# SML (Topology and Homotopy)
# Problem Sheet
**Spring Semester, 2026**

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This problem sheet will continually be updated throughout the semester.

You are welcome to discuss and work on problems in groups, though please indicate that you have worked together. Please show all working and quote any theorems that you use in your submission.

Choose some problems to solve and upload your solution as a pdf to TACT with file name: *Familyname_Firstname_Date.pdf* (e.g. Meidai_Hanako_5-16.pdf).

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## Metric spaces

**Q1.**
Suppose that $\|\cdot\|$ is a norm on $\mathbb{R}^n$ Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that

(a) $d(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

(b) $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$,

(c) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

(d) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

**Q2.**
Let $(X, d)$ be a metric space. Show that

$$d' : X \times X \to \mathbb{R}, \qquad d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

defines a bounded metric on $X$.

**Q3.**
Suppose that $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ are metric spaces. Let $X = X_1 \times X_2 \times \ldots \times X_n$ and define

$$d : X \times X \to \mathbb{R}, \qquad d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \max\big\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\big\}.$$

Show that $(X, d)$ is a metric space.

**Q4.**
Let $X$ be a set.

(a) Show that $B(X, \mathbb{R}) = \big\{ f : X \to \mathbb{R} \mid f \text{ bounded } \big\}$ is a metric space with

$$d(f, g) = \sup_{x \in X} \big|f(x) - g(x)\big|.$$

(b) Explain why $d(f, g)$ from part (a) is *not* a metric for $F(X, \mathbb{R}) = \big\{f : X \to \mathbb{R}\big\}$ in general.

(c) ** Show that with the metric $d$ from part (a), the subspace

$$C_b(X, \mathbb{R}) = \big\{f \in B(X, \mathbb{R}) \mid f \text{ continuous}\big\}$$

is a complete metric space.

**Q5.**
Show that the closure of a set $Y \subset X$ is closed, i.e. $\overline{\overline{Y}} = \overline{Y}$.

**Q6.**
Determine if the following subsets of $\mathbb{R}$ are open, closed, both or neither.

(a) The set of even integers $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$,

(b) The rational numbers $\mathbb{Q} = \big\{ \frac{m}{n} \mid m, n \in \mathbb{Z}\big\}$,

(c) $f^{-1}(\{0\}) = \{x \in \mathbb{R} \mid f(x) = 0\}$ with $f : \mathbb{R} \to \mathbb{R}$ continuous,

**Q7.**
Show that any subset $Y \subset X$ is closed and open when $X$ is given the discrete metric.

**Q8.**
Let $(X, d)$ be a metric space.

(a) Show that $\{x\}$ is closed for any $x \in X$.

(b) If $x, y \in X$ and $x \neq y$, show that there are open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(c) * If $C_1$ and $C_2$ are closed subsets of $X$ with $C_1 \cap C_2 = \emptyset$, show that there are open sets $U_1 \supset C_1$ and $U_2 \supset C_2$ such that $U_1 \cap U_2 = \emptyset$.

**Q9.**
Let $\|\cdot\|$ be a norm for $\mathbb{R}^n$ and consider $\mathbb{R}^n$ as a metric space with $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

(a) Show that addition and scalar multiplication in $\mathbb{R}^n$ is continuous (as a function between metric spaces).

(b) Generalise the previous result to any real vector space $V$ with norm $\|\cdot\|$ and metric $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$.

For this question, you can use the product metric $d : X \times Y \to \mathbb{R}$ given by $d\big((x_1, y_1), (x_2, y_2)\big) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ (see Q19 below).

**Q10.**
Let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be metric spaces and take $X = X_1 \times X_2 \times \ldots X_n$ with the metric from Question 3. Show that for any $j = 1, \ldots, n$, the function

$$p_j : X \to X_j, \qquad p_j(x_1, \ldots, x_n) = x_j \in X_j$$

is continuous.

**Q11.**
Let $(X, d)$ be a metric space and fix $x_0 \in X$. Show that the function $f : X \to \mathbb{R}$, $f(x) = d(x, x_0)$ is uniformly continuous.

**Q12.**
We say that a function $f : (X, d_X) \to (Y, d_Y)$ is Lipschitz if there is a constant $K \geq 0$ such that $d_Y\big(f(x_1), f(x_2)\big) \leq K\, d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

(a) Show that any Lipschitz function is continuous.

(b) Give an example of a function that is continuous but not Lipschitz.

(c) Show that any Lipschitz function is uniformly continuous.

**Q13.**
Show that any discrete metric space is complete.

**Q14.**
** If a complete metric space $\tilde{X}$ contains a dense subspace $X$, we say that $\tilde{X}$ is a completion of $X$. In this exercise, we will construct a completion of a metric space $(X, d)$.

(a) Let $(X, d)$ be a metric space and let $\mathcal{C}(X)$ denote the set of Cauchy sequences in $X$. We now define a relation $\sim$ on $\mathcal{C}(X)$, where

$$(x_n) \sim (y_n) \text{ in } \mathcal{C}(X) \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that $\sim$ is an equivalence relation (reflexive, symmetric and transitive).

(b) Let $\tilde{X}$ denote the set of equivalence classes in $\mathcal{C}(S)$ with $[(x_n)] \in \tilde{X}$ the equivalence class of $(x_n)$. Show that

$$\tilde{d}\big([(x_n)], [(y_n)]\big) = \lim_{n\to\infty} d(x_n, y_n)$$

is a metric on $\tilde{X}$.

(c) Show that $(\tilde{X}, \tilde{d})$ is complete.

(d) Let $(x) \in \mathcal{C}(S)$ denote the constant sequence $x_n = x$ for all $n$. Show that the map $\iota : (X, d) \to (\tilde{X}, \tilde{d})$ with $\iota(x) = [(x)]$ is such that $\tilde{d}(\iota(x), \iota(y)) = d(x, y)$ for all $x, y \in X$. In particular, $\iota$ is continuous and injective.

(e) Show that $\tilde{X}$ is a completion of $X$.

**Q15.**

Let $(X, d)$ be a compact metric space.

(a) If $f : X \to Y$ is continuous, show that the range $f(X) = \{f(x) \mid x \in X\}$ is a compact subset of $Y$.

(b) If $f : X \to \mathbb{R}$ is continuous, show that $\sup_{x \in X} |f(x)| = \max_{x \in X} |f(x)|$.

**Q16.**

** (Weierstrass approximation theorem) Let $C([0,1], \mathbb{R}) = \big\{f : [0,1] \to \mathbb{R} \text{ continuous}\big\}$ and $\mathcal{P}([0,1], \mathbb{R}) = \big\{p : [0,1] \to \mathbb{R} \text{ a polynomial}\big\}$. Show that $\mathcal{P}([0,1], \mathbb{R})$ is dense in $C([0,1], \mathbb{R})$ with respect to the metric $d(f, g) = \sup_{x \in [0,1]} |f(x) - f(y)|$.

(You may wish to look up Bernstein polynomials to help with this question.)

**Q17.**

We say that $f : (X, d_X) \to (Y, d_Y)$ is an isometric map if for any $x_1, x_2 \in X$,

$$d_Y\big(f(x_1),\, f(x_2)\big) = d_X(x_1,\, x_2).$$

Show that any isometric map is injective and continuous.

**Q18.**

Find an example of a compact metric space $(X, d)$ and a map $f : X \to X$ such that $d\big(f(x_1), f(x_2)\big) \leq d(x_1, x_2)$ but with no fixed points.

**Q19.**

Let $(X, d)$ and $(Y, d')$ be metric spaces. Show that the following define metrics on $X \times Y$:

(a) $d_1\big((x_1, y_1), (x_2, y_2)\big) = d(x_1, x_2) + d'(y_1, y_2)$,

(b) $d_2\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{d(x_1, x_2)^2 + d'(y_1, y_2)^2}$.

## Topological spaces

**Q1.**

Find all topologies of the two-point set $X = \{a, b\}$.

**Q2.**

Give an example of an infinite family of open sets $\{U_\alpha\}_{\alpha \in I}$ in a topological space $X$ such that $\bigcap_{\alpha \in I} U_\alpha$ is *not* open.

**Q3.**

Let $X$ be a set.

(a) Show that
$$\mathcal{T}_f = \{U \subset X \mid X \setminus U \text{ either finite or all of } X\}.$$
defines a topology for $X$.

(b) Show that
$$\mathcal{T}_c = \{U \subset X \mid X \setminus U \text{ either countable or all of } X\}.$$
is a topology for $X$.

(c) Is the set
$$\mathcal{T}_\infty = \{U \subset X \mid X \setminus U \text{ either infinite or empty or all of } X\}.$$
a topology for $X$?

**Q4.**

Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{T}_Y)$ a subspace (with subspace topology). Show the following:

(a) If $A \subset Y$ is open in $Y$ and $Y$ is open in $X$, then $A$ is open in $X$.

(b) If $f : X \to Z$ is a continuous function, then $f|_Y : Y \to Z$ is continuous.

**Q5.**

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$.

(a) Show that $\mathrm{Int}(Y) \subset Y \subset \overline{Y} = \overline{\overline{Y}}$.

(b) Define the boundary of $Y$ as the set $\partial Y = \overline{Y} \cap \overline{X \setminus Y}$. Show that
$$\mathrm{Int}(Y) \cap \partial Y = \emptyset, \qquad \mathrm{Int}(Y) \cup \partial Y = \overline{Y}.$$

**Q6.**

Show that the following spaces are homeomorphic.

(a) The open interval $(-1, 1)$ and $\mathbb{R}$,

(b) The circle $S^1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$ and the square $S_\Box = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$.

(c) * The upper-half plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and the open unit ball $B(\mathbf{0}, 1) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

**Q7.**

Find a continuous bijection between topological spaces that is *not* a homeomorphism.

**Q8.**

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$ a collection of open sets such that for each $U \in \mathcal{T}$ and $x \in U$, there is some $B \in \mathcal{B}$ with $x \in B \subset U$. Show that $\mathcal{B}$ is a basis for $\mathcal{T}$.

**Q9.**

Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ be a family of topological spaces.

(a) If $X_\alpha$ is Hausdorff for each $\alpha \in I$, show that $\prod_{\alpha \in I} X_\alpha$ is Hausdorff.

(b) Show that for all $\alpha' \in I$, the projection functions
$$p_{\alpha'} : \prod_{\alpha \in I} X_\alpha \to X_{\alpha'}, \qquad p_{\alpha'}\big((x_\alpha)_{\alpha \in I}\big) = x_{\alpha'}$$
are continuous.

**Q10.**
Let $(X, \mathcal{T})$ be Hausdorff and normal.

(a) Show that any subspace $(Y, \mathcal{T}_Y)$ is Hausdorff and normal.

(b) If $f : X \to Z$ is a homeomorphism, show that $Z$ is Hausdorff and normal.

**Q11.**
Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$ a collection of open sets such that for each $U \in \mathcal{T}$ and $x \in U$, there is some $B \in \mathcal{B}$ with $x \in B \subset U$. Show that $\mathcal{B}$ is a basis for $\mathcal{T}$.

**Q12.**
Let $X = \mathbb{R}$ and take the topology generated by the basis of half-open intervals $\mathcal{B} = \{[a, b) \mid a < b\}$.

(a) Show that the standard topology on $\mathbb{R}$ is coarser than the half-open topology, i.e. $\mathcal{T}_{(a,b)} \subset \mathcal{T}_{[a,b)}$.

(b) Show that $[c, d) \subset \mathbb{R}$ is closed in $\mathcal{T}_{[a,b)}$

(c) Show that a function $f : (\mathbb{R}, \mathcal{T}_{[a,b)}) \to (\mathbb{R}, \mathcal{T}_{(a,b)})$ is continuous if and only if it is right-continuous. That is,

$$\lim_{h \to 0^+} f(x + h) = f(x).$$

**Q13.**
Let $X$ be compact and Hausdorff with $x, y \in X$ such that $x \neq y$. Find a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

**Q14.**
If $(X, \mathcal{T})$ is compact and $Y \subset X$ is closed, show that $(Y, \mathcal{T}_Y)$ is compact in the subspace topology.

**Q15.**
Show that the space $\{0\} \cup \left\{\frac{1}{n} \mid n = 1, 2, \ldots\right\}$ is homeomorphic to $\mathbb{N}^+$, the one-point compactification of $\mathbb{N}$.

**Q16.**

(a) Show that compactness is preserved under homeomorphisms.

(b) Show that local compactness is preserved under homeomorphisms.

(c) Show that connectedness is preserved under homeomorphisms.

(d) Show that path connectedness is preserved under homeomorphisms.

**Q17.**
Show that $(X, \mathcal{T})$ is connected if and only if the only closed and open subsets of $X$ are $\emptyset$ and $X$.

**Q18.**

(a) If $X$ and $Y$ are connected, is $X \cap Y$ connected?

(b) If $X$ is connected and $Y \subset X$, is $Y$ a connected subspace?

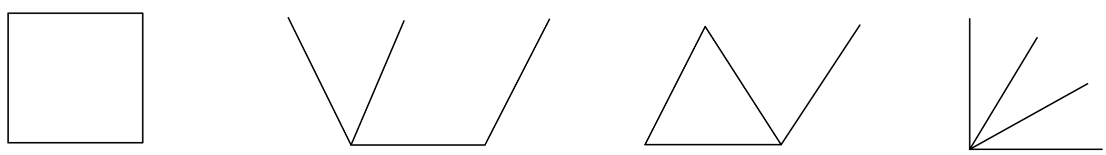
**Q19.**
If $A$ is a connected subspace of $X$, show that $\overline{A}$ is connected.

**Q20.**
We say that a point $x$ in a connected space $X$ is a cut point if $X \setminus \{x\}$ is disconnected.

(a) If $f : X \xrightarrow{\simeq} Y$ is a homeomorphism, show that $x \in X$ is a cut point if and only if $f(x) \in Y$ is a cut point.

(b) Use part (a) to show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

(c) Use part (a) to show that none of the following subspaces of $\mathbb{R}^2$ are homeomorphic



**Q21.**

(a) Let $\{X_1, \ldots, X_n\}$ be connected spaces. Show that $\prod_{j=1}^{n} X_j$ is connected.

(b) * Let $\{X_\alpha\}_{\alpha \in I}$ be a family of connected spaces. Show that $\prod_{\alpha \in I} X_\alpha$ is connected.

(c) Let $\{X_\alpha\}_{\alpha \in I}$ be a family of path connected spaces. Show that $\prod_{\alpha \in I} X_\alpha$ is path connected.

(d) Suppose that $X$ is Hausdorff, locally compact, not compact and connected. Show that the one-point compactification $X^+$ is connected.

**Q22.**

If $(X, \mathcal{T})$ is Hausdorff

(a) Show that the diagonal $\{(x, x) \mid x \in X\} \subset X \times X$ is a closed subset.

(b) Let $(X, \mathcal{T})$ be compact and Hausdorff. Suppose that $\mathcal{T}'$ and $\mathcal{T}''$ are topologies on $X$ such that $\mathcal{T}' \subset \mathcal{T} \subset \mathcal{T}''$ (strict inclusions). Show that $(X, \mathcal{T}')$ is compact but not Hausdorff and $(X, \mathcal{T}'')$ is Hausdorff but not compact.

**Q23.**

Let $\mathcal{O}(n) \subset \mathbb{R}^{n^2}$ denote the set of all $n \times n$ orthogonal matrices (that is, $n \times n$ matrices whose columns form an orthonormal basis of $\mathbb{R}^n$).

(a) Show that $\mathcal{O}(n) \subset \mathbb{R}^{n^2}$ is a closed subspace.

(b) Show that $\mathcal{O}(n) \subset \mathbb{R}^{n^2}$ is a compact subspace.

**Q24.**

Let $(X, \mathcal{T})$ be a topological space and suppose $\sim$ is an equivalence relation on $X$.

(a) If $X$ is compact, show that $X/\sim$ is compact.

(b) If $X$ is connected / path connected, show that $X/\sim$ is connected / path connected.

**Q25.**

Let $I^2 = [0, 1] \times [0, 1] \in \mathbb{R}^2$ denote the product of the unit interval. Describe the following spaces via a quotient of $I^2$ by an equivalence relation.

(a) The cylinder $S^1 \times [0, 1]$,

(b) The torus $\mathbb{T}^2$,

(c) The Möbius strip

(d) The 2-sphere $S^2$.

**Q26.**

(a) Let $\overline{B(\mathbf{0}, 1)} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\} \subset \mathbb{R}^n$ be the closed unit ball. Show that there is a homeomorphism

$$\overline{B(\mathbf{0}, 1)}/\{\partial B(\mathbf{0}; 1) \sim \{\mathrm{pt}\}\} \cong S^n.$$

(b) Let $\Sigma X$ denote the suspension of a topological space $X$. Show that $\Sigma S^n$ is homeomorphic to $S^{n+1}$.

**Q27.**

Give an example of a Hausdorff topological $X$ and an equivalence relation $\sim$ on $X$ such that $X/\sim$ is not Hausdorff.

## Homotopy and the fundamental group + Elements of algebraic topology

**Q1.**

Let $G$ and $H$ be groups and $\phi : G \to H$ a group homomorphism.

(a) Show that $\mathrm{Ran}(\phi)$ (the range/image of $\phi$) is a subgroup of $H$.

(b) Show that $\phi^{-1}(e)$ is a subgroup of $G$ (where $e$ is the identity of $H$).

(c) Show that the Cartesian product $G \times H$ can be made into a group.

**Q2.**

Find a (non-trivial) group homomorphism:

(a) $\phi : \mathbb{Z}_n \to \mathbb{C}^\times$,

(b) $\phi : S_3 \to GL_2(\mathbb{R})$,

(c) $\phi : S_n \to \mathcal{O}(n)$.

**Q3.**

Suppose that $(Y, \mathcal{T}_{\mathrm{disc}})$ has the discrete topology. If $(X, \mathcal{T})$ is a topological space, show that $f, g \in C(X, Y)$ are homotopic if and only if $f = g$.

**Q4.**

Show that homotopy equivalence of topological spaces is an equivalence relation.

**Q5.**

(a) Show that $\mathbb{R}^n \setminus \{\mathbf{0}\} \sim_h S^{n-1}$.

(b) Show that $GL_n(\mathbb{R}) \sim_h \mathcal{O}(n)$.

(c) Show that the Hausdorff property of topological spaces is *not* preserved under homotopy equivalence.

**Q6.**

(a) Show that $\pi_1\big(X \times Y, (x_0, y_0)\big) = \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.

(b) Compute $\pi_1(\mathbb{T}^n)$.

(c) Show that $\pi_n\big(X \times Y, (x_0, y_0)\big) = \pi_n(X, x_0) \oplus \pi_n(Y, y_0)$ for any $n \geq 1$.

**Q7.**

(a) Show that any contractible space is simply connected.

(b) Show that $S^n$ is simply connected for $n \geq 2$.

(c) Show that $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is simply connected for $n \geq 3$.

**Q8.**

Compute the fundamental group of $\mathbb{R}P^n$ for all $n \geq 1$.

**Q9.**

A retraction of $X$ onto a subspace $Y$ is a continuous map $f \in C(X, Y)$ such that $f(y) = y$ for all $y \in Y$. We then say that $Y$ is a retract of $X$.

(a) Show that $S^n$ is a retract of $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$.

(b) Let $f : C(X, Y)$ be a retraction and $j : Y \hookrightarrow X$ the inclusion map. Show that:

 (i) $j_* : \pi_1(Y, y_0) \to \pi_1(X, y_0)$ is injective,

 (ii) $f_* : \pi_1(X, y_0) \to \pi_1(Y, y_0)$ is surjective,

 (iii) If $X$ is simply connected, then $Y$ is simply connected.

**Q10.**

Let $(X, d_X)$ be a compact metric space with $a, b \in X$. Define the metric space of paths from $a$ to $b$,

$$\mathrm{Path}(X, a, b) = \big\{\gamma \in C([0,1], X) \mid \gamma(0) = a,\ \gamma(1) = b\big\}, \qquad d(\alpha, \beta) = \sup_{t \in [0,1]} d_X\big(\alpha(t), \beta(t)\big).$$

If $\alpha, \beta \in \mathrm{Path}(X, a, b)$, show that $\alpha \sim_h \beta$ with fixed endpoints if and only if $\alpha$ and $\beta$ are in the same path component of $\mathrm{Path}(X, a, b)$.

**Q11.**

(a) Find a covering map $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$.

(b) Show that the quotient map $q : S^2 \to \mathbb{R}P^2$ is a covering map.

**Q12.**

Show that for any topological space $(X, \mathcal{T})$, the cone $CX = (X \times [0,1])/(X \times \{0\})$ is contractible.

**Q13.**

Show that if $f \in C(X, Y)$ is a homotopy equivalence, then the induced map $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is a group isomorphism.

**Q14.**

Using that $S^n$ is not contractible, prove the higher-dimensional Brouwer fixed point theorem: if $\overline{B(\mathbf{0}, 1)} \in \mathbb{R}^{n+1}$ is the closed unit ball, then any continuous map $\phi : \overline{B(\mathbf{0}, 1)} \to \overline{B(\mathbf{0}, 1)}$ has a fixed point.

**Q15.**

Show that the degree map $\deg : \pi_n(S^n) \to \mathbb{Z}$ is a group homomorphism (to prove this, you can take representatives $[f], [g] \in \pi_n(S^n)$ such that $f$ and $g$ are smooth).

**Q16.**

Let $X$ and $Y$ be simplicial 1-complexes

(a) Show that $\chi(X \times Y) = \chi(X)\chi(Y)$ (with $\chi$ the Euler characteristic).

(b) Compute $\chi(\mathbb{T}^2)$.

(c) ** Generalise the formula $\chi(X \times Y) = \chi(X)\chi(Y)$ to arbitrary simplicial complexes.

**Q17.**

* Recall the real projective plane $\mathbb{R}P^2 = S^2/\{\mathbf{x} \sim -\mathbf{x}\}$ with quotient map $q : S^2 \to \mathbb{R}P^2$.

(a) Let $N = (0,0,1)$ denote the north pole in $S^2$. Show that there is a homeomorphism between $\mathbb{R}P^2 \setminus \{[N]\}$ and the open Möbius strip $M = S^1 \times (0,1)/\{(x,y) \sim (x, 1-y)\}$.

(b) Show that $\mathbb{R}P^2$ is not orientable.