Lecture 9

Chapter 3: Neoclassical Growth
Part II: Quantitative Analysis

6/15, 2023

The Neoclassical Growth Model

- Last week, we learned that the optimal allocation is determined by

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta$$
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$
$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0$$
$$k_0 : \text{given}$$

- Today, we shall solve the model more explicitly.

Specification

- Functional forms are

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

$$F(K,N) = AK^{\alpha}N^{1-\alpha} \Longrightarrow f(k) = Ak^{\alpha}$$

- Then,

$$\frac{c_t^{-\sigma}}{\beta c_{t+1}^{-\sigma}} = \alpha A k_{t+1}^{\alpha-1} + 1 - \delta$$

$$k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t - c_t$$

$$\lim_{t\to\infty} \beta^t c_t^{-\sigma} k_t = 0$$

$$k_0 : \text{given}$$

Steady State

- Let (k, c) denote the steady state.
- Then, the steady state satisfies

$$\frac{1}{\beta} = \alpha A k^{\alpha-1} + 1 - \delta$$
$$c = Ak^{\alpha} - \delta k$$

- For later use, rewrite them as

$$\alpha A k^{\alpha-1} = \frac{1}{\beta} - 1 + \delta = \frac{1-\beta}{\beta} + \delta$$

$$\frac{c}{k} = Ak^{\alpha-1} - \delta = \frac{\frac{1}{\beta} - 1 + \delta}{\alpha} - \delta = \frac{1-\beta}{\alpha\beta} + \frac{\delta(1-\alpha)}{\alpha}$$

Log-linearization

- Let us first log-linearize the simpler one,

$$k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t - c_t$$

- Log-linearize it around the steady state to obtain

$$dk_{t+1} = \alpha A k^{\alpha-1} dk_t + (1-\delta) dk_t - dc_t$$

$$\Longleftrightarrow$$

$$k \frac{dk_{t+1}}{k} = \underbrace{\alpha A k^{\alpha-1}}_{\frac{1-\beta}{\beta}+\delta} k \frac{dk_t}{k} + (1-\delta) k \frac{dk_t}{k} - c \frac{dc_t}{c}$$

Log-linearization

- Further, divide both sides by k to obtain

$$\hat{k}_{t+1} = \left(\frac{1-\cancel{\beta}}{\beta} + \cancel{\delta} + \cancel{1} - \cancel{\delta}\right)\hat{k}_t - \frac{c}{k}\,\hat{c}_t$$

$$= \frac{1}{\beta}\hat{k}_t - \left(\frac{1-\beta}{\alpha\beta} + \frac{\delta(1-\alpha)}{\alpha}\right)\hat{c}_t$$

Log-linearization

- Thus, the log-linearized equation for
$$k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t - c_t$$
is given by
$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t - \left[\frac{1-\beta}{\alpha\beta} + \frac{\delta(1-\alpha)}{\alpha}\right]\hat{c}_t$$
- Remember:
$$\hat{k}_t = \frac{dk_t}{k} = \frac{k_t - k}{k}, \hat{c}_t = \frac{dc_t}{c} = \frac{c_t - c}{c}$$
  - Thus, variables with hats are measured in percentage deviations from the steady-state values.

Log-linearization

- Let us now log-linearize the other one,

$$\frac{c_t^{-\sigma}}{\beta c_{t+1}^{-\sigma}} = \alpha A k_{t+1}^{\alpha-1} + 1 - \delta$$

- First, to ease our calculation, rewrite it as

$$c_{t+1}^{\sigma} = \beta c_t^{\sigma}[\alpha A k_{t+1}^{\alpha-1} + 1 - \delta]$$

- Then,

$$\sigma c^{\sigma-1} dc_{t+1} = \beta \underbrace{[\alpha A k^{\alpha-1} + 1 - \delta]}_{\frac{1}{\beta}} \sigma c^{\sigma-1} dc_t$$

$$+ \beta c^{\sigma} (\alpha-1) \alpha A k^{\alpha-2} dk_{t+1}$$

Log-linearization

- Further,

$$\sigma c^{\sigma-1} \frac{dc_{t+1}}{c} = \sigma c^{\sigma-1} \frac{dc_t}{c} + \beta c^{\sigma} (\alpha-1)\alpha A k^{\alpha-2} \frac{dk_{t+1}}{k}$$

$$\Leftrightarrow \quad \sigma \hat{c}_{t+1} = \sigma \hat{c}_t + \beta(\alpha-1)\underbrace{\alpha A k^{\alpha-1}}_{\frac{1-\beta}{\beta}+\delta} \hat{k}_{t+1}$$

$$= \sigma \hat{c}_t - (1-\alpha)(1-\beta+\delta\beta)\hat{k}_{t+1}$$

Log-linearization

- Thus, the log-linearized equation for

$$\frac{c_t^{-\sigma}}{\beta c_{t+1}^{-\sigma}} = \alpha A k_{t+1}^{\alpha-1} + 1 - \delta$$

is given by

$$\hat{c}_{t+1} + \frac{1-\alpha}{\sigma}(1 - \beta + \delta\beta)\hat{k}_{t+1} = \hat{c}_t$$

Local Analysis

- Now we have a log-linearized system:

$$\hat{c}_{t+1} + \frac{1-\alpha}{\sigma}(1-\beta+\delta\beta)\hat{k}_{t+1} = \hat{c}_t$$

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t - \left[\frac{1-\beta}{\alpha\beta} + \frac{\delta(1-\alpha)}{\alpha}\right]\hat{c}_t$$

- Let us introduce new parameters:

$$\phi_c = \frac{1-\alpha}{\sigma}(1-\beta+\delta\beta)$$

$$\phi_k = \frac{1-\beta}{\alpha\beta} + \frac{\delta(1-\alpha)}{\alpha}$$

Local Analysis

- Thus, we have a simple-looking system:

$$\hat{c}_{t+1} + \phi_c \hat{k}_{t+1} = \hat{c}_t$$

$$\hat{k}_{t+1} = -\phi_k \hat{c}_t + \frac{1}{\beta} \hat{k}_t$$

- In matrix form,

$$\begin{pmatrix} 1 & \phi_c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\phi_k & 1/\beta \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

- It is easy to verify that $\begin{pmatrix} 1 & \phi_c \\ 0 & 1 \end{pmatrix}$ is invertible.

Local Analysis

- Thus,
$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & \phi_c \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\phi_k & 1/\beta \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$
- Before going to the next page, calculate
$$\begin{pmatrix} 1 & \phi_c \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\phi_k & 1/\beta \end{pmatrix}$$
- Also, try to prove that the steady state of the system is a saddle.
    - We know the result from the phase diagram last week.

Local Analysis

- It is straightforward:

$$\begin{pmatrix}1 & \phi_c \\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 \\ -\phi_k & 1/\beta\end{pmatrix} = \begin{pmatrix}1 & -\phi_c \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ -\phi_k & 1/\beta\end{pmatrix}$$
$$= \begin{pmatrix}1+\phi_c\phi_k & -\phi_c/\beta \\ -\phi_k & 1/\beta\end{pmatrix}$$

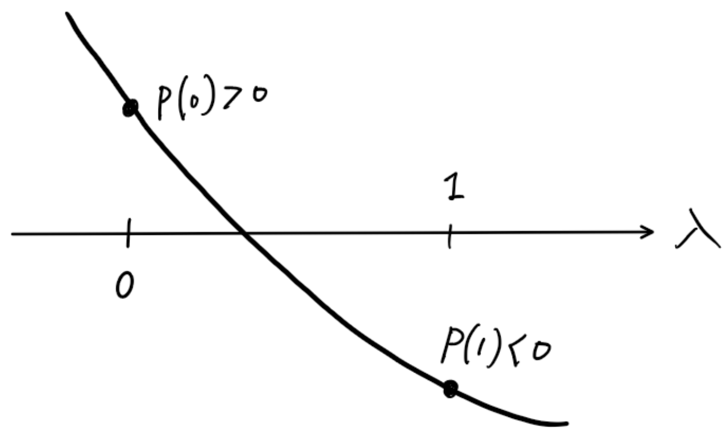
- Consider the characteristic polynomial:

$$p(\lambda) = \lambda^2 - (1 + \phi_c\phi_k + 1/\beta)\lambda + 1/\beta$$

- It is easy to verify that $p(0) = 1/\beta > 0$ and $p(1) = -\phi_c\phi_k < 0$.
- Let us draw a diagram of a quadratic equation with $p(0) > 0$ and $p(1) < 0$.

Local Analysis

- The diagram implies that on the downward region, an eigenvalue is found in between $0 < \lambda < 1$.
- The other root should be found in the region $\lambda > 1$.
- Thus, as shown in the phase diagram last week, the steady state is a **saddle.**



Numerical Analysis

- In what follows, we shall quantify the model to numerically study the model.
- The first step is to specify the parameter values.
- There are two ways:
  - **Estimation**: Formal econometric methods to find the appropriate values of model parameters.
  - **Calibration**: Somewhat informal. Empirical studies outside of the model to find the appropriate values of model parameters.
- Estimation is beyond the scope of this lecture.

Parameters

- Today, we shall borrow the standard parameter values from the literature.
- Let us set one period to be a quarter.
- $\alpha = 0.36$: Consistent with the labor share.
- $\delta = 0.025$: Consistent with 10% per year.
- $\beta = \frac{1}{1+0.01} = 0.99$: Consistent with 4% per year.
- $\sigma = 1$
- $A = 1$

Numerical Analysis

- In what follows, we use Octave (or Matlab).
- First, declare the parameter values.

```
alp = 0.36; % alpha
del = 0.025; % delta
bet = 1/(1+0.01); % beta
sig = 1.0; % sigma
A = 1.0; % TFP
%
phic = (1-alp)*(1-bet+del*bet)/sig;
phik = (1-bet)/(alp*bet) + del*(1-alp)/alp;
```

Numerical Analysis

- In matrix form,

$$\underbrace{\begin{pmatrix} 1 & \phi_c \\ 0 & 1 \end{pmatrix}}_{B_1} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\phi_k & 1/\beta \end{pmatrix}}_{B_2} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

- From this, we obtain $B = B_1^{-1} B_2$.

```
B1 = [1,phic;0,1];
B2 = [1,0;-phik,1/bet];
B = B1\B2; % This is equivalent to inv(B1)*B2
```

Numerical Analysis

- In Matlab (or Octave), the elements of a matrix are written as (raw, column)
- However, E matrix is defined as
$$E = (P_1, P_2) = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix}$$
- See Lecture 3 on this.

```
[E,D] = eig(B);
e11 = E(1,1);
e12 = E(2,1);
e21 = E(1,2);
e22 = E(2,2);
pol = e11/e12
```

Numerical Analysis

- Calculate the eigenvectors to obtain the E-matrix.
- Finally, the saddle path is given by $$\hat{c}_t = \frac{e_{11}}{e_{12}}\hat{k}_t = 0.62\hat{k}_t$$

```
>> Neoclassical

e11 = -0.52576
e12 = -0.85063
e21 =  0.44850
e22 = -0.89378
pol =  0.61808
>> |
```

Policy Function

- Notice that the saddle path $\hat{c}_t = 0.62\hat{k}_t$ gives us a mapping (function) from the current state into the current action.
- This mapping is called the **policy function**.
- The Matlab/Octabe code for this lecture, "Neoclassical.m", is available at TACT.
- The Python counterpart, "PythonCode_Neoclassical.txt", is also available at TACT.

Equilibrium

- Original linear system:

$$\hat{c}_{t+1} + \phi_c \hat{k}_{t+1} = \hat{c}_t$$

$$\hat{k}_{t+1} = -\phi_k \hat{c}_t + \frac{1}{\beta} \hat{k}_t$$

- This system has an infinity of paths (most of them explosive) from an arbitrary initial capital stock.
- TVC allows us to select the saddle path from them:

$$\hat{c}_t = \frac{e_{11}}{e_{12}} \hat{k}_t$$

Equilibrium

- The saddle path satisfies:

$$\hat{c}_t = \frac{e_{11}}{e_{12}} \hat{k}_t$$

$$\hat{k}_{t+1} = -\phi_k \hat{c}_t + \frac{1}{\beta} \hat{k}_t$$

- This system has a unique path from any initial capital stock, leading to the steady state.
- This is the (unique) **equilibrium** (or the solution) of the model. (See Lecture 8)
  - We can now simulate the model.