *Suppose that $W = V(\lambda, I)$ satisfies $YW \subset W$. Then*

$$\mu \in I \Rightarrow \mu - 2 \in I \text{ or } \mu = -\lambda.$$

Proof of Lemma 3.9. Look at

$$Xv_\mu = \frac{\mu - \lambda}{2} v_{\mu+2}.$$

The statement is equivalent to

$$\mu \in I \text{ and } \mu \neq \lambda \Rightarrow \mu + 2 \in I.$$

Suppose $\mu \in I$. Then $v_\mu \in W$. Since $XW \subset W$, $Xv_\mu = \frac{\mu-\lambda}{2} v_{\mu+2} \in W$. By $\mu \neq \lambda$, $v_{\mu+2} \in W$. This means $\mu + 2 \in I$. $\square$

Exercise 3.11. *Prove Lemma 3.10 for Y.*

Both of $\{0\}$ and V are always subrepresentations of V. This fact does not matter the values $\lambda, \mu_0 \in \mathbb{C}$. The subrepresentation $\{0\}$ corresponds to $I = \emptyset$, i.e., $\{0\} = V(\lambda, \emptyset)$.

In order to classify subrepresentations, we want to know other subrepresentations than $\{0\}$ and V.

3.3 For generic parameters

In this subsection, we assume that

$$\pm\lambda \notin \mu_0 + 2\mathbb{Z}.$$

Let $I \subset \mu_0 + 2\mathbb{Z}$ be a nonempty subset. Note that for $\forall\mu \in I$, we have $\lambda \neq \mu$ and $\lambda \neq -\mu$. So, if $\mu \in I$, then $\mu + 2 \in I$ and $\mu - 2 \in I$ by Lemma 3.9. This means that

$$\begin{matrix} \times & \times & \times & \times & \times \\ \mu - 4 \in I & \mu - 2 \in I & \mu \in I & \mu + 2 \in I & \mu + 4 \in I \end{matrix}$$

Theorem 3.12. *Let $\pm\lambda \notin \mu_0 + 2\mathbb{Z}$. Then the list of subrepresentations of $V = V(\lambda, \mu_0 + 2\mathbb{Z})$ is*

(1) $\{0\}$,

(2) V.

Feeling: For generic parameters, representation theory does not depend on the parameters, and the theory is (rather) easy.

3.4 Highest weight submodule

In this subsection, we assume that

$$\lambda \in \mu_0 + 2\mathbb{Z},$$
$$-\lambda \notin \mu_0 + 2\mathbb{Z}.$$

Remark 3.13. $\mu_0 + 2\mathbb{Z} = \lambda + 2\mathbb{Z}$.

Let $I \subset \mu_0 + 2\mathbb{Z}$ be a nonempty subset. Then by Lemmas 3.10 and 3.9, respectively

(a) if $\mu \in I$, then $\mu - 2 \in I$,

(b) if $\mu \in I$ and $\mu \neq \lambda$, then $\mu + 2 \in I$.

Lemma 3.14. *(1) If $\lambda + 2 \in I$, then $I = \lambda + 2\mathbb{Z}$.*

(2) If $\lambda \notin I$, then $I = \emptyset$.

(3) If $\lambda \in I$ and $\lambda + 2 \notin I$, then $I = \lambda + 2\mathbb{Z}_{\leq 0}$.

Proof. Idea of a part of the proof of Lemma 3.14 (1). Suppose $\lambda + 2 \in I$. Then by condition (a), $\lambda, \lambda - 2, \lambda - 4, \ldots \in I$. By condition (b), $\lambda + 4, \lambda + 6, \ldots \in I$. $\square$

$$\begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ & \lambda - 6 \in I & \lambda - 4 \in I & \lambda - 2 \in I & \lambda \in I & \lambda + 2 \in I? \\ & & & & & \text{we don't know.} \end{array}$$

Theorem 3.15. *If $\lambda \in \mu_0 + 2\mathbb{Z}$ and $-\lambda \notin \mu_0 + 2\mathbb{Z}$, then the list of all subrepresentations of $V = V(\lambda, \mu_0 + 2\mathbb{Z}) = V(\lambda, \lambda + 2\mathbb{Z})$ is*

(1) $\{0\}$,

(2) V,

(3) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$.

Illustration of the weights of $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$ is

$$\begin{array}{ccccc} & & & \lambda + 2 & \lambda + 4 \\ \times & \times & \times & \circ & \circ \\ \lambda - 4 & \lambda - 2 & \lambda & & \end{array}$$

$(\times = \text{weights}, \circ = \text{not weights})$

here, λ is called the highest weight of $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$.

Exercise 3.16. *Formulate the list of classification in the case*

$$\begin{array}{c} \lambda \notin \mu_0 + 2\mathbb{Z} \\ -\lambda \in \mu_0 + 2\mathbb{Z}. \end{array}$$

Hint: $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$ is one of the subrepresentations.

Picture of $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$:

$$\begin{array}{cccc} -\lambda - 2 & & & \\ \circ & \times & \times & \times \\ & -\lambda & -\lambda + 2 & -\lambda + 4 \end{array}$$

here, $-\lambda$ is called the lowest weight of $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$.

3.5 Integral weight

In this subsection, we assume that

$$\lambda \in \mu_0 + 2\mathbb{Z},$$
$$-\lambda \in \mu_0 + 2\mathbb{Z}.$$

Remark 3.17.

$$\lambda \in \mathbb{Z},$$
$$\mu_0 + 2\mathbb{Z} \subset \mathbb{Z}.$$

Proof. There exist $n_1, n_2 \in \mathbb{Z}$ such that $\lambda = \mu_0 + 2n_1$ and $-\lambda = \mu_0 + 2n_2$. Thus $\lambda - (-\lambda) = 2(n_1 - n_2)$ and $\lambda = n_1 - n_2 \in \mathbb{Z}$. So, $\mu_0 \in \mathbb{Z}$ and $\mu_0 + 2\mathbb{Z} \subset \mathbb{Z}$. $\square$

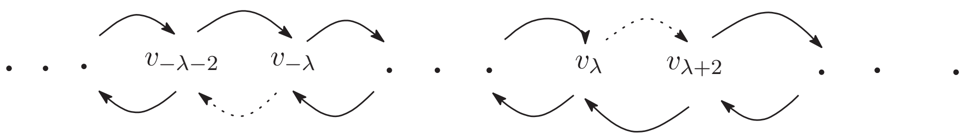
We separate this case into two cases according to $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{Z}_{<0}$.

3.5.1 The case $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda - \mu_0 \in 2\mathbb{Z}$

Recall

$$Xv_\mu \neq 0 \text{ unless } \mu = \lambda,$$
$$Yv_\mu \neq 0 \text{ unless } \mu = -\lambda.$$

Graphical expression



This shows that

- if $\exists \mu \in I$ such that $\mu \in \lambda + 2\mathbb{Z}$ and $\mu > \lambda$, then $-\lambda + 2\mathbb{Z}_{\geq 0} \subset I$,
- if $\exists \mu \in I$ such that $\mu \in \lambda + 2\mathbb{Z}$ and $\mu < -\lambda$, then $\lambda + 2\mathbb{Z}_{\leq 0} \subset I$,
- if $\exists \mu \in I$ such that $\mu \in \lambda + 2\mathbb{Z}$ and $-\lambda \leq \mu \leq \lambda$, then $[-\lambda, \lambda] \subset I$, where
$$[-\lambda, \lambda] := \{\mu \in \lambda + 2\mathbb{Z} \mid -\lambda \leq \mu \leq \lambda\}.$$

Theorem 3.18. *Suppose $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda - \mu_0 \in 2\mathbb{Z}$, then list of all subrepresentations of $V = V(\lambda, \mu_0 + 2\mathbb{Z}) = V(\lambda, \lambda + 2\mathbb{Z})$ is*

(1) $\{0\}$*: zero,*

(2) $V = V(\lambda, \lambda + 2\mathbb{Z})$*: whole,*

(3) $V(\lambda, [-\lambda, \lambda])$*: finite dimensional representation,*

(4) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$*: highest weight representation,*

(5) $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$*: lowest weight representation.*

We note that $\dim V(\lambda, [-\lambda, \lambda]) = \lambda + 1$.

Exercise 3.19. *Give a proof of Theorem 3.18.*

3.5.2 The case $\lambda \in \mathbb{Z}_{<0}$ and $\lambda - \mu_0 \in 2\mathbb{Z}$

Theorem 3.20. *Suppose* $\lambda \in \mathbb{Z}_{<0}$ *and* $\lambda - \mu_0 \in 2\mathbb{Z}$*, then the list of all subrepresentations of* $V = V(\lambda, \mu_0 + 2\mathbb{Z}) = V(\lambda, \lambda + 2\mathbb{Z})$ *is*

(1) $\{0\} = V(\lambda, \emptyset)$,

(2) $V = V(\lambda, \lambda + 2\mathbb{Z})$,

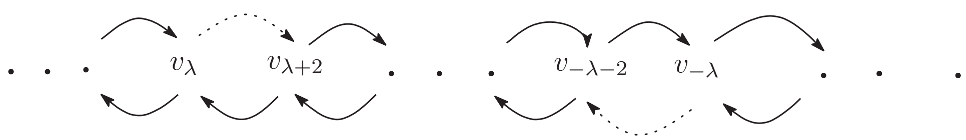
(3) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$,

(4) $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$,

(5) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0}) \bigoplus V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0}) = V(\lambda, (\lambda + 2\mathbb{Z}_{\leq 0}) \bigcup (-\lambda + 2\mathbb{Z}_{\geq 0}))$.

See Remark 3.25.

We give a graphical expression



Exercise 3.21. *Prove it.*

3.6 Irreducible/indecomposable

Definition 3.22.
- A representation (of $\mathfrak{sl}(2, \mathbb{C})$) is called *reducible* if it has proper nonzero subrepresentation.
- A representation is called *decomposable* if it is a direct sum of two proper subrepresentations.
- A representation is called *irreducible* if it is not reducible.

- A representation is called *indecomposable* if it is not decomposable.

Remark 3.23.
- *irreducible* $\Rightarrow$ *indecomposable.*
- *irreducible* $\not\Leftarrow$ *indecomposable.*

Theorem 3.24. $V(\lambda, \mu_0 + 2\mathbb{Z})$ *is irreducible* $\Leftrightarrow \pm\lambda \notin \mu_0 + 2\mathbb{Z}$.

Proof. It is from Theorems 3.12, 3.15, Exercise 3.16, Theorem 3.18 and 3.20. $\square$

Remark 3.25. *The representation of type (5) in Theorem 3.20 is an example of a decomposable representation. In the special case $\lambda = -1$, the representation of type (5) is*

$$V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0} = \{-1, -3, -5, \ldots\}) \bigoplus V(\lambda, -\lambda + 2\mathbb{Z} = \{1, 3, 5, \ldots\}) = V.$$

This means that, if $\lambda = -1$, then (2)=(5) in Theorem 3.20, so we should omit either (2) or (5) in the case $\lambda = -1$, in order to obtain the complete list.

Exercise 3.26. *Any other representation $V(\lambda, \mu_0 + 2\mathbb{Z})$ than $V(-1, 1 + 2\mathbb{Z})$ is indecomposable.*

Theorem 3.27. *The list of all irreducible subrepresentation of $V(\lambda, \mu_0 + 2\mathbb{Z})$ (of $\mathfrak{sl}(2, \mathbb{C})$) is*

(1) $V(\lambda, \mu_0 + 2\mathbb{Z})$*;* $\pm\lambda \notin \mu_0 + 2\mathbb{Z}$,

(2) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$*;* $\lambda \notin \mathbb{Z}_{\geq 0}$*: highest weight representation,*

(3) $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$*;* $\lambda \notin \mathbb{Z}_{\geq 0}$*: lowest weight representation,*

(4) $V(\lambda, [-\lambda, \lambda])$*;* $\lambda \in \mathbb{Z}_{\geq 0}$*: finite dimensional representation.*

Exercise 3.28. *Prove Theorem 3.27.*

Remark 3.29. *For a compact (finite) group, a indecomposable representation (over $\mathbb{C}$) is irreducible. We are in the different context, so that the list in Theorem 3.24 is different from the list in Theorem 3.27.*