Goal: The classification of irreducible unitary representations of $SL(2,\mathbb{R})$.

Aim: Explain the standard technique of the theory of Lie groups.

Textbook: Howe-Tan, Non-Abelian Harmonic Analysis, Springer, 1992.

In the textbook, Chapter II and Chapter III §1, they use higher techniques, such as Schwartz distributions and formal vectors.

In my lecture, instead: Based on Linear algebra and on Calculus. Combination of these two will be applied to Representation theory and Lie groups.

1 Leibniz rule (Lie algebra $\mathfrak{sl}(2,\mathbb{C})$)

Recall the calculus of one variable

$$x : \text{ variable}$$
$$\partial = \tfrac{d}{dx} : \ \partial f = f' = \tfrac{df}{dx}.$$

Definition 1.1. Let $\lambda \in \mathbb{C}$ be a constant. We define operators

$$\begin{aligned} X &:= -\partial, \\ H &:= -2x\partial + \lambda, \\ Y &:= x^2\partial - \lambda x. \end{aligned}$$

Exercise 1.2. *Prove*

(1) $HX - XH = 2X$*,*

(2) $HY - YH = -2Y$*,*

(3) $XY - YX = H$*.*

Proof. (2): It is enough to prove

$$HYf - YHf = -2Yf. \tag{1.1}$$

We have

$$Yf = x^2\partial f - \lambda x f = x^2 f' - \lambda x f$$

and

$$\begin{aligned} HYf &= (-2x\partial + \lambda)Yf = -2x\partial(Yf) + \lambda Yf \\ &= -2x\partial(x^2 f' - \lambda x f) + \lambda(x^2 f' - \lambda x f) \\ &= -2x(x^2 f - \lambda x f)' + \lambda(x^2 f' - \lambda x f) \\ &= -2x\big\{(2xf' + x^2 f'') - \lambda(f + xf')\big\} + \lambda(x^2 f' - \lambda x f) \\ &= -4x^2 f' - 2x^3 f'' + 2\lambda x f + 2x^2\lambda f' + \lambda x^2 f' - \lambda^2 x f. \end{aligned}$$

Therefore,

$$Hf = -2xf' + \lambda f$$

and

$$\begin{aligned} YHf &= x^2\partial(HF) - \lambda x(Hf) \\ &= x^2(-2xf' + \lambda f)' - \lambda x(-2xf' + \lambda f) \\ &= x^2\big\{-2(f' + xf'') + \lambda f'\big\} - \lambda x(-2xf' + \lambda f) \\ &= -2x^2 f' - 2x^3 f'' + \lambda x^2 f'' + \lambda x^2 f' + 2\lambda x^2 f' - \lambda x f. \end{aligned}$$

Thus, the left-hand side of (1.1) is

$$HYf - YHf = -2x^2 f' + 2\lambda x f,$$

and the right-hand side of (1.1) is

$$-2Yf = -2(x^2\partial f - \lambda x f) = -2x^2 f' + 2\lambda x f.$$

$\square$

I will leave (1) and (3) for one of report problems.

These three relations are often called the $\mathfrak{sl}_2$-*relation*.

Exercise 1.3. *Compute the following operator*

$$C = 2XY + 2YX + H^2.$$

Proof. We will compute using the $\mathfrak{sl}_2$-relation.

$$\begin{aligned} C &= 2XY + 2YX + H^2 \qquad \text{by Ex. 1.2 (3)} \\ &= 2(YX + H) + 2YX + H^2 \\ &= 4YX + H^2 + 2H. \end{aligned}$$

Now, we use the definition of X, H, Y. Then, we obtian

$$\begin{aligned} C &= 4(x^2\partial - \lambda x)(-\partial) + (-2x\partial + \lambda)^2 + 2(-2x\partial + \lambda) \\ &= (-4x^2\partial^2 + 4\lambda x\partial) + \left\{4(x\partial)^2 - 4\lambda x\partial + \lambda^2\right\} + (-4x\partial + 2\lambda). \end{aligned}$$

Note 1.4. $(x\partial)^2 \neq x^2\partial^2$. *In fact,* $x\partial f = xf'$ *and* $\partial x f = (xf)' = f + xf'$, *then*

$$\partial x = x\partial + 1. \tag{1.2}$$

By (1.2),

$$(x\partial)^2 = x\partial x\partial = x(x\partial + 1)\partial = xx\partial\partial + x\partial = x^2\partial^2 + x\partial.$$

Now, we come back to the computation of C

$$\begin{aligned} C &= -4x^2\partial^2 + 4(x\partial)^2 + \lambda^2 - 4x\partial + 2\lambda \\ &= -4x^2\partial^2 + 4(x^2\partial^2 + x\partial) + \lambda^2 - 4x\partial + 2\lambda \\ &= \lambda^2 + 2\lambda. \end{aligned}$$

$\square$

Summarizing above: $\left\{\begin{array}{l} X = -\partial \\ H = -2x\partial + \lambda \\ Y = x^2\partial - \lambda x \end{array}\right\}$ then $C = 2XY + 2YX + H^2 = \lambda^2 + 2\lambda$ is a constant.

Note 1.5. $\lambda^2 + 2\lambda = (\lambda+1)^2 - 1$, $1 =$*the half sum of positive roots for* $\mathfrak{sl}(2,\mathbb{C})$.

Fact 1.6 (Not proved here)**.** *The* $\mathfrak{sl}_2$*-relation and* $C = \lambda^2 + 2\lambda$ *are essentially all the relation of* X, H, Y *defined here. In other words,*

$$\mathbb{C}[X,Y,H] \simeq \mathbb{C}\langle x,h,y\rangle / \left(\begin{array}{c} hx - xh - 2x, hy - yh + 2y, xy - yx - h, \\ 2xy + 2yx + h^2 - \lambda^2 - 2\lambda \end{array} \right),$$

where $(\cdots)$ *means the two-sided ideal of the non-commutative free algebra* $\mathbb{C}\langle x,h,y\rangle$.