SML (Mathematics of Quantum Information Theory)
Problem Sheet

Spring Semester, 2025

This problem sheet will continually be updated throughout the semester.

You are welcome to discuss and work on problems in groups, though please indicate that you have worked together. Please show all working and quote any theorems that you use in your submission.

Choose some problems to solve and upload your solution as a pdf to TACT with file name: *Familyname_Firstname_Date.pdf* (e.g. Meidai_Hanako_6-16.pdf).

Linear Algebra review

1. (a) Let $\langle \cdot \mid \cdot \rangle$ be a complex-valued inner-product of a complex vector space $\mathcal{H}$. Prove the Cauchy–Schwarz inequality

$$\big|\langle \psi \mid \phi \rangle\big| \leq \|\psi\|\,\|\phi\|, \qquad \psi, \phi \in \mathcal{H}.$$

(b) Use part (a) to show the triangle inequality for the norm $\|\psi\| = \sqrt{\langle \psi \mid \psi \rangle}$,

$$\|\psi + \phi\| \leq \|\psi\| + \|\phi\|.$$

2. Show that if $\psi_n \to \psi$ in $\mathcal{H}$, then for any $\phi \in \mathcal{H}$, $\langle \phi \mid \psi_n \rangle \to \langle \phi \mid \psi \rangle$ in $\mathbb{C}$.

3. ** Let $\ell^2(\mathbb{N})$ be the space

$$\ell^2(\mathbb{N}) = \Big\{ (x_n)_{n\geq 0} \,:\, x_n \in \mathbb{C} \text{ for all } n \text{ and } \sum_{n\geq 0} |x_n|^2 < \infty \Big\}.$$

Show that $\ell^2(\mathbb{N})$ is infinite-dimensional.

4. (a) Let $\{e_j\}_{j\in J}$ be an orthonormal basis of a Hilbert space. Show that the decomposition of any vector as a linear combination of basis vectors is unique.

(b) Let $\mid \psi \rangle = \sum_j \mid e_j \rangle\langle e_j \mid \psi \rangle = \sum_j \psi_j \mid e_j \rangle$ and $\mid \phi \rangle = \sum_j \mid e_j \rangle\langle e_j \mid \phi \rangle = \sum_j \phi_j \mid e_j \rangle$ with $\psi_j = \langle e_j \mid \psi \rangle$ and $\phi_j = \langle e_j \mid \phi \rangle \in \mathbb{C}$. Show that

$$\langle \psi \mid \phi \rangle = \sum_j \overline{\psi_j} \phi_j.$$

5. Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $\psi, \phi \in \mathcal{H}$. Show that

(a) $\operatorname{Rank}\big(|\psi\rangle\langle\phi|\big) = 1$, (b) $\big(|\psi\rangle\langle\phi|\big)^* = |\phi\rangle\langle\psi|$.

6. Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

(a) Show that for a linear operator $U : \mathcal{H} \to \mathcal{H}$

$$U^*U = UU^* = \mathbf{1}_{\mathcal{H}} \quad \Longleftrightarrow \quad \langle U\psi \mid U\phi \rangle = \langle \psi \mid \phi \rangle \text{ for all } \psi, \phi \in \mathcal{H}.$$

(b) If $\{|e_j\rangle\}_{j=1}^n$ and $\{|\tilde{e}_j\rangle\}_{j=1}^n$ are orthonormal bases of H, show that there is a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $U|e_j\rangle = |\tilde{e}_j\rangle$ for all j. Conversely, if $\{|e_j\rangle\}_{j=1}^n$ is an orthonormal basis and U is unitary, show that $\{|Ue_j\rangle\}_{j=1}^n$ is an orthonormal basis.

(c) If $H = H^*$ with spectral decomposition $H = \sum_j \lambda_j P_{\lambda_j}$, show that the operator $V = e^{iH} = \sum_j e^{i\lambda_j} P_{\lambda_j}$ is unitary.

7. Let $\{|e_j\rangle\}_{j=1}^n$ be an orthonormal basis of a Hilbert space $\mathcal{H} \cong \mathbb{C}^n$. Define the linear operator $S : \mathcal{H} \to \mathcal{H}$ such that $S|e_j\rangle = |e_{j+1}\rangle$ for $j \leq n-1$, $S|e_n\rangle = |e_1\rangle$ and extended linearly.

(a) Find the matrix representation of S with respect to the orthonormal basis $\{|e_j\rangle\}_{j=1}^n$.

(b) Show that $S^n = \mathbf{1}_n$.

(c) Find all eigenvalues and eigenvectors of S in the case that $n = 3$ and $\mathcal{H} \cong \mathbb{C}^3$.

8. Suppose that for all $j = 1, \dots, n$ there is a continuous map $[a, b] \ni t \mapsto |\psi_j(t)\rangle \in \mathcal{H}$ such that $\{|\psi_j(t)\rangle\}_{j=1}^n$ is an orthonormal system for all $t \in [a, b]$. Let $P(t) = \sum_{j=1}^n |\psi_j(t)\rangle\langle\psi_j(t)|$ be the corresponding subspace projection. Show that for any $|\phi_1\rangle$, $|\phi_2\rangle \in \mathcal{H}$, the maps

$$[a, b] \ni t \mapsto \langle\phi_1 \mid P(t)\phi_2\rangle \in \mathbb{C}, \qquad [a, b] \ni t \mapsto P(t)|\phi_1\rangle \in \mathcal{H}$$

are continuous.

9. Consider the 2×2 matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Verify that for any $j, k \in \{1, 2, 3\}$,

$$\sigma_j^* = \sigma_j, \qquad \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{j,k}\,\mathbf{1}_2,$$

where $\delta_{j,k} = 0$ if $i \neq j$ and $\delta_{j,k} = 1$ if $j = k$.

(b) Find all eigenvalues and normalised eigenvectors of σ_j, for $j = 1, 2, 3$.

(c) Write σ_1, σ_2 and σ_3 in diagonal form,

$$\sigma_j = \lambda_1^{(j)}|e_1^{(j)}\rangle\langle e_1^{(j)}| + \lambda_2^{(j)}|e_2^{(j)}\rangle\langle e_2^{(j)}| \qquad \lambda_2^{(j)},\, \lambda_2^{(j)} \in \mathbb{R}, \qquad j \in \{1, 2, 3\},$$

and $\{|e_1^{(j)}\rangle, |e_2^{(j)}\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$.

10. Let A and B be diagonalisable operators on a finite-dimensional Hilbert space $\mathcal{H}$.

(a) Suppose that eigenvalues of A and B are non-degenerate. Show that if $[A, B] = 0$, then A and B are simultaneously diagonalisable, i.e. there is an orthonormal basis $\{|e_j\rangle\}_{j=1}^n$ that diagonalises both A and B.

(b) ** Prove the same result as part (a) without the assumption of non-degenerate eigenvalues.

11. Let A and B be linear operators on $\mathcal{H}$ such that $A^2 = \mathbf{1}_\mathcal{H}$ and $B^2 = -\mathbf{1}_\mathcal{H}$. Show that for any $t \in \mathbb{R}$

(a) $\exp(itA) = \cos(t)\mathbf{1} + i\sin(t)A$ (b) $\exp(tB) = \cos(t)\mathbf{1} + \sin(t)B$.

12. *** Let $\mathcal{H}$ be an infinite dimensional (separable) Hilbert space with orthonormal basis $\{e_j\}_{j\geq 1}$. Define the sequence of projections $\{P_N\}_{N\geq 0}$ as

$$P_N = \sum_{j=1}^N |e_j\rangle\langle e_j|$$

(a) Show that for any $|\psi\rangle \in \mathcal{H}$, $\big\|P_N|\psi\rangle - |\psi\rangle\big\| \to 0$ as $N \to \infty$.

(b) Show that it does **not** hold that

$$\|P_N - \mathbf{1}_\mathcal{H}\| \xrightarrow{N\to\infty} 0, \qquad \text{where } \|A\| = \sup_{\|\psi\|=1} \big\|A|\psi\rangle\big\|.$$

If you like, you may consider the case $\mathcal{H} = \ell^2(\mathbb{N})$, which has the canonical orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ such that $e_j(n) = \delta_{j,n}$.

13. Let $\mathcal{H}$ be a finite-dimensional space and $\mathcal{L}(\mathcal{H}) = \{A \mid A : \mathcal{H} \to \mathcal{H} \text{ linear}\}$ the space of linear operators.

(a) Show that the sesquilinear form,

$$\langle\cdot,\cdot\rangle : \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \to \mathbb{C}, \qquad \langle A, B\rangle = \mathrm{Tr}(A^*B)$$

defines an inner-product (called the Hilbert–Schmidt inner product) on $\mathcal{L}(\mathcal{H})$.

(b) If $\dim(\mathcal{H}) = n$ show that $\dim\big(\mathcal{L}(\mathcal{H})\big) = n^2$.

(c) Find an orthonormal basis for $\mathcal{L}(\mathcal{H})$ with respect to the Hilbert–Schmidt inner product.

14. Let $\mathcal{H}$ be a finite-dimensional space and $\mathrm{Tr} : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ the trace. Show that:

(a) $\mathrm{Tr}(UAU^*) = \mathrm{Tr}(A)$ for all $A \in \mathcal{L}(\mathcal{H})$ and U unitary,

(b) The trace is independent of the choice of orthonormal basis,

(c) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in \mathcal{L}(\mathcal{H})$.

Postulates of quantum mechanics

1. Let $a|\psi\rangle + b|\phi\rangle \in \mathcal{H}$ be a superposition of orthogonal states, $a, b \in \mathbb{C}$ and $\|a\psi + b\phi\| = 1$. Show that for an observable $A = A^*$, it may occur that

$$\langle A\rangle_{a\psi+b\phi} \neq \langle A\rangle_{a\psi+e^{i\theta}b\phi}, \quad \theta \in \mathbb{R}.$$

2. ** Suppose that $H(t)$ is a self-adjoint operator and $U(t)$ is a solution to the initial value problem

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H(t)U(t), \qquad U(0) = \mathbf{1}.$$

Show that $U(t)$ is unitary and is a unique solution.

3. Solve the Schrödinger equation and find the time evolution for the case where $H(t) = H$ is a Pauli matrix, σ_1, σ_2 and σ_3.

4. Let $|+\rangle = \frac{1}{\sqrt{2}}(1,1)$, $|-\rangle = \frac{1}{\sqrt{2}}(1,-1) \in \mathbb{C}^2$. Calculate the probability to measure $+1$ for the observables σ_y and σ_z in the following states

(a) $|+\rangle$, (b) $|-\rangle$, (c) $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ (d) $\rho = \frac{1}{2}(|+\rangle\langle+|) + \frac{1}{2}(|-\rangle\langle-|)$

5. Let $\mathcal{H} = \mathbb{C}^3$ and consider the observable

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Show that the possible outcomes of a measurement of A are $+1$ and -1.

(b) Let $|\psi\rangle = \frac{1}{\sqrt{3}}(1,1,1)$. Compute

(i) $\mathbf{P}_\psi(+1)$, (ii) $\mathbf{P}_\psi(-1)$, (iii) $\langle A\rangle_\psi$, (iv) $\Delta_\psi(A)$.

6. (Nuclear magnetic resonance)** Let $\omega_0, \omega, g \in \mathbb{R}$ and define the time-dependent Hamiltonian

$$H(t) = \frac{\omega_0}{2}\sigma_3 + g\big(\cos(\omega t)\sigma_1 + \sin(\omega t)\sigma_2\big) \in M_2(\mathbb{C}).$$

(a) Show that the state $|\psi(t)\rangle \in \mathbb{C}^2$ satisfies the Schrödinger equation $i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$ if and only if

$$i\frac{d}{dt}|\phi(t)\rangle = \Big(\frac{\omega_0 - \omega}{2}\sigma_3 + g\sigma_1\Big)|\phi(t)\rangle, \qquad \text{where } |\phi(t)\rangle = \exp\big(\tfrac{1}{2}i\omega t\sigma_3\big)|\psi(t)\rangle. \tag{1}$$

(b) Find the general solution $|\phi(t)\rangle$ to Equation (1) and therefore a solution $|\psi(t)\rangle$ to the Schrödinger equation.

(c) Qualitatively describe the solution $|\phi(t)\rangle$ in the case that

(i) $|g|$ small, (ii) $|\omega - \omega_0| \gg |g|$, (iii) $\omega = \omega_0$

7. Let $\rho \in \mathrm{Dens}(\mathcal{H})$ be a density operator.

(a) Show that $\langle \psi \mid (\rho - \rho^2)\psi\rangle \geq 0$.

(b) Show that ρ is a pure state $\rho = |\psi\rangle\langle\psi|$ if and only if $\rho^2 = \rho$.

(c) Show that ρ describes a non-pure state if and only if $\mathrm{Tr}(\rho^2) < 1$.

8. Let $\mathcal{H} = \mathbb{C}^2$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and consider the operator

$$\rho_{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} 1 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 \end{pmatrix}.$$

(a) Show that $\rho_{\mathbf{x}} \in \mathrm{Dens}(\mathbb{C}^2)$ if and only if $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq 1$.

(b) Show that $\rho_{\mathbf{x}}$ is a pure state if and only if $\|\mathbf{x}\| = 1$.

9. For any $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, define the matrix $\mathbf{a} \cdot \sigma = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$.

(a) Show that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$(\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma) = (\mathbf{a}\cdot\mathbf{b})\mathbf{1}_2 + i(\mathbf{a}\times\mathbf{b})\cdot\sigma,$$

where $\mathbf{a}\cdot\mathbf{b} = \sum_j a_j b_j$ is the dot product and

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \in \mathbb{R}^3$$

is the cross product of vectors in $\mathbb{R}^3$.

(b) Let $\hat{n}(\theta,\phi) = \big(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\big) \in \mathbb{R}^3$ and $D_{\hat{n}}(\alpha) = \exp\big(-i\frac{\alpha}{2}\hat{n}(\theta,\phi)\cdot\sigma\big)$. Show that for all $\alpha, \beta \in \mathbb{R}$

$$D_{\hat{n}}(\alpha)D_{\hat{n}}(\beta) = D_{\hat{n}}(\alpha+\beta)$$

10. Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\mathcal{W} \subset \mathcal{H}$ a subspace. Suppose that there is a linear operator $U : \mathcal{W} \to \mathcal{H}$ such that $\langle w_1 \mid w_2\rangle = \langle Uw_1 \mid Uw_2\rangle$ for all $|w_1\rangle, |w_2\rangle \in \mathcal{W}$. Show that there exists a unitary operator $U' : \mathcal{H} \to \mathcal{H}$ such that $U'|w\rangle = U|w\rangle$ for all $|w\rangle \in \mathcal{W}$ (that is, U' extends U to $\mathcal{H}$).

11. Let $A = A^*$ be a generic self-adjoint operator on $\mathcal{H} = \mathbb{C}^2$ with spectral decomposition $A = \lambda_1|e_1\rangle\langle e_1| + \lambda_2|e_2\rangle\langle e_2|$.

(a) For $|\psi\rangle = a|0\rangle + b|1\rangle$ a generic pure state, compute

(i) $\mathbf{P}_\psi(\lambda_1)$, (ii) $\mathbf{P}_\psi(\lambda_2)$, (iii) $\langle A\rangle_\psi$, (iv) $\Delta_\psi(A)$.

(b) Let $\rho_{\mathbf{x}} = \frac{1}{2}\big(\mathbf{1}_2 + \mathbf{x}\cdot\sigma\big) = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 \end{pmatrix}$ with $\mathbf{x} \in \mathbb{R}^3$ a point in the Bloch ball, $\|\mathbf{x}\| \leq 1$. Calculate

(i) $\mathbf{P}_{\rho_{\mathbf{x}}}(\lambda_1)$, (ii) $\mathbf{P}_{\rho_{\mathbf{x}}}(\lambda_2)$, (iii) $\langle A\rangle_{\rho_{\mathbf{x}}}$, (iv) $\Delta_{\rho_{\mathbf{x}}}(A)$.

12. Show that, up to the equivalence $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$, any Qubit $|\psi\rangle \in \mathbb{C}^2$ can be written

$$|\psi\rangle \sim |\psi(\theta,\phi)\rangle = e^{-i\frac{\phi}{2}}\cos\big(\tfrac{\theta}{2}\big)|0\rangle + e^{i\frac{\phi}{2}}\sin\big(\tfrac{\theta}{2}\big)|1\rangle, \quad \theta,\phi \in \mathbb{R}.$$

13. Consider the self-adjoint and unitary operators A and B on $\mathbb{C}^2$, where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ -1 & -\sqrt{3} \end{pmatrix}.$$

(a) Write A and B in the form $\hat{n}(\theta,\phi)\cdot\sigma$ with $\hat{n}(\theta,\phi) = \big(\sin(\theta)\cos(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\theta)\big) \in S_{\mathbb{R}^3}$ a point on the Bloch sphere.

(b) Compute the ± 1 eigenvectors of A and B.

Tensor products and entanglement

1. (a) Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \in \mathbb{C}^2$. Write out the tensor products $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly in terms of the orthonormal basis built from $|0\rangle$ and $|1\rangle$.

   (b) Show that $\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\big) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can not be written as a single product $|\psi\rangle \otimes |\phi\rangle$ with $|\psi\rangle \in \mathbb{C}^2$, $|\phi\rangle \in \mathbb{C}^2$.

   (c) More generally, show that a 2-Qubit $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ is entangled if and only if

$$\det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \neq 0.$$

2. Find square matrices A and B of the same size such that under the Kronecker product $A \otimes B \neq B \otimes A$.

3. (a) Let $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$, $|\eta_1\rangle, |\eta_2\rangle \in \mathcal{K}$. Show that

$$|\psi_1 \otimes \eta_1\rangle\langle\psi_2 \otimes \eta_2| = \big(|\psi_1\rangle\langle\psi_2|\big) \otimes \big(|\eta_1\rangle\langle\eta_2|\big).$$

   (b) If $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$, show that $(A \otimes B)^* = A^* \otimes B^*$. Show that the tensor product of projections is a projection and the tensor product of unitaries is a unitary.

   (c) Show that the matrix (with respect to the standard orthonormal basis of $\mathbb{C}^4$)

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^4)$$

   can *not* be written in the form $A \otimes B$ with $A, B \in \mathcal{L}(\mathbb{C}^2)$.

4. Suppose that $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are diagonalisable. Show that $A \otimes B$ is diagonalisable. Also, describe the spectrum/eigenvalues $\sigma(A \otimes B)$ in terms of the spectrum/eigenvalues of A and B.

5. Let $H \in \mathcal{L}(\mathbb{C}^2)$ be the Hadamard operator, $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

   (a) Compute $H^{\otimes 2}$ explicitly, both in terms of the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and the Kronecker product of matrices.

   (b) Show that $H^{\otimes n} \in \mathcal{L}\big((\mathbb{C}^2)^{\otimes n}\big) \cong \mathcal{L}\big(\mathbb{C}^{2^n}\big)$ can be written

$$H^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x,y=0}^{2^n-1} (-1)^{x\cdot y} |x\rangle\langle y|, \qquad (-1)^{x\cdot y} = (-1)^{x_0 y_0} \cdots (-1)^{x_{n-1} y_{n-1}},$$

   where $|x\rangle = |x_{n-1} \cdots x_1 x_0\rangle$ and $|y\rangle = |y_{n-1} \cdots y_1 y_0\rangle$ is the binary decomposition of basis elements in $\mathbb{C}^{2^n} \cong (\mathbb{C}^2)^{\otimes n}$.

6. (a) Let $\mathbb{C}^4 \cong \mathbb{C}^2_A \otimes \mathbb{C}^2_B$. Given a (4×4)-matrix M, write the matrix representation of $\mathrm{Tr}^{\mathbb{C}^2_A}(M) \in \mathcal{L}(\mathbb{C}^2_B)$.

   (b) For any $M \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$, show that

$$\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{Tr}^{\mathcal{K}}(M)\right) = \mathrm{Tr}_{\mathcal{K}}\left(\mathrm{Tr}^{\mathcal{H}}(M)\right) = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{K}}(M) \in \mathbb{C}.$$

   (c) If $M = M^* \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$, show that $\mathrm{Tr}^{\mathcal{H}}(M)$ and $\mathrm{Tr}^{\mathcal{K}}(M)$ are self-adjoint.

7. Suppose that $\rho \in \mathrm{Dens}(\mathcal{H} \otimes \mathcal{K})$, $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Show that

$$\mathrm{Tr}^{\mathcal{K}}(\rho) \in \mathrm{Dens}(\mathcal{H}), \qquad \mathrm{Tr}^{\mathcal{H}}(\rho) \in \mathrm{Dens}(\mathcal{K}), \qquad \langle A \otimes \mathbf{1}_{\mathcal{K}}\rangle_\rho = \langle A\rangle_{\mathrm{Tr}^{\mathcal{K}}(\rho)}, \qquad \langle \mathbf{1}_{\mathcal{H}} \otimes B\rangle_\rho = \langle B\rangle_{\mathrm{Tr}^{\mathcal{H}}(\rho)}.$$

8. Define the Bell states in $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big),$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big).$$

   (a) Show that the Bell states form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(b) Let $\mathrm{Tr}^{\mathbb{C}^2} : \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^2)$ denote the partial trace onto an operator on the first tensor. Show that the density matrices

$$\mathrm{Tr}^{\mathbb{C}^2}(|\Phi^+\rangle\langle\Phi^+|),\ \mathrm{Tr}^{\mathbb{C}^2}(|\Phi^-\rangle\langle\Phi^-|),\ \mathrm{Tr}^{\mathbb{C}^2}(|\Psi^+\rangle\langle\Psi^+|),\ \mathrm{Tr}^{\mathbb{C}^2}(|\Psi^-\rangle\langle\Psi^-|) \in \mathrm{Dens}(\mathbb{C}^2)$$

describe non-pure states in $\mathbb{C}^2$.

(c) For an operator $A \in \mathcal{L}(\mathbb{C}^2)$, show that

$$\langle A \otimes \mathbf{1}_2 \rangle_{\Phi^+} = \langle A \otimes \mathbf{1}_2 \rangle_{\Phi^-} = \langle A \otimes \mathbf{1}_2 \rangle_{\Psi^+} = \langle A \otimes \mathbf{1}_2 \rangle_{\Psi^+}$$

9. Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be a pure state. Show that $|\psi\rangle$ is a product state if and only if $\mathrm{Tr}^{\mathcal{K}}(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}^{\mathcal{H}}(|\psi\rangle\langle\psi|)$ are pure states.

10. Let $\rho_A \in \mathrm{Dens}(\mathcal{H}_A)$ be a density operator. Find an auxiliary Hilbert space $\mathcal{H}_B$ and a pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that the partial trace $\mathrm{Tr}^{\mathcal{H}_B}(|\psi_{AB}\rangle\langle\psi_{AB}|) = \rho_A$.

11. (General measurements) We say that a collection of linear operators $\{M_m\}_{m=1}^K \subset \mathcal{L}(\mathcal{H})$ are a set of measurement operators if $\sum_m M_m^* M_m = \mathbf{1}_{\mathcal{H}}$. Given a pure state $|\psi\rangle$, we say $\mathbf{P}_\psi(m) = \langle M_m^* M_m \rangle_\psi$ is the probability to obtain an outcome m for $m = 1, \ldots, K$.

(a) Show that measurement operators can be constructed from any self-adjoint operator $A = A^*$.

(b) Let $\mathcal{M} \cong \mathbb{C}^K$ be a Hilbert space with an orthonormal basis $\{|m\rangle\}_{m=1}^K$. Fix also a pure state $|\phi_{\mathcal{M}}\rangle \in \mathcal{M}$ and define

$$U : \mathcal{H} \otimes \mathrm{span}\{|\phi_{\mathcal{M}}\rangle\} \to \mathcal{H} \otimes \mathcal{M}, \qquad U\big(|\psi\rangle \otimes |\phi_{\mathcal{M}}\rangle\big) = \sum_{m=1}^K (M_m|\psi\rangle) \otimes |m\rangle.$$

Show that U can be extended to a unitary operator $U' \in \mathcal{L}(\mathcal{H} \otimes \mathcal{M})$ (you can use Q10 from the Postulates of quantum mechanics section).

(c) Let $P_m = \mathbf{1}_{\mathcal{H}} \otimes |m\rangle\langle m| \in \mathcal{L}(\mathcal{H} \otimes \mathcal{M})$. Show that

$$\langle U^* P_m U \rangle_{|\psi \otimes \phi_{\mathcal{M}}\rangle} = \langle \psi \mid M_m^* M_m \psi \rangle = \mathbf{P}_\psi(m).$$

(d) Suppose that the eigenvalue 1 of P_m is measured via the pure state $U|\psi \otimes \phi_{\mathcal{M}}\rangle$. Show that this state collapses to

$$U|\psi \otimes \phi_{\mathcal{M}}\rangle \xrightarrow{\text{measurement}} \Big(\frac{M_m|\psi\rangle}{\sqrt{\langle \psi \mid M_m^* M_m \psi \rangle}} \Big) \otimes |m\rangle.$$

12. Show that a *quantum copier* does not exist. That is, there is no linear map

$$K : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad K\big(|\psi\rangle \otimes |\eta\rangle\big) = |\psi\rangle \otimes |\psi\rangle.$$

13. Find the Schmidt decomposition of the following vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\text{(a)} \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad \text{(b)} \quad \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \qquad \text{(c)} \quad \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

14. Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ be a pure state.

(a) Show that $|\psi\rangle$ is a product state if and only if the Schmidt number (number of non-zero terms in the sum of the Schmidt decomposition) is 1.

(b) Show that the Schmidt number of $|\psi\rangle$ is the same as the rank of the reduced density operator $\rho_{\mathcal{H}} = \mathrm{Tr}^{\mathcal{H}}(|\psi\rangle\langle\psi|) \in \mathrm{Dens}(\mathcal{H})$.

15. Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and consider the operator T with matrix representation (in the canonical basis)

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Find the matrix representation of T and T^* in $\mathbb{C}^8$. Show that $T^2 = (T^*)^2 = 0$ and check if $TT^* = T^*T$ or $TT^* \neq T^*T$.

(b) Define the Hamiltonian $H = T + T^*$. Show that for any integer $m \geq 0$,

$$H^{2m} = K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1}_2 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_2, \qquad H^{2m+1} = H.$$

(c) Show that the time evolution operator

$$\exp(-itH) = (\mathbf{1} - K) + \cos(t)K - i\sin(t)H.$$

(d) Suppose that at $t = 0$ $|\psi(0)\rangle = |010\rangle$. Compute the time evolution $|\psi(t)\rangle$ for any $t > 0$.

(e) Suppose we take a measurement of the first Qubit (first tensor product) of $|\psi(t)\rangle$. What is the probability of measuring $|0\rangle$ as the first qubit? What does the pure state $|\psi(t)\rangle$ collapse to after such a measurement?

16. (a) Show that for any self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$ and pure state $|\psi\rangle \in \mathcal{H}$,

$$\langle\psi \mid A\psi\rangle \leq \sqrt{\langle\psi \mid A^2\psi\rangle}.$$

When is $\langle\psi \mid A\psi\rangle = \sqrt{\langle\psi \mid A^2\psi\rangle}$?

(b) Suppose $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $A = QS + RS + RT - QT$, where

$$Q = \hat{n}_Q \cdot \sigma \otimes \mathbf{1}_2, \qquad R = \hat{n}_R \cdot \sigma \otimes \mathbf{1}_2, \qquad S = \mathbf{1}_2 \otimes \hat{n}_S \cdot \sigma, \qquad T = \mathbf{1}_2 \otimes \hat{n}_T \cdot \sigma,$$

where $\hat{n}_Q, \hat{n}_R, \hat{n}_S, \hat{n}_T \in S_{\mathbb{R}^3}$ are unit vectors in $\mathbb{R}^3$. Show that

$$A^2 = 4(\mathbf{1}_2 \otimes \mathbf{1}_2) + [\hat{n}_Q \cdot \sigma, \hat{n}_R \cdot \sigma] \otimes [\hat{n}_S \cdot \sigma, \hat{n}_T \cdot \sigma].$$

(c) Use the results of (a) and (b) to show Tsirelson's inequality,

$$\langle\psi \,|\, (QS + RS - RT - QT)\psi\rangle \leq 2\sqrt{2}$$

for any pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Quantum circuits and algorithms

1. Show that the *classical* Toffoli gate, $\mathbf{TOF}(x_1, x_2, x_3) = (x_1, x_2, x_1x_2 \oplus x_3)$ can be constructed from the *classical* logic gates **ID**, **AND**, **XOR** and **COPY**.

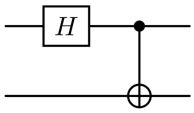
2. (a) Show that there exists $\xi, \alpha, \beta, \gamma \in \mathbb{R}$ such that any single Qubit gate U can be written as

$$U = e^{i\xi} R_z(\alpha) R_y(\beta) R_z(\gamma), \qquad R_z(\alpha) = e^{-\frac{i\alpha}{2}\sigma_z}, \quad R_y(\beta) = e^{-\frac{i\beta}{2}\sigma_y}.$$

   (b) If U is single Qubit gate, show that there is a $\xi \in \mathbb{R}$ and unitary operators A, B, C constructed from R_y and R_z such that $ABC = \mathbf{1}$ and

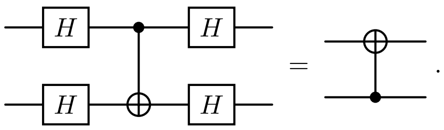
$$U = e^{i\xi} AXBXC.$$

3. Show that the circuit

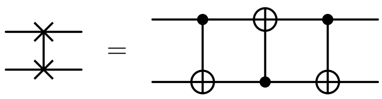


   will transform a state $|x_1x_0\rangle$ in the computational basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ to a Bell state.

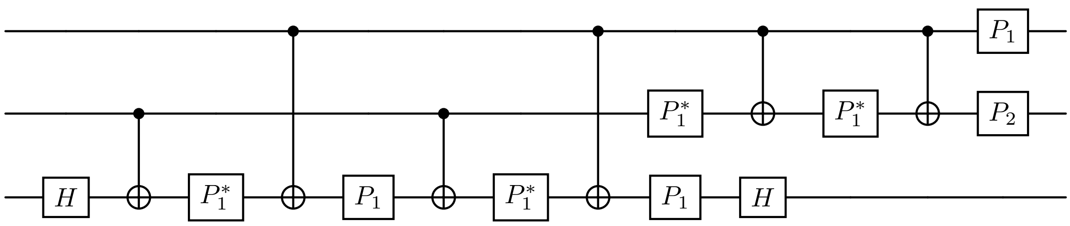
4. (a) Write the following equality of diagrams as an equality of operators and show it is true,



   (b) Show that the swap gate $S|x_1x_0\rangle = |x_0x_1\rangle$ can be written as a combination of controlled-NOT gates,



5. Show that the quantum TOFFOLI gate can be decomposed as follows:



   where $P_1 = P(\frac{\pi}{4})$ and $P_2 = P(\frac{\pi}{2})$.

6. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a unitary matrix. Find a quantum circuit comprised of single Qubit gates and controlled-NOT gates to implement the two-level unitary

$$\tilde{V} = \begin{pmatrix} 1 &&&&&&& \\ & a &&&&& b & \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ & c &&&&& d & \\ &&&&&&& 1 \end{pmatrix},$$

   (Note: you can use the quantum Toffoli and controlled-V gates in your circuit so long as you explain that they can be decomposed as a combination of single-Qubit and CNOT gates, see Q5).

7. Let $V \in \mathcal{L}(\mathbb{C}^2)$ be a self-adjoint unitary operator.

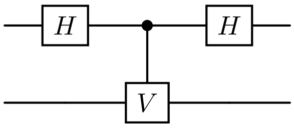
(a) Show that for any pure state $|\psi\rangle \in \mathbb{C}^2$,

$$|+_V\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle + V|\psi\rangle), \qquad |-_V\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle - V|\psi\rangle)$$

are eigenvectors of V.

(b) Consider the 2-Qubit system $\mathbb{C}^2 \otimes \mathbb{C}^2$ and quantum circuit,

$$(H \otimes \mathbf{1}_2)C^1(V)(H \otimes \mathbf{1}_2) \;=$$



,

where H is the Hadamard gate.

i. Suppose the input state for the circuit is $|0\rangle \otimes |\psi\rangle$. Compute the output state.

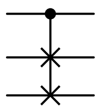
ii. Suppose we measure the observable $|1\rangle\langle 1| \otimes \mathbf{1}_2$ in the output state. Show that this measurement will collapse the output state into $|\pm_V\rangle$ depending on whether the measurement value is 0 or 1.

8. Let $\mathcal{H} = (\mathbb{C}^2)^{\otimes 4}$ be a 4-Qubit system with canonical basis $\{|x\rangle\}_{x=0}^{15}$. Design a quantum circuit that implements addition modulo 16. Namely, design a circuit that implements the map

$$U_+ : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad U_+ : |x\rangle \otimes |y\rangle \mapsto |x\rangle \otimes |x + y \bmod 16\rangle,$$

for $x, y \in \{0, 1, \ldots, 15\}$. You may describe your circuit as a product of unitary operators, a circuit diagram or via pseudo-code.

9. For a 3-Qubit system, we define the FREDKIN gate,



(a) Show that this gate can be written on the canonical/computational basis

$$F\big(|x_2\rangle \otimes |x_1\rangle \otimes |x_0\rangle\big) = |x_2\rangle \otimes |x_1 \oplus x_2(x_1 \oplus x_0)\rangle \otimes |x_0 \oplus x_2(x_1 \oplus x_0)\rangle.$$

(b) Show that the FREDKIN gate can be written as a combination of controlled-NOT gates and Q-TOF.

(c) Write down the matrix representation of F in the canonical basis.

10. Let $N > 1$ be some fixed positive integer and $1 \leq x \leq N - 1$ such that $x^r = 1 \bmod N$ for some $r \geq 2$. Show that the operator $U_x : \mathbb{C}^N \to \mathbb{C}^N$ defined on the canonical basis $\{|k\rangle\}_{k=0}^{N-1}$ such that $U_x|k\rangle = |xk \bmod N\rangle$ is unitary.

11. Let $S \subset \{0, 1, \ldots, 2^n - 1\}$ be a non-empty subset and define the function $f_S : \{0, 1, \ldots, 2^n - 1\} \to \{0, 1\}$, where $f_S(x) = 1$ if $x \in S$ and $f_S(x) = 0$ if $x \notin S$. Show that for the pure states $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \in \mathbb{C}^2$,

$$U_{f_S}(|\psi\rangle \otimes |-\rangle) = (\mathbf{1} - 2P_S)|\psi\rangle \otimes |-\rangle, \qquad P_S = \sum_{x \in S} |x\rangle\langle x|.$$

Elements of quantum information theory

1. (a) Show that the partial trace $\mathrm{Tr}^{\mathcal{K}} : \mathcal{L}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{L}(\mathcal{H})$ is an example of a quantum channel.

   (b) Show that the transpose of a matrix, considered as a linear map $\mathcal{L}(\mathbb{C}^n) \ni A \mapsto A^T \in \mathcal{L}(\mathbb{C}^n)$, is *not* a completely positive map.

2. Let $\mathcal{H} = \mathbb{C}^2$, $p \in [0,1]$ and consider the following quantum channels.

$$\Phi_X(\rho) = pX\rho X^* + (1-p)\rho, \qquad \Phi_Y(\rho) = pX\rho Y^* + (1-p)\rho, \qquad \Phi_Z(\rho) = pZ\rho Z^* + (1-p)\rho,$$

   where $\rho \in \mathrm{Dens}(\mathbb{C}^2)$ and X, Y, Z are the Pauli matrices. By the equivalence $\rho = \rho_{\mathbf{x}}$ for some $\mathbf{x} \in \mathbb{R}^3$ with $\|\mathbf{x}\| \leq 1$, show the effect of these channels on points in the Bloch ball that represent mixed states in $\mathbb{C}^2$.

3. Let $\mathcal{H}_B$ be a finite-dimensional Hilbert space and $\rho_B \in \mathrm{Dens}(\mathcal{H})$. Then for any operator $V \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $V^*V \leq \mathbf{1}_{A\otimes B}$, show that

$$\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A), \qquad \Phi(T) = \mathrm{Tr}^B \big(V(T \otimes \rho_B)V^* \big) \in \mathcal{L}(\mathcal{H}_A),$$

   defines a quantum operation.

4. Show that the composition of two quantum operations is also a quantum operation.

5. ** Find the Kraus operators/operator-sum representation of the depolarising channel, where for $p \in [0,1]$, and $T \in \mathrm{Dens}(\mathbb{C}^d)$,

$$\Phi(\rho) = \frac{p}{d}\mathbf{1}_d + (1-p)\rho.$$

6. Show that the quantum trace distance on states is a metric. That is for any $\rho, \nu, \omega \in \mathrm{Dens}(\mathcal{H})$, $D(\rho,\nu) \geq 0$, $D(\rho,\nu) = 0$ if and only if $\rho = \nu$, $D(\rho,\nu) = D(\nu,\rho)$ and

$$D(\rho,\nu) \leq D(\rho,\omega) + D(\omega,\nu).$$

7. Compute the trace distance $D(\rho,\nu)$, where

$$\text{(a)} \quad \rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \quad \nu = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|,$$
$$\text{(b)} \quad \rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \quad \nu = \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|.$$

8. Let $\{p_i\}_{i=1}^n$ be a probability distribution and $\{\rho_i, \nu_i\}_{i=1}^n \subset \mathrm{Dens}(\mathcal{H})$.

   (a) Show that the trace distance is jointly convex on density operators,

$$D\Big(\sum_i p_i\rho_i, \sum_i p_i\nu_i\Big) \leq \sum_i p_i D(\rho_i,\nu_i).$$

   (b) Show that the fidelity is jointly concave on density operators,

$$F\Big(\sum_i p_i\rho_i, \sum_i p_i\nu_i\Big) \geq \sum_i p_i F(\rho_i,\nu_i).$$

9. Let $|\psi\rangle$ and $|\phi\rangle$ be pure states in $\mathcal{H}$ with density operators ρ_ψ and ρ_ϕ. Show that

$$D\big(\rho_\psi, \nu_\phi\big) = \sqrt{1 - F(\rho_\psi,\nu_\phi)^2}.$$

10. Show that the entropy of a state is additive under a tensor product,

$$S(\rho \otimes \nu) = S(\rho) + S(\nu), \qquad \rho \in \mathrm{Dens}(\mathcal{H}),\ \nu \in \mathrm{Dens}(\mathcal{K}).$$

11. ** Let $\mathbf{P} = \{p_j\}_{j=1}^n$ be a probability distribution and $\{\rho_j\}_{j=1}^n \subset \mathrm{Dens}(\mathcal{H})$. Show that

$$\sum_{j=1}^n p_j S(\rho_j) \leq S\Big(\sum_{j=1}^n p_j\rho_j\Big) \leq \sum_{j=1}^n p_j S(\rho_j) + H(\mathbf{P}).$$