Algebraic Topology: Problem Set 1

Due: Thursday, June 5.

In Lecture 6, we constructed, for every map $f\colon (X,x) \to (Y,y)$ of pointed k-spaces, the following sequence of maps of pointed k-spaces:

$$(\Omega(Y,y),\bar{y}) \xrightarrow{\partial} (F(f,y),(x,\bar{y})) \xrightarrow{i} (X,x) \xrightarrow{f} (Y,y)$$

We define a sequence of maps of pointed k-spaces

$$(\Omega(C,c),\bar{c}) \xrightarrow{g} (A,a) \xrightarrow{h} (B,b) \xrightarrow{k} (C,c)$$

to be a *fiber sequence* if there exists a map $f\colon (X,x) \to (Y,y)$ of pointed k-spaces and a homotopy commutative diagram of pointed k-spaces

$$\begin{array}{ccccccc}
(\Omega(Y,y),\bar{y}) & \xrightarrow{\partial} & (F(f,y),(x,\bar{y})) & \xrightarrow{i} & (X,x) & \xrightarrow{f} & (Y,y) \\
\downarrow{\scriptstyle \Omega(\psi)} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\
(\Omega(C,c),\bar{c}) & \xrightarrow{g} & (A,a) & \xrightarrow{h} & (B,b) & \xrightarrow{k} & (C,c)
\end{array}$$

such that the vertical maps η, φ, and ψ are homotopy equivalences. Here $\Omega(\psi)$ denotes the map of loop spaces induced by the map ψ. Prove that, for every map $f\colon (X,x) \to (Y,y)$ of pointed k-spaces, the sequence of maps of pointed k-spaces

$$(\Omega(X,x),\bar{x}) \xrightarrow{-\Omega(f)} (\Omega(Y,y),\bar{y}) \xrightarrow{\partial} (F(f,y),(x,\bar{y})) \xrightarrow{i} (X,x),$$

where the map $-\Omega(f)$ is defined by

$$-\Omega(f)(\omega)(t) = f(\omega(-t)),$$

is a fiber sequence.