Lecture 8

Chapter 3: Neoclassical Growth
Part I: Global Analysis

6/1, 2023

The Solow Model: A Brief Review

- It is good idea to start with the Solow model.
- Let K_t and N_t denote capital and labor.
- The aggregate output Y_t is determined by

$$Y_t = F(K_t, N_t)$$

- **Neoclassical assumptions**:
  - $F_1 > 0, F_2 > 0, F_{11} < 0, F_{22} < 0$
  - Inada conditions:
    - $\lim_{K \to 0} F_1 = \infty, \lim_{K \to \infty} F_1 = 0$
    - $\lim_{N \to 0} F_2 = \infty, \lim_{N \to \infty} F_2 = 0$
  - Constant Returns to Scale

The Solow Model: A Brief Review

- Most widely used specification is Cobb-Douglas:
$$Y_t = AK_t^{\alpha} N_t^{1-\alpha}$$
- α satisfies $1 < \alpha < 1$
- A is referred to as Total Factor Productivity (TFP).
- Capital accumulation:
$$K_{t+1} = K_t + I_t - \delta K_t$$
- I_t is capital formation (investment) in t.
- δ is the depreciation rate ($0 < \delta < 1$).

The Solow Model: A Brief Review

- One household represents the entire economy (the representative household assumption).
- Household has two roles:
  - Supplier of capital (via saving)
  - Supplier of labor
- Savings are <u>exogenously</u> determined by
$$S_t = sY_t$$
- s is the saving rate ($0 < s < 1$)
- Equivalently, consumption decision is <u>exogenous</u>:
$$C_t = (1 - s)Y_t$$

The Solow Model: A Brief Review

- Household supplies labor input inelastically. In other words, labor supply is exogenous. Thus,
$$\text{Labor input} = \text{Population}$$
- Population is assumed to grow exogenously:
$$N_{t+1} = nN_t$$
- $n > 1$ is the **gross** population growth rate.
  - Also referred to as the population growth **factor**.
  - You might prefer $N_{t+1} = (1+n)N_t$ instead.
    - This is fine, too.
    - This is just a matter of taste.

The Solow Model: A Brief Review

- We consider a closed economy without government.
- Without government and foreign countries, output is either consumed or invested:

$$Y_t = C_t + I_t$$

    - This is the goods market equilibrium condition.
- As usual, we can transform the goods market equilibrium condition into the capital market equilibrium condition:

$$I_t = Y_t - C_t = S_t$$

The Solow Model: A Brief Review

- Model summary:

$$Y_t = F(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha}$$
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$Y_t = C_t + I_t \Leftrightarrow I_t = S_t$$
$$S_t = sY_t$$
$$N_{t+1} = nN_t$$

- We can reduce the system to a two-dimensional nonlinear system:

$$\begin{cases} K_{t+1} = sAK_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t \\ N_{t+1} = nN_t \end{cases}$$

The Neoclassical Growth Model

- Consider the equations again:

$$Y_t = F(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha}$$
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$Y_t = C_t + I_t \Leftrightarrow I_t = S_t$$
$$S_t = sY_t$$
$$N_{t+1} = nN_t$$

- We shall continue to use the first three equations.
- We will drop the fourth equation.
- Only to simplify the analysis, we shall assume no population growth: $n = 1 \Rightarrow N_{t+1} = N_t$.

The Neoclassical Growth Model

- Consider the equations once again:

$$Y_t = F(K_t, N_t)$$
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$Y_t = C_t + I_t$$

- Eliminate Y_t and I_t to obtain a single equation:

$$K_{t+1} = F(K_t, N_t) - C_t + (1-\delta)K_t$$

- Divide both sides by population N_t to obtain

$$\frac{N_{t+1}}{Nt} \times \frac{K_{t+1}}{N_{t+1}} = F\left(\frac{K_t}{N_t}, 1\right) - \frac{C_t}{N_t} + (1-\delta)\frac{K_t}{N_t}$$

The Neoclassical Growth Model

- Consider
$$\frac{N_{t+1}}{Nt} \times \frac{K_{t+1}}{N_{t+1}} = F\left(\frac{K_t}{N_t}, 1\right) - \frac{C_t}{N_t} + (1-\delta)\frac{K_t}{N_t}$$
- Let $k_t = K_t/N_t$ denote the capital-labor ratio.
- Let $c_t = C_t/N_t$ denote consumption per capita.
- Because $N_{t+1} = N_t$ (by assumption), we obtain
$$k_{t+1} = F(k_t, 1) - c_t + (1-\delta)k_t$$
- It is convenient to define $f(k_t) = F(k_t, 1)$. Then,
$$k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$$

The Neoclassical Growth Model

- Instead of imposing $S_t = sY_t$, we shall find the optimal sequence of consumption by solving the infinite-horizon utility-maximization problem:
$$\max_{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
subject to
$$k_{t+1} = f(k_t) - c_t + (1-\delta)k_t \text{ for } t = 0,1,\ldots$$
- The initial condition k_0 is given (parameter).
- Likewise, k_t cannot be chosen in period t. k_t is a **state variable**.

The Neoclassical Growth Model

- The current-value Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}]\}$$

- FOCs are :

$$c_t : u'(c_t) - \lambda_t = 0 \;\text{ for } t = 0,1,\ldots$$

$$k_{t+1} : -\lambda_t + \beta\lambda_{t+1}[f'(k_{t+1}) + 1 - \delta] = 0 \;\text{ for } t = 0,1,\ldots$$

$$\lambda_t : k_{t+1} = f(k_t) - c_t + (1-\delta)k_t \;\text{ for } t = 0,1,\ldots$$

$$\text{TVC} : \lim_{t\to\infty} \beta^t \lambda_t k_t = 0$$

The Neoclassical Growth Model

- Eliminate the multipliers to obtain

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

$$\lim_{t\to\infty} \beta^t \lambda_t k_t = 0$$

$$k_0 : \text{given}$$

- The optimal allocation is determined by the solution to the above system of nonlinear difference equations.

Global Analysis

- Let us start with finding the steady states.
- Let $k_{t+1} = k_t = k$ and $c_{t+1} = c_t = c$ in the system.
- Then, a steady state (k, c) is a solution to

$$\frac{1}{\beta} = f'(k) + 1 - \delta$$

$$c = f(k) - \delta k$$

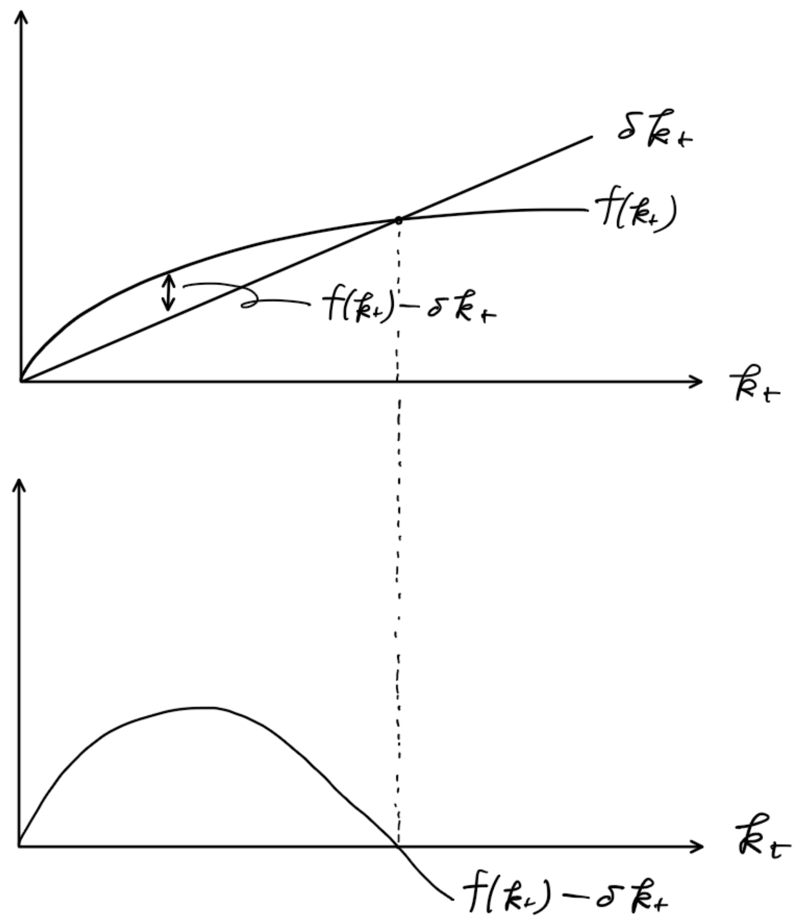
- The first equation determines the value of k. Then, given this value, we can calculate the value c from the second equation.
- Thus, there is a unique steady state.

Global Analysis

- First, consider
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$
- Subtract k_t from both sides to obtain
$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$$
- Thus, k_t in increasing over time ($k_{t+1} > k_t$) if and only if
$$f(k_t) - \delta k_t - c_t > 0 \Leftrightarrow c_t < f(k_t) - \delta k_t$$
- Let us draw a diagram.

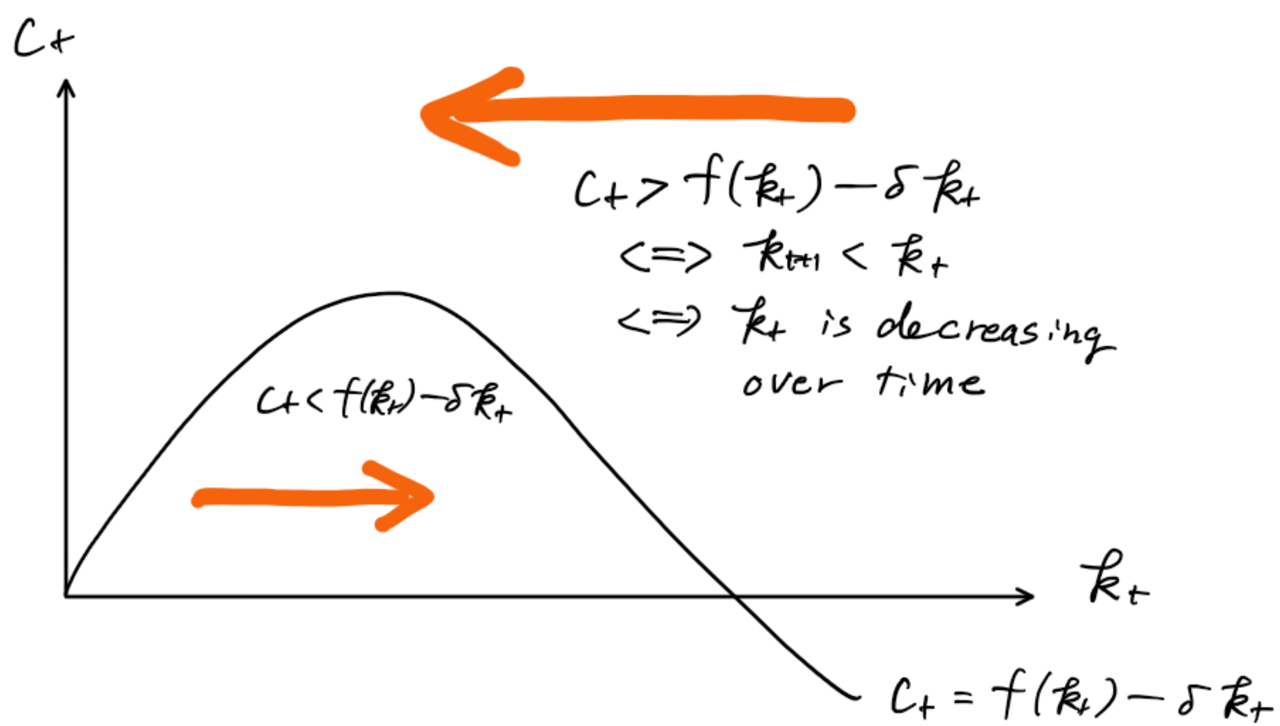
Global Analysis

- To verify the shape of $f(k_t) - \delta k_t$, draw $f(k_t)$ and δk_t separately on the same plane.
- The shape of $f(k_t) - \delta k_t$ is below.



Global Analysis

- k_t in increasing over time ($k_{t+1} > k_t$) if and only if
$$c_t < f(k_t) - \delta k_t$$



Global Analysis

- Next, consider the Euler equation:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta$$

- This equation does not contain $c_{t+1} - c_t$ term.
- Let us therefore consider the condition

$$c_{t+1} > c_t \Leftrightarrow u'(c_{t+1}) < u'(c_t)$$

$$\Leftrightarrow \frac{u'(c_t)}{u'(c_{t+1})} > 1 \Leftrightarrow \frac{u'(c_t)}{\beta u'(c_{t+1})} > \frac{1}{\beta}$$

Global Analysis

- Thus,

$$c_{t+1} > c_t \Leftrightarrow \frac{u'(c_t)}{\beta u'(c_{t+1})} > \frac{1}{\beta}$$
$$\Leftrightarrow f'(k_{t+1}) + 1 - \delta > \frac{1}{\beta}$$

- Therefore,

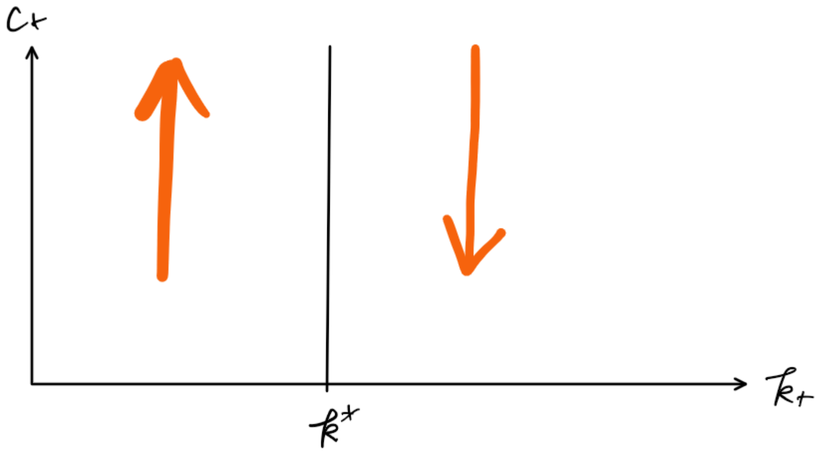
$$c_t > c_{t-1} \Leftrightarrow f'(k_t) + 1 - \delta > \frac{1}{\beta}$$

- Thus, c_t is increasing over time if and only if

$$f'(k_t) > \frac{1}{\beta} - 1 + \delta$$

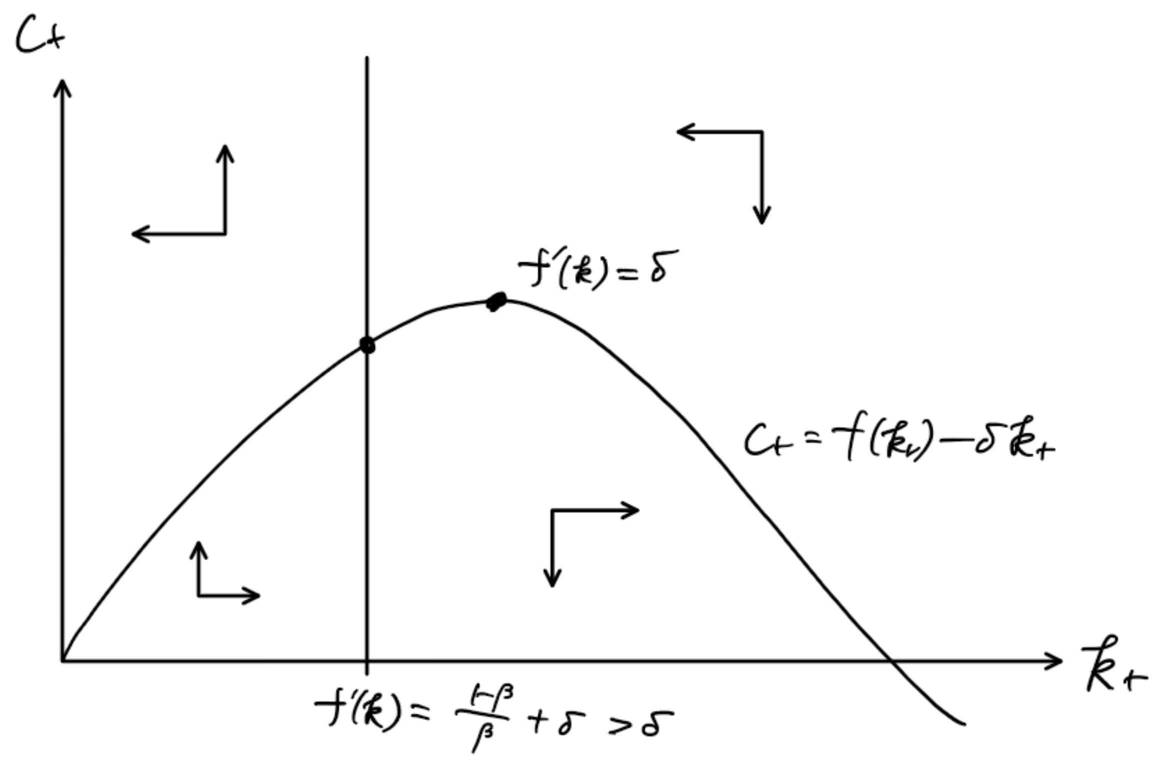
Global Analysis

- Let k^* solve
$$f'(k^*) = \frac{1}{\beta} - 1 + \delta$$
- Then, $f'' < 0$ implies
$$f'(k_t) > \frac{1}{\beta} - 1 + \delta$$
$$\Leftrightarrow k_t < k^*$$
- Thus, c_t increases over time if and only if
$$k_t < k^*$$



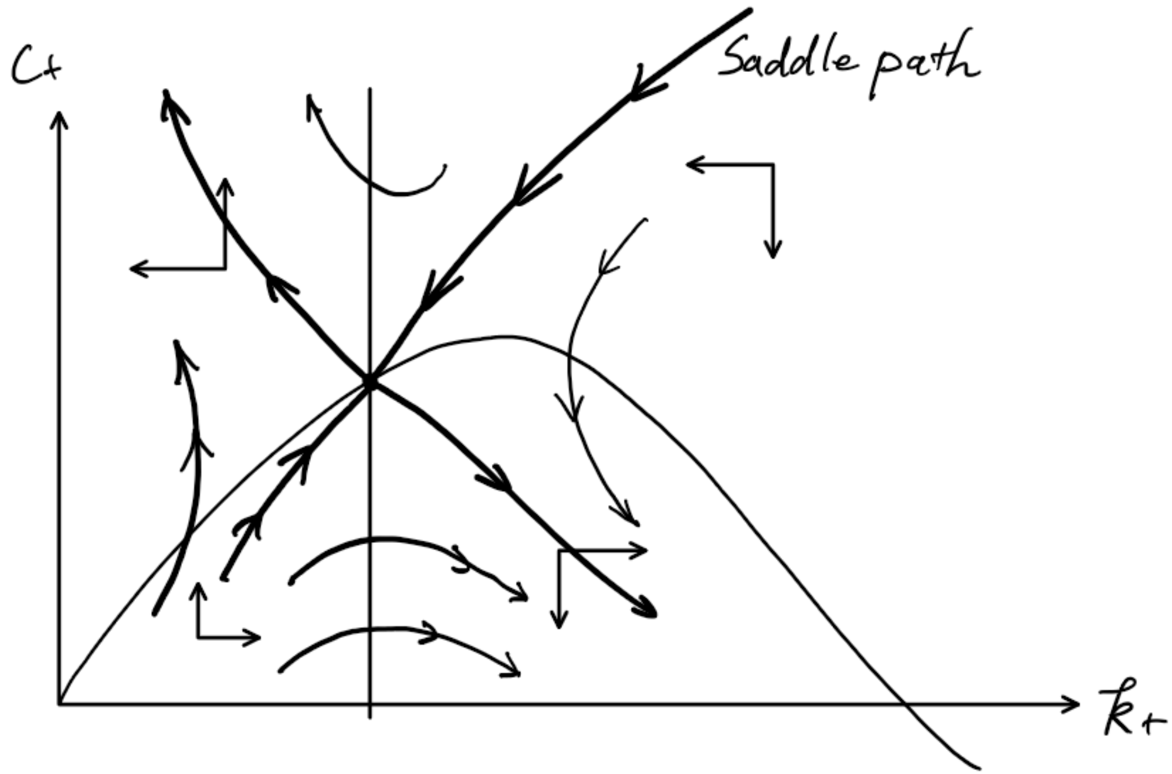
Global Analysis

- There are 4 regions.



Global Analysis

- The steady-state is a saddle.



Global Analysis

- Every trajectory in the phase diagram satisfies the Euler equation and the budget constraint:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta$$
$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$$

- Finally, we impose the transversality condition and the initial condition

$$\lim_{t \to \infty} \beta^t \lambda_t k_t = 0$$
$$k_0 : \text{given}$$

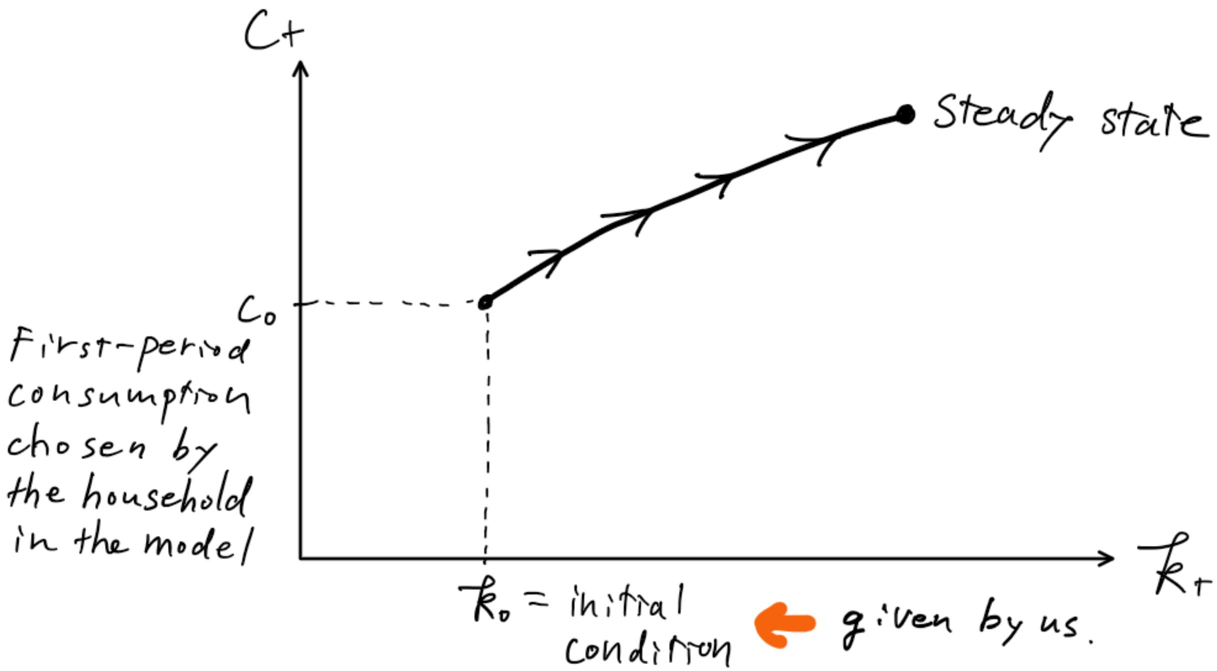
- This gives us a point on the saddle path.

Saddle Path = Unique Solution

- Consider the transversality condition
$$\lim_{t\to\infty} \beta^t \lambda_t k_t = \lim_{t\to\infty} \beta^t u'(c_t) k_t = 0$$
- Notice that all trajectories other than the saddle path will eventually hit an axis or diverge.
- For example, consider a path leading eventually to $k_t \to 0$. Output will be zero and thus $c_t \to 0 \Rightarrow u'(c_t) \to \infty$, violating the transversality condition.
- The only path that never violate the transversality condition is the saddle path.

Saddle Path = Unique Solution

- Optimal sequences $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$ are <u>uniquely determined</u> for each k_0.



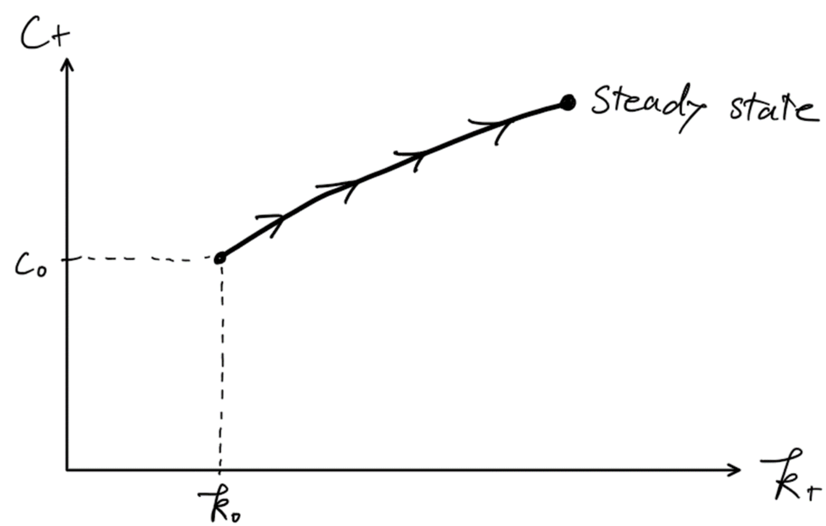
Blanchard and Kahn (1980)

- Consider the local dynamics around a steady state.
- Suppose we obtain the eigenvalues of the linearized system.
- **Theorem**: Let $\#_\lambda$ denote the number of explosive eigenvalues denote. Let $\#_c$ the number of jump variables. Then,

$$\#_\lambda = \#_c \Longrightarrow \text{Solution path is unique}$$
$$\#_\lambda < \#_c \Longrightarrow \text{Solution path is indeterminate}$$
$$\#_\lambda > \#_c \Longrightarrow \text{Solution path does not exist}$$

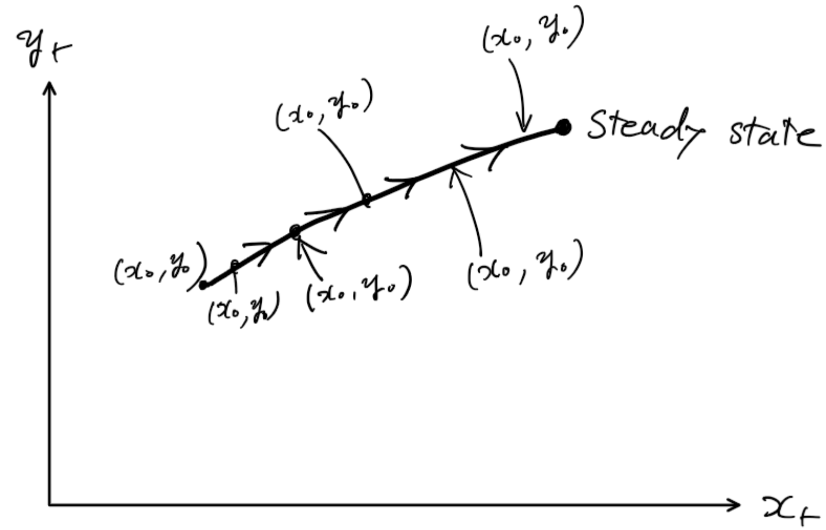
$\#_\lambda = \#_c = 1$

- In this model, the solution path is unique because k_t is a **state variable** and c_t is a **jump variable** (or, **control variable**).
- As you can see, a state variable needs an initial value, while a jump variable is determined within the model.



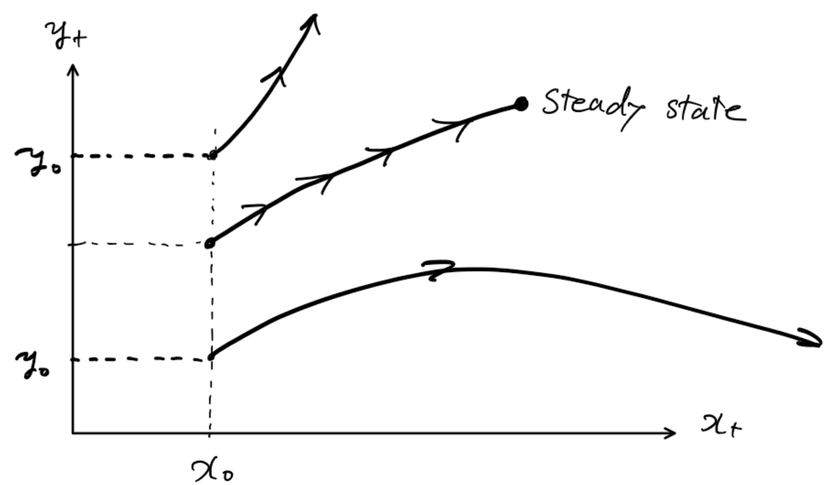
$\#_\lambda = 1 < \#_c = 2$

- Suppose that in some model, the steady state is a saddle, but there are <u>two jumpers</u>.
- Because (x_0, y_0) must be chosen by the household and any point on the saddle path is fine, there is an <u>infinity</u> of solution paths.



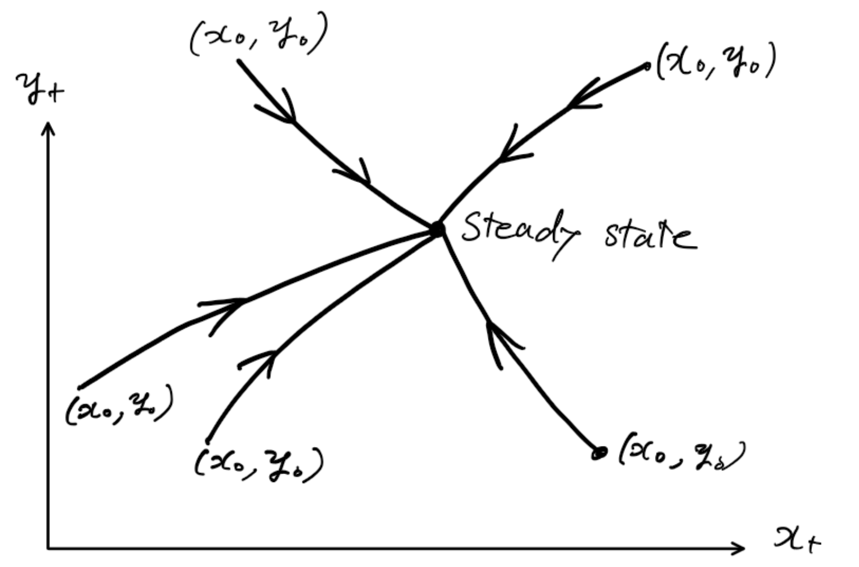
$\#_\lambda = 1 > \#_c = 0$

- Suppose that in some model, the steady state is a saddle, but there is no jumper.
- Because (x_0, y_0) must be chosen outside of the model (by us), the probability that the initial point happens to be on the saddle path is zero.
- Thus, the model explodes.



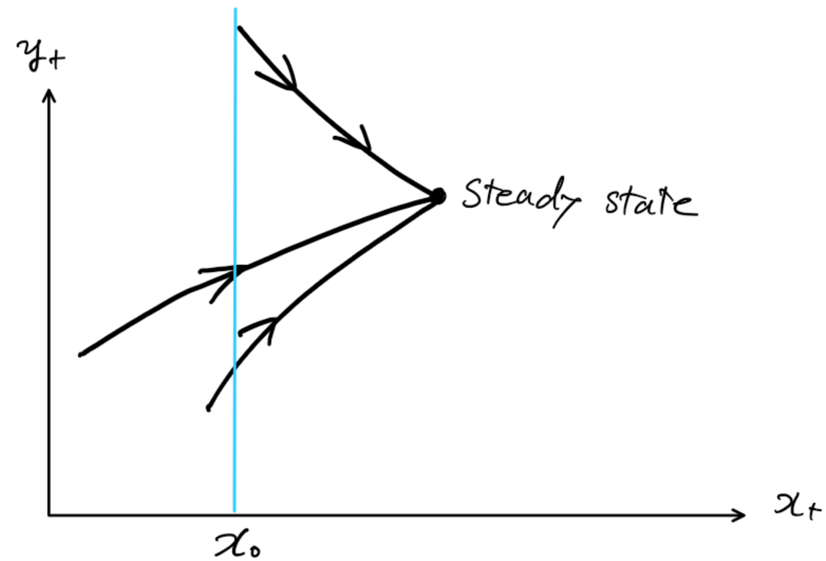
$\#_\lambda = \#_c = 0$

- Suppose that in some model, the steady state is a sink, and there is no jumper.
- Because every trajectory leads to the steady state, any initial point chosen outside of the model is fine.
- The solution path is unique for each initial point (x_0, y_0).



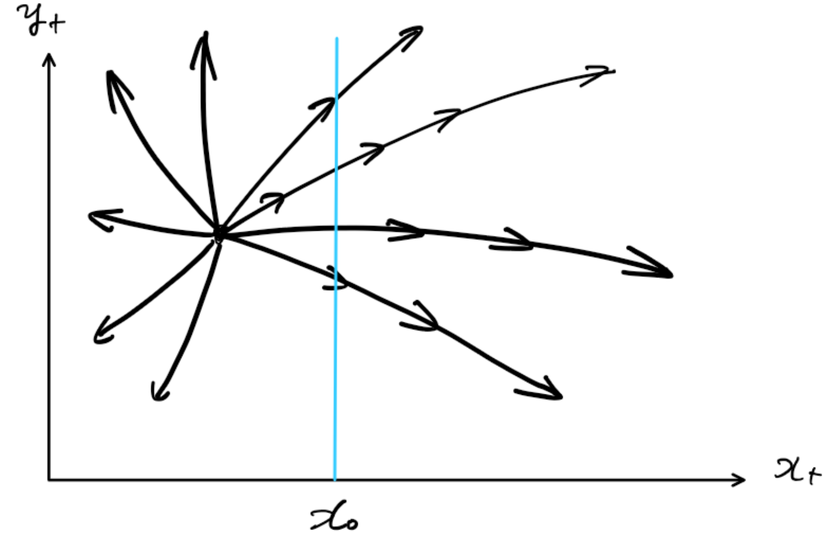
$\#_\lambda = 0 < \#_c = 1$

- Suppose that in some model, the steady state is a sink, and there is one jumper.
- Because every trajectory leads to the steady state, given x_0, there is an infinity of initial value y_0 for the household.
- Thus, the solution path is indeterminate.



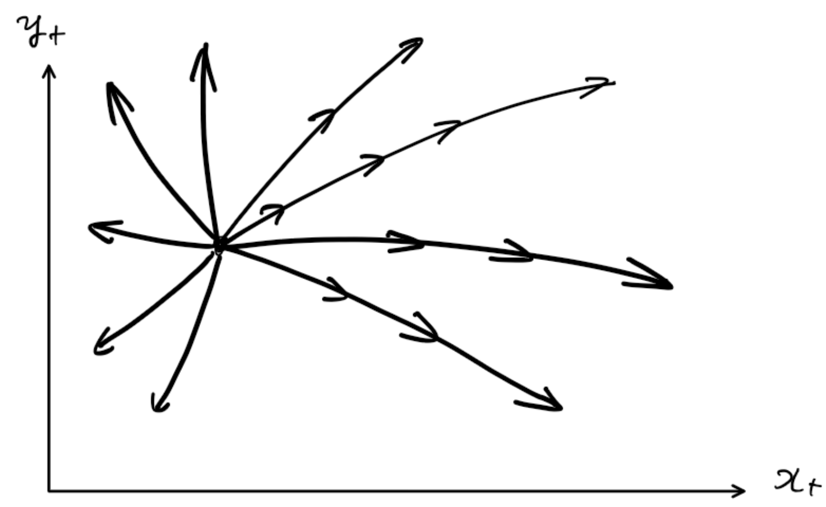
$$\#_\lambda = 2 > \#_c = 1$$

- Suppose that in some model, the steady state is a <u>source</u>, and there is one jumper.
- Given any initial value x_0, every path is explosive.



$\#_\lambda = \#_c = 2$

- Suppose that in some model, the steady state is a source, and there are two jumpers.
- The initial point (x_0, y_0) is entirely chosen by the household.
- The household optimally chooses (x_0, y_0) to be exactly at the steady state, and stay there forever.



Summary

- The Blanchard-Kahn condition is quite useful for checking whether your model has a unique solution.
- In general, any non-monetary model without external effects (such as public goods and environments) has a unique solution or a unique equilibrium.
- In many **monetary models**, however, we often encounter both multiple steady states and local indeterminacy.