Lecture 11

Chapter 5: General Equilibrium
Part I: Real Economy

6/29, 2023

Goals

- We have studied difference equations, dynamic optimization, and dynamic programming so far.
- This completes our (painful?) journey for acquiring mathematical skills.
- This week and the next, we finally move on to macro**economics**.
  - In the modern literature, macroeconomics is part of applied general equilibrium analysis.
  - Today we focus on how <u>market equilibrium</u> is described.
  - For the first time, we shall introduce **prices**.

Optimal Allocation

- Consider the following allocation problem:

$$\max_{\{c_t\},\{L_t\},\{K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(L_t)]$$

subject to:

$$K_{t+1} = F(K_t, L_t) + (1-\delta)K_t - C_t$$

- This should look familiar to you.
- Formulate the Lagrangian or the Bellman equation to derive the FOCs.

Optimal Allocation

- The optimal allocation is a sequence of c_t, L_t, and K_t satisfying the following difference equations,

$$\frac{v'(L_t)}{u'(c_t)} = F_2(K_t, L_t),$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = F_1(K_{t+1}, L_{t+1}) + 1 - \delta,$$

$$K_{t+1} = F(K_t, L_t) + (1 - \delta)K_t - C_t$$

and the transversality condition.

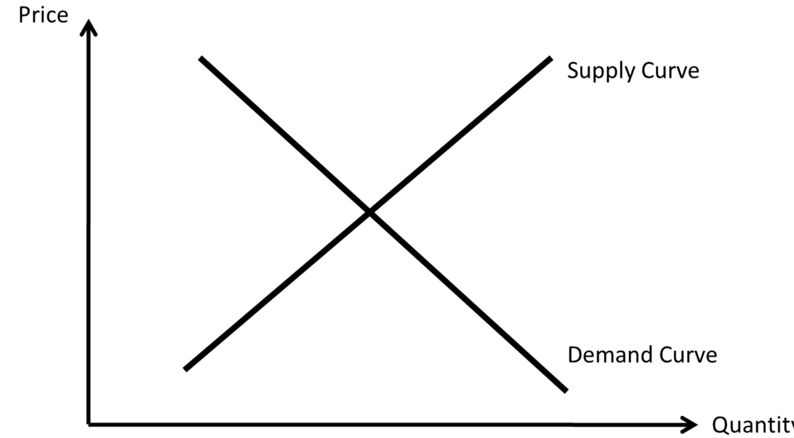
Optimal Allocation

- Observe that there is no **price** in this economy.
  - Because there is only one agent in this economy and he/she owns everything, there is no exchange at all and therefore price mechanism is not needed.
- This allocation is by construction **socially optimal**.
  - Allocation determined by the hypothetical (super-intelligent) social planner. He/she does not need to exist in the real world. What is important is that we can compute and study the property of this allocation.
- The key question is whether the market economy can do as well as the hypothetical planner does.

Competitive Economy

Static Economy

- Consider the undergrad micro theory.
- We need three components:
  - Demand function:
    $q^D = D(p)$
  - Supply function:
    $q^S = S(p)$
  - Equilibrium condition:
    $D(p) = S(p)$



Competitive Economy

- Now, let us introduce **markets**:
  - **Goods market**: consumption goods are traded.
  - **Labor market**: labor service is traded.
  - **Capital market**: capital is traded.
    - The same market can be understood as **asset market**.
- These are the same as undergrad macro.
- We shall assume that all markets are perfectly competitive, as in undergrad micro.
- Thus, all agents are **price takers** in all markets.

Competitive Economy

- In the real world, there are many products and services in the goods market. However, we shall simplify this dimension by assuming a single final consumption good.
  - The price of the good in period t is P_t yen.
- Likewise, we assume that there is a single type of labor service and a single type of capital.
  - The price of labor service (wage rate) per hour in period t is W_t yen.

Competitive Economy

- A **price** is an exchange rate of two objects.
  - An apple of 100 yen means that we give up 100 units of money for an apple.
- Thus, we need to model **money** (= currency) to study the price of the consumption good.
  - Money is the topic for the next week.
  - Today, we shall avoid the complicated issues surrounding the theory of money by focusing on a non-monetary economy.

Competitive Economy

- In a **real economy**, we do not model money.
- As a result, the general price level P_t cannot be determined (because the purchasing power of money $1/P_t$ cannot be determined without money), meaning that P_t can be any real number.
- In the real world, the wage rate is measured in terms of money, such as W_t yen per hour. Without money, we can only define the **real wage rate**,

$$w_t = \frac{W_t}{P_t}$$

Competitive Economy

- Note how the **real wage rate** w_t is measured.
  - Not in terms of yen.
  - This is in terms of units of the consumption good (such as "12.5 apples per hour").
- Similarly, the **real rental price of capital** r_t is measured in terms of units of the consumption good.

Competitive Economy

- There are two types of agents:
  - Households: unit mass, homogeneous
  - Firms: unit mass, homogeneous
- Unit mass?
  - The number of individuals is a <u>natural number</u>.
    - Not differentiable, and the number is too large.
  - A mass is a <u>quantity</u>, which is a real number.
    - Differentiable, and the size can be normalized to one.
    - How many people? = <u>infinity</u> within an interval of 0 and 1.
- A model with a representative agent = A model with an infinity of homogeneous agents.

Capital Market = Asset Market

- Firms demand capital in the capital market
  - Equivalently, firms supply asset in the asset market. These activities are same thing.
- Likewise, households demand and purchase asset in the asset market.
  - Equivalently, households supply capital in the capital market.

Firms

- Production technology: $F(K_t, L_t)$
- F satisfies the neoclassical assumptions:
  - $F_1 > 0, F_2 > 0, F_{11} < 0, F_{22} < 0$
  - Inada conditions:
    - $\lim_{K\to 0} F_1 = \infty, \lim_{K\to\infty} F_1 = 0$
    - $\lim_{L\to 0} F_2 = \infty, \lim_{L\to\infty} F_2 = 0$
  - Constant Returns to Scale

Firms

- The firm's profit in yen is
$$P_t F(K_t, L_t) - P_t r_t K_t - P_t w_t L_t$$
- Because our model cannot determine P_t, divide the profit by P_t to obtain
$$F(K_t, L_t) - r_t K_t - w_t L_t$$
- Thus, the profit maximization problem is given by
$$\max_{K_t, L_t} F(K_t, L_t) - r_t K_t - w_t L_t$$
for all t.
- Note that the problem is static even though the model has an infinite horizon.

Firms

- From the FOCs, firm's demands for capital and labor K_t, L_t satisfy:

$$F_1(K_t, L_t) = r_t$$
$$F_2(K_t, L_t) = w_t$$

- Euler's Theorem: If $F(K, L)$ satisfies CRS, then

$$F(K, L) = F_1(K, L)K + F_2(K, L)L$$

- Euler's Theorem and FOCs imply **zero profit**:

$$F(K_t, L_t) - r_t K_t - w_t L_t = 0$$

Firms

- Implications of zero profit:
  - No incentive for anyone outside of the market to enter.
  - No incentive for anyone inside the market to exit.
- In other words, our model cannot determine the number of firms.
- Thus, we usually assume either
  - There is one perfectly competitive firm.
  - There is a unit mass of homogeneous firms.
- These two representations are the same.

Households

- A typical household's problem:

$$\max_{\{c_t\},\{h_t\},\{a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]$$

subject to:

$$a_t + w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

- The household takes the sequences of the wage rate $\{w_t\}_{t=0}^{\infty}$ and the asset price $\{1/R_{t+1}\}_{t=0}^{\infty}$ as given (price taker).

Individual Rationality

- Formulate the Lagrangian, derive the FOCs, and eliminate the multiplier.
- Demand for c_t, a_{t+1} and labor supply h_t satisfy:

$$\frac{v'(h_t)}{u'(c_t)} = w_t,$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = R_{t+1},$$

$$a_t + w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

Market Clearing

- Market clearing conditions:
  Goods market: $F(K_t, L_t) = c_t + I_t$
  Labor market: $h_t = L_t$
  Capital market: $\frac{1}{R_{t+1}} a_{t+1} = K_{t+1}$
- Equilibrium rate of interest:

$$R_t = 1 + r_t - \delta = 1 + F_1 - \delta$$

  - RHS is the **technological rate of returns on capital**: One unit of income is saved in the form of capital. One unit is the principal, interest r_t is from this period's production, and δ is depreciated.

General Equilibrium

- Capital market clearing implies

$$a_t = R_t K_t = (1 + r_t - \delta) K_t$$

- Substitute it into the household's budget:

$$a_t + w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

$$\Leftrightarrow$$

$$r_t K_t + (1 - \delta) K_t + w_t h_t = c_t + K_{t+1}$$

$$\Leftrightarrow$$

$$K_{t+1} = F(K_t, L_t) + (1 - \delta) K_t - C_t$$

- Identical to the resource constraint on page 3.

General Equilibrium

- Labor market and capital market clearing imply:

$$\frac{v'(L_t)}{u'(c_t)} = w_t = F_2(K_t, L_t),$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = R_{t+1} = 1 + F_1(K_t, L_t\ ) - \delta.$$

- These are identical to the conditions for the optimal allocation on page 4.
- The two models generate the same allocation.
- The competitive economy is **Pareto optimal**.

Welfare Theorem

- **First Welfare Theorem**: Every competitive equilibrium is Pareto optimal.
- **Second Welfare Theorem**: Every Pareto optimum can be decentralized as a competitive equilibrium.
- For conditions for these theorems, see any microeconomics textbook.
  - If your model includes scale effects, externality, public goods, fixed costs, imperfect competition, and market imperfections (such as asymmetric information), then the welfare theorem fails.

General Equilibrium with Government

Government Consumption

- Government provides public goods G_t. Public goods usually increase our utility or production:
  - $U(c_t, h_t, G_t)$
  - $F(K_t, L_t, G_t)$
- For many research questions, it suffices to treat G_t as **government consumption**.
  - G_t does not enter utility or production function.
  - Government consumption is a complete waste.
  - Sounds strange, but this way we can avoid the tricky discussion of how G_t improves U and F.
    - This could destroy your research.

Tax

- Let T_t be the amount of tax.
- Household budget under **lump-sum tax**:
$$a_t + w_t h_t - T_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$
- Household budget under **distortionary taxes**:
$$T_t = \boldsymbol{\tau_I} w_t h_t + \boldsymbol{\tau_C} c_t + \boldsymbol{\tau_K} a_t$$
- Distortionary taxes change the relative prices of actions (labor supply, consumption, wealth).
  - For many research questions, it suffices to assume lump-sum tax.

Government's Budget Constraint

- Typically, the government is not an optimizer.
- However, it faces the budget constraint.
- If tax is the only source of revenue, then the government budget constraint is
$$G_t = T_t$$
- This is called the balanced budget.

GE under Balanced Budget

- Market clearing conditions:

$$F(K_t, L_t) = c_t + I_t + G_t$$

$$h_t = L_t$$

$$\frac{1}{R_{t+1}} a_{t+1} = K_{t+1}$$

- Budget constraints:

$$a_t + w_t h_t - T_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

$$G_t = T_t$$

- <u>One new equation and two new variables</u>.

GE under Balanced Budget

- One new equation and two new variables:
$$G_t = T_t$$
- We need to specify either $\{G_t\}_{t=0}^{\infty}$ or $\{T_t\}_{t=0}^{\infty}$.
- If $\{G_t\}$ is specified, then $\{T_t\}$ is endogenous.
- If $\{T_t\}$ is specified, then $\{G_t\}$ is endogenous.
- If $\{G_t\}$ and one of the two distortionary tax rates are specified, then the other distortionary tax rate is endogenous.

Budget Deficits and Debt

- Let us relax the assumption of balanced budget.
- Suppose that the government participate in the asset market.
- b_t is the government debt outstanding in period t.
- r_t^b is the interest on bonds.
- $b_{t+1} - b_t$ is the newly issued bonds.
- **Government's budget constraint** is:

$$G_t + r_t^b b_t = T_t + b_{t+1} - b_t$$

Budget Deficits and Debt

- **Primary budget deficit**: $G_t - T_t = D_t$
- Primary budget **surplus**: $T_t - G_t = -D_t$
- An alternative definition of deficit:
$$b_{t+1} - b_t = G_t - T_t + r_t^b b_t$$
- This definition includes interest obligation $r_t^b b_t$.
- Sometimes the definition of deficit matters, but not always.

Arbitrage Condition

- There are two **assets**, capital and bonds.
- For the two assets to **coexist**, the rates of returns (= asset prices) must be the same:

$$R_t = 1 + r_t - \delta = 1 + r_t^b$$

- In any equilibrium, individuals are indifferent between the two assets.
  - Only the sum $K_t + b_t$ is determined by the demand side.
  - Equilibrium level of b_t is determined by the supply of bonds. ⇒ Equilibrium level of K_t is determined.

General Equilibrium

- Market clearing conditions:

$$F(K_t, L_t) = c_t + I_t + G_t$$
$$h_t = L_t$$
$$\frac{1}{R_{t+1}} a_{t+1} = K_{t+1} + b_{t+1}$$

- Budget constraints:

$$a_t + w_t h_t - T_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$
$$G_t + r_t^b b_t = T_t + b_{t+1} - b_t$$

General Equilibrium

- Substitute $\frac{1}{R_{t+1}} a_{t+1} = K_{t+1} + b_{t+1}$ into budget:

$$R_t(K_t + b_t) + w_t h_t - T_t = c_t + K_{t+1} + b_{t+1}$$
$$\Leftrightarrow$$
$$Y_t + (1-\delta)K_t + \left(1 + r_t^b\right)b_t - T_t$$
$$= c_t + K_{t+1} + b_{t+1}$$
$$\Leftrightarrow$$
$$Y_t - T_t - c_t \equiv \boldsymbol{S_t} = \boldsymbol{I_t} + \boldsymbol{D_t}$$

- This result is the same as in undergrad macro.

General Equilibrium

- Now we have one new equation and 3 new variables (G_t, T_t, b_t):
$$G_t + r_t b_t = T_t + b_{t+1} - b_t$$
- Equivalently, we have one new equation and 2 new variables (D_t, b_t):
$$b_{t+1} = D_t + (1 + r_t) b_t$$
- We typically specify the sequence of primary deficits, $\{D_t\}_{t=0}^{\infty}$, as exogeneous and let the sequence of debt as endogenous.
  - Specifications depend on your research questions.

Ricardian Equivalence

- Economy without capital:

$$\max_{\{c_t\},\{h_t\},\{b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]$$

subject to:

$$b_t + w_t h_t - T_t = c_t + \frac{b_{t+1}}{1 + r_{t+1}}$$

- The government's budget constraint:

$$G_t + r_t b_t = T_t + b_{t+1} - b_t$$

- $\{G_t\}_{t=0}^{\infty}$ is exogenous.

Ricardian Equivalence

- **Theorem**: Suppose that the sequence of government spending $\{G_t\}_{t=0}^{\infty}$ is exogenously given. Then, the timing of lump-sum tax does not matter for the equilibrium allocation and welfare.
- To fix idea, consider a scenario in which the government reduces T_t to $T_t - \Delta$ without changing its spending plan $\{G_t\}_{t=0}^{\infty}$.
- This tax-cut must be financed by the same amount of bonds.
- The bonds must be financed by future tax.

Ricardian Equivalence

- From the government's budget constraint,
$$G_t + r_t b_t = T_t - \Delta + b_{t+1} - b_t$$
- Thus, b_{t+1} must the increase by Δ.
- In the next period,
$$G_{t+1} + (1 + r_{t+1})(b_{t+1} + \Delta) = (T_{t+1} + \Omega) + b_{t+2}$$
- Suppose T_{t+1} must be increased by Ω to offset the budget impact of Δ. Then,
$$\Omega = (1 + r_{t+1})\Delta$$

Ricardian Equivalence

- Now consider the household's budget constraint:

$$b_t + w_t h_t - T_t = c_t + \frac{b_{t+1}}{1 + r_{t+1}}$$

- The intertemporal budget constraint is

$$\sum_{t=0}^{\infty} \frac{c_t}{\Pi_{i=0}^{t}(1 + r_i)} = w_0 h_0 - T_0 + \cdots + \frac{w_t h_t - T_t + \Delta}{\Pi_{i=0}^{t}(1 + r_i)} + \frac{w_{t+1} h_{t+1} - T_{t+1} - \Omega}{\Pi_{i=0}^{t+1}(1 + r_i)} + \cdots$$

Ricardian Equivalence

- Substitute $\Omega = (1 + r_{t+1})\Delta$ into the above:

$$\sum_{t=0}^{\infty} \frac{c_t}{\prod_{i=0}^{t}(1+r_i)} = w_0 h_0 - T_0 + \cdots + \frac{w_t h_t - T_t + \Delta}{\prod_{i=0}^{t}(1+r_i)} + \frac{w_{t+1} h_{t+1} - T_{t+1} - (1+r_{t+1})\Delta}{\prod_{i=0}^{t+1}(1+r_i)} + \cdots$$

- The terms on Δ disappears!

Ricardian Equivalence

- Thus, the household’s intertemporal budget constraint is **invariant** to any government intervention that change the timing of lump-sum taxation.
- The same budget constraint, the same consumption path.
- Therefore, this policy will never stimulate consumption.

Fiscal Stimulus

- Given the Ricardian equivalence theorem, fiscal stimulus must change either
  - relative prices (by distortionary taxes), or
  - household wealth (by permanent tax changes).
- Under the balanced budget, a permanent increase in G_t will increase output because…
  - There is a permanent increase in T_t
  - This will reduce the household's lifetime wealth.
  - The household wants to (needs to?) work harder.
  - Output goes up.

The Laffer Curve

- Household's problem:

$$\max_{\{c_t\},\{h_t\},\{a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]$$

subject to:

$$a_t + (1-\tau)w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

- τ: labor income tax rate.
- This is a **distortionary tax**.

The Laffer Curve

- Demand for c_t, a_{t+1} and labor supply h_t satisfy:

$$\frac{v'(h_t)}{u'(c_t)} = (1-\tau)w_t,$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = R_{t+1},$$

$$a_t + (1-\tau)w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$

The Laffer Curve

- Market clearing conditions:

$$F(K_t, L_t) = c_t + I_t + G_t$$
$$h_t = L_t$$
$$\frac{1}{R_{t+1}} a_{t+1} = K_{t+1} + b_{t+1}$$

- Budget constraints:

$$a_t + (1-\tau) w_t h_t = c_t + \frac{a_{t+1}}{R_{t+1}}$$
$$G_t + r_t^b b_t = \tau w_t h_t + b_{t+1} - b_t$$

The Laffer Curve

- Functional form:
$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$
- Factor prices:
$$r_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha}$$
$$w_t = (1-\alpha) A K_t^{\alpha} L_t^{-\alpha}$$
- Thus, in any steady state,
$$r = \alpha A k^{\alpha-1}$$
$$w = (1-\alpha) A k^{\alpha}$$
- $k = K/L$ is the capital-labor ratio.

The Laffer Curve

- Functional forms:

$$u(c) = \ln c\,, \qquad v(h) = \phi \frac{h_t^{1+\mu}}{1+\mu}$$

- Household's optimization:

$$\phi h_t^{\mu} c_t = (1-\tau) w_t, \qquad \frac{c_{t+1}}{c_t} = \beta R_{t+1},$$

- In any steady-state,

$$\phi h^{\mu} c = (1-\tau)(1-\alpha) A k^{\alpha}$$
$$\frac{1}{\beta} = \alpha A k^{\alpha-1} + 1 - \delta$$

The Laffer Curve

- The resource constraint (p.35):
$$Y_t + (1-\delta)K_t + \left(1 + r_t^b\right)b_t - \tau w_t h_t$$
$$= c_t + K_{t+1} + b_{t+1}$$
- The government budget
$$G_t + r_t b_t = \tau w_t h_t + b_{t+1} - b_t$$
- They imply
$$Y_t + (1-\delta)K_t = c_t + K_{t+1} + G_t$$
- Assuming a constant G, in any steady-state,
$$Y = c + \delta K + G$$

The Laffer Curve

- In any steady-state, $Y = c + \delta K + G$
- Divide both sides by L

$$Ak^{\alpha} = \frac{c}{L} + \delta k + \frac{G}{L}$$

- Remember that $\frac{1}{\beta} = \alpha A k^{\alpha - 1} + 1 - \delta$ determines k.
- Thus, (with $g = G/L$)

$$\frac{c}{L} = Ak^{\alpha} - \delta k - g$$

- So far, the equilibrium is independent of τ.

The Laffer Curve

- Given k, we can find the steady-state labor supply by

$$L = \left[\frac{(1-\tau)(1-\alpha)Ak^{\alpha}}{\phi(Ak^{\alpha} - \delta k - g)}\right]^{\frac{1}{1+\mu}}$$

- The steady-state labor supply depends on the labor income tax (as expected).

The Laffer Curve

- The relationship between the government's tax revenue and the tax rate is called the **Laffer curve**:

$$\tau w L = \tau(1-\alpha)Ak^{\alpha}\left[\frac{(1-\tau)(1-\alpha)Ak^{\alpha}}{\phi(Ak^{\alpha}-\delta k-g)}\right]^{\frac{1}{1+\mu}}$$

$$= \tau(1-\alpha)y\left[\frac{(1-\tau)(1-\alpha)}{\phi\left(1-\delta\frac{k}{y}-\frac{g}{y}\right)}\right]^{\frac{1}{1+\mu}} = \Omega(\tau)$$

- Evidently, $\Omega(0)=\Omega(1)=0$.

The Laffer Curve

- Note that the revenue is measured in units of consumption goods, not in Yen or Dollars.
- For numerical analysis, it is convenient to measure the Laffer curve as a fraction of GDP:

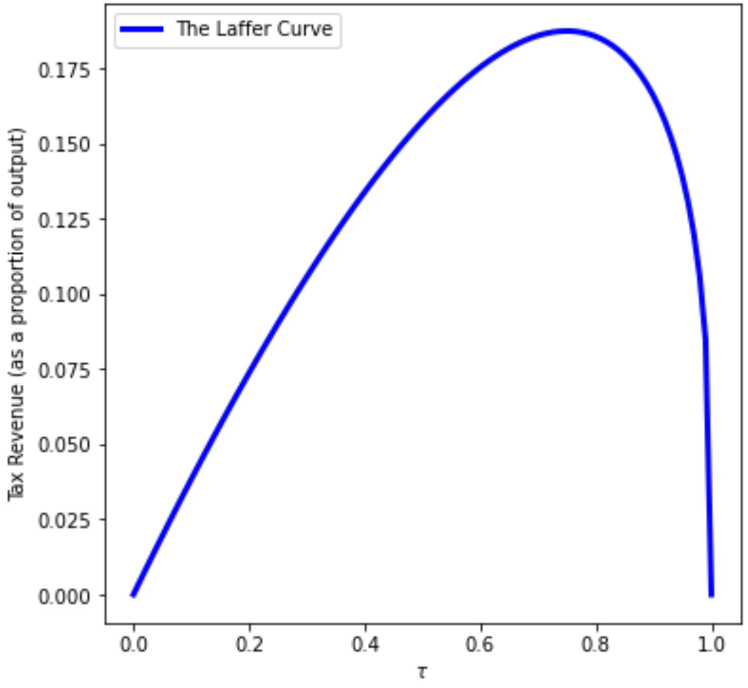
$$\frac{\tau w L}{y} = \tau(1-\alpha)\left[\frac{(1-\tau)(1-\alpha)}{\phi\left(1-\delta\frac{k}{y}-\frac{g}{y}\right)}\right]^{\frac{1}{1+\mu}}$$

- Parameter values:

$$\beta = 0.91, \delta = 0.1, \alpha = 0.4, \phi = 3.46, \mu = 2, A = 1, \frac{g}{y} = \frac{1}{5}$$

The Laffer Curve

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import root
plt.rcParams['figure.figsize'] = (6,6)

# Parameter values

β = 0.91
δ = 0.1
α = 0.4
φ = 3.46
μ = 2
A = 1
R = 1/β -1 +δ
kl = (R/(α*A))**(1/(α-1))
y = A*(kl**α)
g = y/5
taulessΩ = (1-α)*A*(kl**α)/(φ*A*(kl**α)-φ*δ*kl-φ*g)

# The Laffer curve
def laffer(tau):
    return tau*(1-α)*A*(kl**α)*(((1-tau)*taulessΩ)**(1/(1+μ)))/y

# Drawing a diagram

grid = np.linspace(0, 1, 100)
fig, ax = plt.subplots()

eq = laffer(grid)

ax.plot(grid, eq, 'b-', lw=3, label='The Laffer Curve')

ax.set_xlabel('$τ$')
ax.set_ylabel('Tax Revenue (as a proportion of output)')
ax.legend()

plt.show()
```



Further Readings

- Trabandt and Uhlig, “The Laffer Curve Revisited,” *Journal of Monetary Economics*, 2011.